Problems on Low-dimensional Topology

Edited by T. Ohtsuki[1]

This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in June 2–4, 2010.

Contents

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1 Surjective homomorphisms between 2-bridge knot groups

(Masaaki Suzuki)

Let K be a knot in S^3 and $G(K)$ the fundamental group of the complement $S^3 - K$. We write $K_1 \geq K_2$ if there exists a surjective homomorphism from $G(K_1)$ onto $G(K_2)$. It is well known that this relation $\geq$ is a partial order on the set of prime knots. This partial order is determined on the set of prime knots with up to 10 crossings in [50]. Furthermore this result is extended to the set of prime knots with up to 11 crossings in [30].

In this section, we focus on surjective homomorphisms between 2-bridge knot groups. Schubert showed that 2-bridge knots are classified by Schubert normal forms, which are rational numbers. Let $K(r)$ denote the 2-bridge knot corresponding to a rational number r. Ohtsuki-Riley-Sakuma [73] gave a systematic construction which, for a given 2-bridge knot $K(r)$, provides surjective homomorphisms from $K(\tilde{r})$ onto $K(r)$. On the other hand, Lee-Sakuma [52] showed that the converse statement holds in a sense, that is, they describes all upper-meridian-pair-preserving surjective homomorphisms between 2-bridge knot groups.

The following is a very simple question about this subject.

Question 1.1 (M. Suzuki). *Does there exist a 2-bridge knot which surjects simultaneously onto $G(3_1)$ and $G(4_1)$?*

The author confirmed that there exists no such 2-bridge knot with up to 20 crossings. The above problem is one of the simplest model of the following problem.

Problem 1.2 (M. Suzuki). *For given two rational numbers r and r', determine whether there exists a 2-bridge knot $K(\tilde{r})$ such that $K(\tilde{r}) \geq K(r)$ and $K(\tilde{r}) \geq K(r')$. In other words, find an algorithm to determine whether there exists such a 2-bridge knot $K(\tilde{r})$ or not.*

If it is possible to answer the above problem, we can describe the Hasse diagram of 2-bridge knots with respect to the partial order $\geq$.

2 Extensions of Burau representation of the braid groups

(Hiroshi Matsuda)

Burau [9] introduced a representation $\varphi_n \colon B_n \to M(n; \mathbb{Z}[t, t^{-1}])$ of the braid group B_n defined by

$$\varphi_n(\sigma_i) = \begin{pmatrix} I_{i-1} & O & O & O \\ O & 1-t & t & O \\ O & 1 & 0 & O \\ O & O & O & I_{n-(i+1)} \end{pmatrix},$$

where $\sigma_1, \cdots, \sigma_{n-1}$ denote the usual generators of B_n. He constructed from φ_n a knot invariant which is essentially equal to the Alexander polynomial $\Delta_K(t)$ of a knot K.

I propose to extend φ_n by choosing a non-commutative algebra Ω instead of $\mathbb{Z}[t, t^{-1}]$. Consider a mapping defined by

$$\psi_n(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & O & O & O \\ \hline O & \alpha & \beta & O \\ O & \gamma & \delta & O \\ \hline O & O & O & I_{n-(i+1)} \end{array} \right),$$

where $\alpha, \beta, \gamma, \delta \in \Omega$. We suppose that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is invertible in $M(2, \Omega)$. The above mapping can be extended to a representation $\psi_n \colon B_n \to M(n; \Omega)$ if $\alpha, \beta, \gamma, \delta$ satisfy some polynomial equations derived from the relation $\psi_3(\sigma_1)\psi_3(\sigma_2)\psi_3(\sigma_1) = \psi_3(\sigma_2)\psi_3(\sigma_1)\psi_3(\sigma_2)$. When the inverses of α and β exist, those polynomial equations are simplified as follows,

$$\begin{cases} \gamma = \alpha^{-1}\beta^{-1}\alpha(1-\alpha), \\ \delta = 1 - \alpha^{-1}\beta^{-1}\alpha\beta, \\ \beta\alpha^{-1}\beta^{-1}\alpha - \alpha^{-1}\beta^{-1}\alpha\beta - \alpha + \beta^{-1}\alpha\beta = 0. \end{cases} \tag{1}$$

(In general, we do not have to assume the existence of α^{-1} and β^{-1}.)

Problem 2.1 (H. Matsuda). *Choosing your favorite algebra Ω with unit 1, construct a knot invariant from the above representation ψ_n.*

When $\Omega = M(2; \mathbb{C})$, it is shown in [58] that there is a 6-parameter family of solutions of (1) and we can construct a knot invariant from the corresponding representation ψ_n, though it might be equal to the product $\Delta_K(t_1)\Delta_K(t_2)$ of two copies of the Alexander polynomial.

3   A conjugation sub-quandle of $\mathrm{PSL}(2, \mathbb{F}_q)$

(Yuichi Kabaya)

Let p be a prime, and let $\mathbb{F}_q$ be the finite field of order $q = p^n$. We define the quandle X_q to be the set $(\mathbb{F}_q^2 \setminus \{0\})/\{\pm 1\}$ with the binary operation given by

$$\begin{pmatrix} c' & d' \end{pmatrix} * \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} c' & d' \end{pmatrix} \begin{pmatrix} 1 + cd & d^2 \\ -c^2 & 1 - cd \end{pmatrix}.$$

We see that X_q can be regarded as a sub-quandle of the conjugation quandle of $\mathrm{PSL}(2, \mathbb{F}_q)$, as follows. A *conjugation quandle* of a group is the group with the binary operation $x * y = y^{-1}xy$. Let $\mathrm{PSL}(2, \mathbb{F}_q)$ be the projective special linear group over $\mathbb{F}_q$, and put $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let X_q' be the sub-quandle $\{g^{-1}hg \,|\, g \in \mathrm{PSL}(2, \mathbb{F}_q)\}$ of the conjugation quandle of $\mathrm{PSL}(2, \mathbb{F}_q)$. Then, X_q' can naturally be identified with $Z(h)\backslash\mathrm{PSL}(2, \mathbb{F}_q)$, where $Z(h)$ denotes the centralizer of h. Further,

since $Z(h) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$, $Z(h)$ is equal to the stabilizer of $\begin{pmatrix} 0 & 1 \end{pmatrix}$ with respect to the right action of $\mathrm{PSL}(2, \mathbb{F}_q)$ on $\mathbb{F}_q^2$. Thus, we have an isomorphism $X_q \to X_q'$ taking $\begin{pmatrix} 0 & 1 \end{pmatrix} g$ to $g^{-1}hg$. That is, putting $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this is the isomorphism taking $\begin{pmatrix} c & d \end{pmatrix}$ to $\begin{pmatrix} 1 + cd & d^2 \\ -c^2 & 1 - cd \end{pmatrix}$.

Many knots admit non-trivial colorings by X_q. Let K be a hyperbolic knot. Then we have a holonomy representation $\varphi : \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathbb{H}^3/\varphi(\pi_1(S^3 \setminus K))$ is isometric to $S^3 \setminus K$. In particular, φ takes each meridian to a parabolic element of $\mathrm{PSL}(2, \mathbb{C})$, which is conjugate to h. Since the holonomy representation φ is characterized by a finite number of algebraic equations in $\mathrm{PSL}(2, \mathbb{C})$, φ is conjugate to a representation into $\mathrm{PSL}(2, \mathcal{F})$ for some finite extension field $\mathcal{F}$ of $\mathbb{Q}$. Further, in many cases,[2] φ is conjugate to a representation into $\mathrm{PSL}(2, \mathcal{O})$, where $\mathcal{O}$ is the ring of algebraic integers of $\mathcal{F}$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ over $(p) \subset \mathbb{Z}$. Then the residue field $\mathcal{O}/\mathfrak{p}$ is an extension of $\mathbb{F}_p$, and hence it is isomorphic to $\mathbb{F}_q$ with some $q = p^n$. This induces a representation $\pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{F}_q)$. Further, it induces a X_q-coloring of a knot diagram D of K, where a X_q-coloring of a knot diagram D is a map from the set of arcs of D to X_q which satisfies the relation shown in Figure 1 at each crossing of D. We can naturally regard a X_q-coloring of D as a quandle homomorphism from the knot quandle of K to X_q.
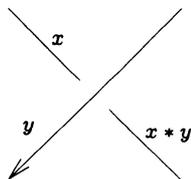
Figure 1: Definition of a coloring at a crossing

Problem 3.1 (Y. Kabaya). *Compute the quandle (co)homology of X_q.*

We have a natural map $H^*(\mathrm{PSL}(2, \mathbb{F}_q)) \to H_Q^*(X_q)$. It is known [69] that "vol $+ \sqrt{-1}\,\mathrm{CS}$" of a 3-manifold can be presented by using a 3-cocycle of $\mathrm{PSL}(2, \mathbb{C})$, where "vol" and "CS" denote the hyperbolic volume and the Chern-Simons invariant; see also [35] for such a presentation using a quandle 3-cocycle. We expect that invariants like "mod p version" of "vol $+ \sqrt{-1}\,\mathrm{CS}$" are obtained from cocycles of X_q.

When $q = 2$, X_2 is isomorphic to the dihedral quandle R_3. When $q = 3$, X_3 is isomorphic to the Alexander quandle $\mathbb{Z}[T^{\pm 1}]/(2, T^2 + T + 1)$.

We define the quandle $\tilde{X}_q$ to be the set $\mathbb{F}_q^2 \setminus \{0\}$ with the same binary operation as X_q. The quotient map $\tilde{X}_q \to X_q$ is a central extension of X_q by $\mathbb{Z}/2\mathbb{Z}$; see [14] for the definition of quandle extensions. This extension seems to be non-trivial.

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[2]For example, if the knot complement has no closed essential surface, the holonomy representation φ is conjugate to a representation into $\mathrm{PSL}(2, \mathcal{O})$; see *e.g.* [57, Theorem 5.2.5], [87, Theorem 6.7.5]. Note also that, even if φ was not conjugate to a representation into $\mathrm{PSL}(2, \mathcal{O})$, φ induces a representation $\pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{F}_q)$ in a similar way, for all but finitely many p.
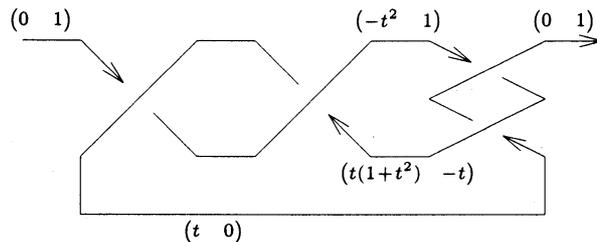
Figure 2: A general non-trivial X_q-coloring of a diagram of the figure-eight knot

For example, we show a general non-trivial X_q-coloring of a diagram of the figure-eight knot in Figure 2. Without loss of generality, we can put colors of two arcs to be $(0\ \ 1)$ and $(t\ \ 0)$ with some non-zero t, as shown in Figure 2. Then, the defining relations of coloring at the left two crossings determine the colors of the other arcs. Further, the defining relations of coloring at the right two crossings give

$$\begin{cases} \pm(-t^2\ \ 1) = \big(t(1+t^2)\ \ -t\big)*(0\ \ 1) = \big(t(1+t^2)\ \ t^3\big), \\ \pm(t\ \ 0) = (0\ \ 1)*\big(t(1+t^2)\ \ -t\big) = \big(-t^2(1+t^2)^2\ \ 1+t^2+t^4\big). \end{cases}$$

They are rewritten

$$\begin{cases} t^2 \pm t + 1 = 0, \\ t^2 \mp t + 1 = 0. \end{cases}$$

Hence, $t^2 \pm t + 1 = 0$. Since its discriminant is -3, it has solutions in $\mathbb{F}_p$ when $\left(\frac{-3}{p}\right) = 1$ or $p = 3$, that is, when $p \equiv 0, 1 \bmod 3$. In this case, we have non-trivial X_p-colorings. Otherwise, $t^2 \pm t + 1$ is irreducible in $\mathbb{F}_p$, and we have $\mathcal{O}/\mathfrak{p} = \mathbb{F}_{p^2}$. In this case, $t^2 \pm t + 1 = 0$ has solutions in $\mathbb{F}_{p^2}$, and we have non-trivial X_{p^2}-colorings.

Similarly, a general non-trivial $\tilde{X}_q$-coloring of the diagram of the figure-eight knot is given by

$$\begin{cases} t^2 + t + 1 = 0, \\ t^2 - t + 1 = 0. \end{cases}$$

Hence, $2t = 0$. Therefore, when p is an odd prime, we have no non-trivial $\tilde{X}_q$-colorings, and any non-trivial X_q-coloring does not lift to an $\tilde{X}_q$-coloring. This suggests that $H_Q^2(X_q; \mathbb{Z}/2\mathbb{Z})$ is non-trivial, since the quandle cocycle invariant derived from a cocycle of this cohomology can be regarded as an obstruction of such lifting.

From the viewpoint of a relation to "vol$+\sqrt{-1}$ CS", it might be better to consider cocycles of PSL$(2, \mathbb{F}_{p^m})$ and quandle cocycle invariants associated to them for all m at the same time fixing p. The behavior of the numbers of representations of a knot group to PSL$(2, \mathbb{F}_{p^m})$ can be described by the congruence zeta function (Weil conjecture); see [84]. It might be a problem to describe the behavior of quandle cocycle invariants associated to such a family of cocycles.

4 C_k-concordance among string links

(Jean-Baptiste Meilhan, Akira Yasuhara)

A C_k-*move* is a local move on (string) links as illustrated in Figure 3, in particular, a C_1-move is a crossing change [26]. (The C_k-move can also be defined by using the theory of claspers [26].) The C_k-move generates an equivalence relation on (string) links, called C_k-*equivalence*. The C_k-equivalence (string) links share all finite type invariants of degree $\leq k - 1$ [26].
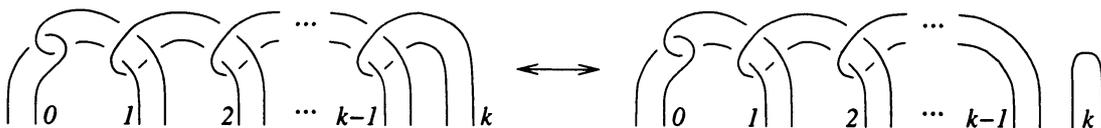
Figure 3: A C_k-move involves $k + 1$ strands of a link, labeled here by integers between 0 and k.

Two n-string links L, L' are *concordant* if there is an embedding

$$f : (\sqcup_{i=1}^n [0,1]_i) \times [0,1] \longrightarrow (D^2 \times [0,1]) \times [0,1]$$

such that $f((\sqcup_{i=1}^n[0,1]_i) \times \{0\}) = L \times \{0\}$ and $f((\sqcup_{i=1}^n[0,1]_i) \times \{1\}) = L' \times \{1\}$, and such that $f(\partial(\sqcup_{i=1}^n[0,1]_i) \times [0,1]) = (\partial L) \times [0,1]$. Here, $\sqcup_{i=1}^n[0,1]_i$ is the disjoint union of n copies of $[0,1]$. Two n-string links L, L' are C_k-*concordant* if there is a sequence $L = L_0, L_1, ..., L_m = L'$ such that for each $i \geq 1$, L_i and L' are either C_k-equivalent or concordant. The C_k-concordant string links share all finite type, concordance invariants of degree $\leq k-1$. It seems to be natural to ask the following.

Question 4.1 (J.-B. Meilhan, A. Yasuhara). *Let L, L' be n-string links. Is it true that L and L' are C_k-concordant if and only if they share all finite type concordance invariants of degree $\leq k - 1$?*

It is known that the Milnor invariants are concordance invariants and the Milnor invariants of length $\leq k$ are C_k-equivalence invariants [26]. Habegger and Masbaum showed that all *rational* finite type concordance invariants of string links are given by Milnor invariants via the Kontsevich integral [25]. It is first essentially shown by Ng [70] that, for an integer $k \geq 3$, two knots (1-string links) K and K' are C_k-concordant if and only if $\mathrm{Arf}(K) = \mathrm{Arf}(K')$. Any knot (1-string link) is C_2-equivalent to the trivial one [68]. Two n-string links are C_2-concordant if and only if they share all invariants $\mu(ij)$ $(1 \leq i < j \leq n)$ [68]. Thus we have an affirmative answer to Question 4.1 for $n = 1$ or $k = 2$. Hence we may assume that $n \geq 2$ and $k \geq 3$.

Remark 4.2 ([61]).

(1) Two n-string links are C_3-concordant if and only if they share all invariants Arf_i $(1 \leq i \leq n)$, the Milnor invariants of length ≤ 3, and $\mu(jiij)$ mod 2 $(1 \leq i < j \leq n)$. Here Arf_i is the Arf invariant for the closure of the ith-component of a string link.

(2) Two n-string links are C_5-concordant if and only if they share all invariants

Arf$_i$ $(1 \leq i \leq n)$, the Milnor invariants of length ≤ 5, $\mu(jiiiij)$ mod 2, $(1 \leq i, j \leq n)$, and $\mu(kijjik)$ mod 2 $(1 \leq i, j < k \leq n)$.
(3) For $k = 4, 6$, two n-string links are C_k-concordant if and only if they share all invariants Arf$_i$ $(1 \leq i \leq n)$ and the Milnor invariants of length $\leq k$.

The classification of string links up to C_3-concordance is given by concordance finite type invariants of degree ≤ 2, and hence rational concordance finite type invariants of degree 2, together with the invariants Arf$_i$, do not suffice to determine all concordance finite type invariants of degree 2 [61, Remark 5.14].

Remark 4.2 (1), (3) give an affirmative answer to Question 4.1 for $k = 3, 4$ and 6. So far we have not known if $\mu(jiiiij)$ (mod 2) and $\mu(kijjik)$ (mod 2) are finite type invariants of degree ≤ 4 while they are of degree 5. In general, $\mu(i_0, ..., i_m, i_m, ..., i_0)$ (mod 2) $(i_0, ..., i_m \in \{1, ..., n\})$ are C_{2m+1}-equivalence invariants [61, Remark 5.4]. Hence we have the following question.

Question 4.3 (J.-B. Meilhan, A. Yasuhara). *For an integer $m \geq 3$, are $\mu(i_0, ..., i_m, i_m, ..., i_0)$ (mod 2) $(i_0, ..., i_m \in \{1, ..., n\})$ finite type invariants of degree $\leq 2m$?*

5 Planar algebras

(Nobuya Sato)

The notion of planar algebra was introduced by Jones [38]; for introductions to planar algebras, see [65, 80]. The definition of (subfactor) planar algebras do not need any knowledge of subfactors. In the study of planar algebras, it is a problem to give a presentation by generators and relations for every planar algebra.

For example, it is known [65] that, putting $q = \exp\left(\frac{2\pi\sqrt{-1}}{8n-4}\right)$, the D_{2n} subfactor planar algebra is generated by a single element S with $4n - 4$ legs, subject to the following relations,
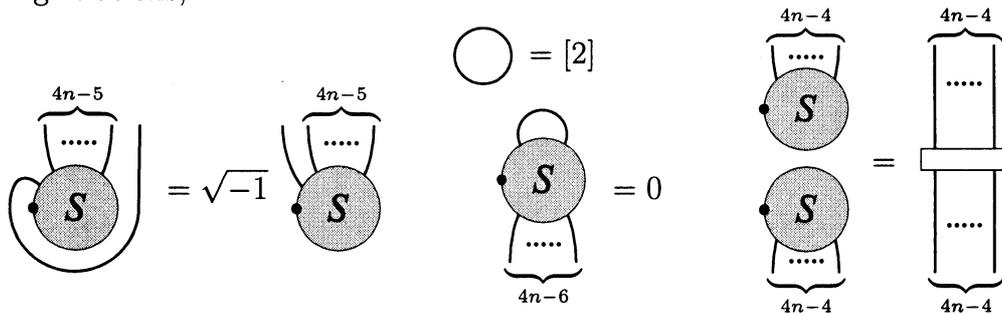
where the box denotes the Jones-Wenzl idempotent, and we put $[n] = (q^n - q^{-n})/(q - q^{-1})$. We define a crossing in this planar algebra by a defining relation of the linear skein of the Kauffman bracket,
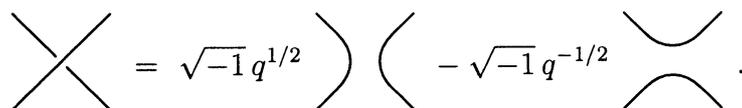
Then, it is shown [65] that S satisfies the following relation, called *partial braiding*,

$$= \quad = \quad -$$

The D_{2n} subfactor planar algebra guarantees that the linear skein involving such S is well defined, and the space of closed diagrams of this linear skein is 1-dimensional. Further, we define two projectors by

$$P = \frac{1}{2}\left(\quad + \quad \right), \qquad Q = \frac{1}{2}\left(\quad - \quad \right).$$

By substituting P and Q into a knot diagram of a knot, we obtain an invariant of this knot, which is shown to be equal to a colored Jones polynomial at a root of unity [66]. Further, it is shown in [66] that, for an ℓ-component link L and a positive integer k, we obtain an invariant of L as the sum of all ways of substituting k copies of P and $\ell - k$ copies of Q into the components of L.

Problem 5.1 (N. Sato). *Find more examples of the skein theory of subfactor planar algebras with partial braidings. Then, investigate link invariants associated with them. What can we say for them?*

As other examples, it is known [4] that the E_6 (resp. E_8) subfactor planar algebra is generated by a single element with 6 legs (resp. 10 legs) subject to some relations. Since these planar algebra do not have braidings, we can not define link invariants from them. However, we can define the state-sum invariants of 3-manifolds from the E_6 and E_8 subfactors by associating a tetrahedron of a triangulation of a 3-manifold with $6j$-symbol of such subfactors; for details, see *e.g.* [89].

Problem 5.2 (N. Sato). *Give a skein theoretic construction of the state-sum invariant of 3-manifolds for the E_6 and E_8 subfactors.*

6 The Rasmussen invariant of a knot

(Tetsuya Abe)

In [76], Rasmussen introduced a smooth concordance invariant of a knot K, now called the Rasmussen invariant $s(K)$. This gives a lower bound for the 4-ball genus $g_*(K)$ of a knot K as follows,

$$|s(K)| \leq 2g_*(K).$$

In particular, if $s(K) \neq 0$, then we can prove that K is not (smoothly) slice (*i.e.*, $g_*(K) > 0$). A motivation for studying the Rasmussen invariant is to find and

describe non-slice knots with trivial Alexander polynomial systematically by calculating the Rasmussen invariant, noting that a knot with trivial Alexander polynomial is always topologically slice[3] ([15]).

We briefly review the definition of the Rasmussen invariant. For a diagram D of a knot K, Lee [53] introduced Lee's complex $C_*^{\text{Lee}}(D)$ and its homology $H_*^{\text{Lee}}(K)$, and proved that

$$H_*^{\text{Lee}}(K) \cong \begin{cases} \mathbb{Q}^2 & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Further, Lee [53] constructed two concrete cycles[4] $f_o(D)$ and $f_{\bar{o}}(D)$ of $C_*^{\text{Lee}}(D)$ from an orientation o and its opposite orientation $\bar{o}$ of D, and proved that $[f_o(D)]$ and $[f_{\bar{o}}(D)]$ form a basis of $H_0^{\text{Lee}}(K)$, where $[\,\cdot\,]$ denotes its homology class. Rasmussen [76] defined a filtration grading $u \colon C_*^{\text{Lee}}(D) \setminus \{0\} \to \{\text{odd integers}\}$ which induces a descending filtration on $C_*^{\text{Lee}}(D)$,

$$C_*^{\text{Lee}}(D) \supset \cdots \supset \mathcal{F}^{2k-1}C_*^{\text{Lee}}(D) \supset \mathcal{F}^{2k+1}C_*^{\text{Lee}}(D) \supset \mathcal{F}^{2k+3}C_*^{\text{Lee}}(D) \supset \cdots.$$

Then the filtration grading s on a non-zero element $[x] \in H_*^{\text{Lee}}(K)$ is defined by

$$s([x]) = \max\big\{q(y) \mid [x] = [y]\big\},$$

which induces a filtration on $H_*^{\text{Lee}}(K)$. It is shown in [76] that this filtration is of the following form,

$$\mathbb{Q}^2 \cong H_0^{\text{Lee}}(D) = \mathcal{F}^{2m-1}H_0^{\text{Lee}}(D) \supsetneq \mathcal{F}^{2m+1}H_0^{\text{Lee}}(D) \supsetneq \mathcal{F}^{2m+3}H_0^{\text{Lee}}(D) = \{0\},$$

for some m, and that $[f_o(D)], [f_{\bar{o}}(D)] \in \mathcal{F}^{2m-1}H_0^{\text{Lee}}(D) \setminus \mathcal{F}^{2m+1}H_0^{\text{Lee}}(D)$. The *Rasmussen invariant* $s(K)$ of K is defined to be $2m$, i.e., $s(K) = s([f_o(D)]) + 1 = s([f_{\bar{o}}(D)]) + 1$. It is shown in [76] that $s(K)$ does not dependent on the choice of a diagram D of K.

It is algorithmically possible to compute $s(K)$ by definition for each K, but, in practice, it is hard to compute $s(K)$ from the definition by hand for all but some simplest knots, since the computational complexity is of exponential order of the number of crossings of a diagram of K. It is a problem to find efficient methods to compute $s(K)$. There are some known methods to compute $s(K)$ so far.

- $s(K)$ can be computed from the Khovanov homology (see [76]), and there are fast computer programs to compute the Khovanov homology.

- For any alternating knot K, $s(K)$ is equal to the signature of K [76].

- For any positive knot K, $s(K)$ can be computed from the definition; see [76].

- For any strongly quasipositive knot K, it is known [55, Theorem 4][5] (see also [83]) that $s(K) = 2g_*(K) = 2g(K)$, where $g(K)$ denotes the genus of K, and

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[3]A knot in S^3 is (smoothly) *slice* if it bounds a smooth disc in B^4 whose boundary is S^3. A knot in S^3 is *topologically slice* if it bounds a locally flat topological disc in B^4.

[4]In [76], these are denoted by s_o and $s_{\bar{o}}$ respectively.

[5]It is known [79, Theorem 4.90] that the assumption of [55, Theorem 4] is equivalent to the condition that K is strongly quasipositive. Though the conclusion of [55, Theorem 4] is $\tau(K) = g_*(K) = g(K)$, we can replace $\tau(K)$ with $s(K)/2$, since the resulting formula can be proved in the same way.

we can determine $s(K)$ from $g_*(K)$ or $g(K)$ when we know the value of $g_*(K)$ or $g(K)$.

- There is a table [12] of the values of $s(K)$ for knots with up to 11 crossings.

A lower bound of $s(K)$ is given by the slice-Bennequin inequality [75, 83]

$$w(D) - O(D) + 1 \leq s(K)$$

for any diagram D of K, where $w(D)$ denotes the writhe of D and $O(D)$ denotes the number of Seifert circles of D; see also [45, 46, 56] for its refinements.

(a)   **(b)**
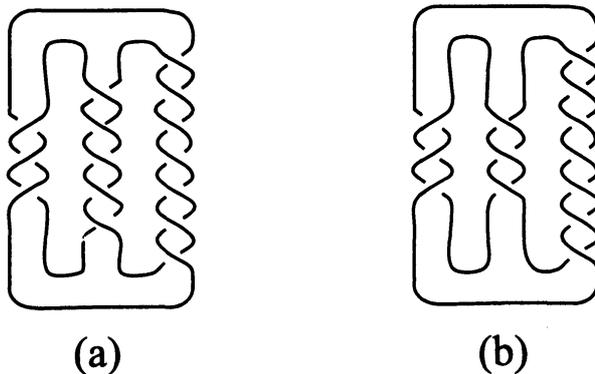
Figure 4: The pretzel knots $P(3, -5, -7)$ and $P(3, -3, -7)$

We consider the Rasmussen invariant of the pretzel knot $P(a, b, c)$ of type (a, b, c) (see Figure 4) for odd integers a, b, c. Note that $|s(P(a, b, c))| \leq 2g_*(s(P(a, b, c))) \leq 2g(s(P(a, b, c))) \leq 2$. When a, b, c are positive odd integers, $P(a, b, c)$ is alternating, and its Rasmussen invariant is equal to its signature. Further, since $s(\overline{K}) = -s(K)$ for the mirror image of K, $s(P(-a, -b, -c)) = -s(P(a, b, c))$. Hence, it is sufficient to study the following case; it is shown in [2][6] that for positive odd integers a, b, c,

$$s(P(a, -b, -c)) = \begin{cases} 0 & \text{if } a \geq \min\{b, c\}, \\ 2 & \text{if } a < \min\{b, c\}. \end{cases}$$

Problem 6.1 (T. Abe). *Let a, b, c be positive odd integers, and let D be the standard diagram of the pretzel knot $P(a, -b, -c)$. Then, give an explicit presentation of a cycle $f(D) \in C_*^{\text{Lee}}(D)$ such that $[f_o(D)] = [f(D)]$ and $q(f(D)) = 1$.*

We expect that such a cycle has particular properties, and such properties would be useful to compute $s(K)$ for a general knot K.

When $ab + bc + ca = -1$, the Alexander polynomial of $P(a, b, c)$ is trivial. In fact,[7] there are infinitely many such pretzel knots.

The cycle $f_2(D)$ given in [3] satisfies that $q(f_2(D)) \geq -1$. In particular, when $a \geq \min\{b, c\}$,

$$0 = s(P(a, -b, -c)) \geq q(f_2(D)) + 1 \geq 0,$$

---

[6]It has partially been shown in [85].

[7]When $|\ell|$ is a divisor of $k(k+1)$, $(a, b, c) = (2k+1, 2\ell-2k-1, 2k(k+1)/\ell-2k-1)$ satisfies that $ab+bc+ca = -1$.

and the equalities of this formula hold, and, hence, $f_2(D)$ is a solution to Problem 6.1. So, it is a problem to find a solution when $a < \min\{b, c\}$.

For example, $s(P(3, -5, -7))$ is computed in [83] by finding a strongly quasipositive diagram of $P(3, -5, -7)$ (see [83, Figure 1]) which is different from the standard diagram shown in Figure 4 (a). It means that in [83] the difficulty of the computation is resolved by finding an appropriate diagram of the knot. An intention of Problem 6.1 is to resolve the difficulty by finding an appropriate cycle for a given diagram.

7 Exceptional surgeries on knots

(Kazuhiro Ichihara, In Dae Jong)

Let K be a knot in the 3-sphere S^3 and γ a slope (i.e., an isotopy class of nontrivial simple closed curves) on the boundary of a tubular neighborhood $N(K)$ of K. Then, the *Dehn surgery on K along a slope γ* is defined as an operation to create a (new) 3-manifold as follows: Take the exterior $E(K)$ of the knot K (i.e., removing the interior of $N(K)$), and glue a solid torus V to $E(K)$ so that a simple closed curve representing γ bounds a meridian disk in V. The manifold obtained by the Dehn surgery on K along a slope γ is denoted by $K(\gamma)$. It is well-known that slopes on the boundary torus $\partial E(K)$ are parameterized by $\mathbb{Q} \cup \{1/0\}$ by using the standard meridian-longitude system for K. Thus, when γ corresponds to $r \in \mathbb{Q} \cup \{1/0\}$, we call the Dehn surgery along a slope γ the *r-Dehn surgery* or *r-surgery* for brevity, and denote $K(\gamma)$ by $K(r)$. See [47], [77] for basic references.

A knot with the complement which admits a complete hyperbolic structure of finite volume is called a *hyperbolic knot*. It is well-known that, on a hyperbolic knot, Dehn surgeries yielding non-hyperbolic manifolds are only finitely many. This fact was originally established by Thurston in [87] as a part of Hyperbolic Dehn Surgery Theorem. Thereby a Dehn surgery on a hyperbolic knot yielding a non-hyperbolic manifold is called *exceptional*. In view of this, it is an interesting and challenging problem to determine and classify all exceptional surgeries on hyperbolic knots in S^3, where we recall that, as a consequence of the affirmative solution to the Geometrization Conjecture, exceptional surgeries are classified into the following three types; a Seifert fibered surgery, a toroidal surgery, a reducible surgery. A Dehn surgery is said to be *Seifert fibered / toroidal / reducible* if which yields a Seifert fibered / toroidal / reducible manifold respectively.

In this section, we address two problems and review known facts about the problems.

Problem 7.1 (K. Ichihara, I. D. Jong). *Determine and classify exceptional surgeries on alternating knots in the 3-sphere.*

A knot in S^3 is called *alternating* if it admits a diagram with alternatively arranged over-crossings and under-crossings running along it. Note that Menasco showed in [62] that an alternating knot in S^3 is hyperbolic unless it is a $(2, p)$-torus

knot $T_{2,p}$. Here $T_{a,b}$ denotes the torus knot of type (a, b), that is, the knot isotopic to the (a, b)-curve on the standard embedded torus in S^3.

Typical example of alternating knots are given by *two-bridge knots*, i.e., knots with bridge index two. On two-bridge knots, exceptional surgeries are completely classified by Brittenham and Wu in [8].

Concerning reducible surgeries, it was shown by Menasco and Thistlethwaite in [63, Corollary 1.1] that a hyperbolic alternating knot has no reducible surgery.

Concerning toroidal surgeries, in [74], based on the result obtained in [63], Patton claimed that if an alternating knot admits a toroidal surgery, then it is either a 2-bridge knot or a 3-strand pretzel knot. This is also achieved as [7, Lemmas 3.1 and 3.3]. Thus, together with the results by Brittenham and Wu in [8] and Wu in [91], toroidal surgeries on alternating knots have been completely classified.

Concerning Seifert fibered surgeries, Delman and Roberts showed in [13] that no Dehn surgeries on a hyperbolic alternating knot yield 3-manifolds with finite fundamental groups. Note that irreducible 3-manifolds with finite fundamental groups must be Seifert fibered. Also, in the forthcoming paper [34], the authors with Shigeru Mizushima showed that any alternating knot other than the trefoil yields no Dehn surgeries yielding 3-manifolds which are toroidal and Seifert fibered, extending the result obtained by Motegi in [67].

On the other hand, Lackenby in [51] showed that a hyperbolic alternating knot admitting a "sufficiently complicated" alternating diagram has no exceptional surgeries. Further all exceptional surgeries on alternating knots are integral surgeries, which was shown by the first author in [31]. Namely, if r-surgery on a hyperbolic alternating knot is exceptional, then $r \in \mathbb{Z}$.

Problem 7.2 (K. Ichihara, I. D. Jong). *Determine and classify exceptional surgeries on Montesinos knots in the 3-sphere.*

A *Montesinos knot* of type $(R_1, R_2, \ldots, R_l)$, denoted by $M(R_1, R_2, \ldots, R_l)$, is defined as a knot admitting a diagram obtained by putting rational tangles $R_1, R_2, \ldots, R_l$ together in a circle. Remark that, here and in the sequel, we abuse R_i to denote a rational number (or, an irreducible fraction) or the corresponding rational tangle depending on the context. The minimal number of such rational tangles is called the *length* of a Montesinos knot. In particular, a Montesinos knot K is called a *pretzel knot* of type $(a_1, a_2, \ldots, a_l)$, denoted by $P(a_1, a_2, \ldots, a_l)$, if the rational tangles in K are of the form $1/a_1, 1/a_2, \ldots, 1/a_l$.

It is already known which Montesinos knots are non-hyperbolic. If the length of a Montesinos knot is less than three, then it actually is a two-bridge knot. Then Menasco in [62] showed that non-hyperbolic two-bridge knots are just the torus knots $T_{2,x}$. Otherwise, non-trivial non-hyperbolic Montesinos knots are just $P(-2, 3, 3)$ and $P(-2, 3, 5)$, which are actually $T_{3,4}$ and $T_{3,5}$ respectively. This was originally shown by Oertel in [72, Corollary 5] together with the result in the unpublished monograph in [5] by Bonahon and Siebenmann, recently version-uped as [6].

A Montesinos knot of length $l < 3$ is a two-bridge knot and thus exceptional surgeries on such knots are completely determined in [8]. On the other hand, it is

shown by Wu in [90] that a Montesinos knot of length $l > 3$ admits no exceptional surgery. Therefore the remaining is the case where the length is just three.

In [90], Wu showed that there are no reducible surgeries on hyperbolic Montesinos knots. It was also obtained by Wu in [91] that a complete classification of toroidal surgeries on Montesinos knots of length three. Furthermore, the authors gave a complete classification of surgeries to give 3-manifolds with cyclic or finite fundamental groups on Montesinos knots in [32], and showed that there are no toroidal Seifert surgeries on on Montesinos knots other than the trefoil in [33]. Therefore the remaining open is to determine and classify all atoroidal Seifert fibered surgeries with infinite fundamental groups on Montesinos knots of length three.

Recently in [92], Wu have further obtained new restriction on Seifert fibered surgeries on Montesinos knots. In fact, it was shown that if $\frac{1}{q_1-1} + \frac{1}{q_2-1} + \frac{1}{q_3-1} \leq 1$, then a Montesinos knot of type $(p_1/q_1, p_2/q_2, p_3/q_3)$ admits no atoroidal Seifert fibered surgery. As a consequence, we have a classification of exceptional Dehn surgeries on all Montesinos knots except those $M(p_1/q_1, p_2/q_2, p_3/q_3)$ with $q_1 = 2$, or $(q_1, q_2) = (3, 3)$, or $(q_1, q_2, q_3) = (3, 4, 5)$ for $q_1 \leq q_2 \leq q_3$. Further restrictions on p_i and the surgery slopes for these cases were also given. See [92, Theorems 8.2 and 8.3] for details.

It is already known which Montesinos knots are alternating. Actually, by the claim given in [86, Section 4, 2nd paragraph], it is shown that a Montesinos knot is alternating if and only if its reduced Montesinos diagram, introduced in [54], is alternating. This means that a Montesinos knot is alternating if and only if it is expressed as $M(R_1, R_2, \ldots, R_l)$ with all R_i's having the same sign.

Very recently, in [34], for alternating Montesinos knots other than two-bridge knots, we have obtained that there are no Seifert fibered surgeries.

8 Minimal dilatation of closed surfaces

(Eiko Kin, Mitsuhiko Takasawa)

Let $\mathrm{Mod}(\Sigma)$ be the mapping class group on an orientable surface Σ. An element $\phi \in \mathrm{Mod}(\Sigma)$ which contains a pseudo-Anosov homeomorphism $\Phi : \Sigma \to \Sigma$ as a representative is called a pseudo-Anosov mapping class. One of the numerical invariants for pseudo-Anosov mapping classes is the *dilatation*. The dilatation $\lambda(\phi) > 1$ is defined to be the expansion constant $\lambda(\Phi)$ for the invariant foliation of Φ.

Fixing the surface Σ, the dilatation $\lambda(\phi)$ for a pseudo-Anosov element $\phi \in \mathrm{Mod}(\Sigma)$ is known to be an algebraic integer with a bounded degree depending only on the topological type of Σ. For each $c > 1$, the set of dilatations $\lambda(\phi)$ for $\phi \in \mathrm{Mod}(\Sigma)$ bounded by c from above is finite (Ivanov). In particular the set

$$\mathrm{Dil}(\Sigma) = \{\lambda(\phi) \mid \text{pseudo-Anosov } \phi \in \mathrm{Mod}(\Sigma)\}$$

achieves a minimum $\delta(\Sigma)$. We denote by δ_g, the minimal dilatation $\delta(\Sigma_g)$ for a closed surface Σ_g of genus g.

Penner proved that $\log \delta_g \asymp 1/g$. The following question was posed by McMullen.

Question 8.1 (McMullen [60]). *Does* $\lim\limits_{g \to \infty} g \log \delta_g$ *exist?* $(\Longleftrightarrow$ *Does* $\lim\limits_{g \to \infty} |\chi(\Sigma_g)| \log \delta_g$ *exist?) What is its value?*

On the question above, it is known (Hironaka [29], Aaber-Dunfield [1], Kin-Takasawa [49]) that

$$\limsup_{g \to \infty} g \log \delta_g \leq \log(\tfrac{3+\sqrt{5}}{2}) \quad (= \log(1 + \text{golden ratio})).$$

Question 8.2 (E. Kin, M. Takasawa). *Is it true that*

$$\lim_{g \to \infty} g \log \delta_g = \log(\tfrac{3+\sqrt{5}}{2})?$$

9 The magic "magic manifold"

(Eiko Kin, Mitsuhiko Takasawa)

The magic manifold N, which is the exterior of the 3-component link shown in Figure 5, is an intriguing example. Many pseudo-Anosov homeomorphisms with the smallest known dilatation occur as the monodromies of Dehn fillings of N [1, 29, 48, 49]. For example, we have: For each $n \geq 9$ (resp. $n = 3, 4, 5, 7, 8$), the pseudo-Anosov homeomorphism on an n-punctured disk D_n with the smallest known dilatation (resp. smallest dilatation) occurs as the monodromy on a fiber for a Dehn filling of N [48].
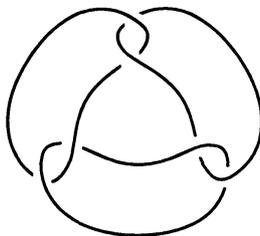
Figure 5: The 3-component link, whose exterior is the magic manifold

We pose the following question.

Question 9.1 (E. Kin, M. Takasawa). *Does the magic manifold N satisfy the following properties (1),(2)?*

(1) *There exist Dehn fillings of N giving an infinite sequence of fiberings over S^1, with closed fibers Σ_{g_i} of genus $g_i \geq 2$ with $g_i \to \infty$, and with monodromy Φ_i so that $\delta_{g_i} = \lambda(\Phi_i)$.*

(2) *There exist Dehn fillings of N giving an infinite sequence of fiberings over S^1, with fibers D_{n_i} having n_i punctures with $n_i \to \infty$, and with monodromy Φ_i so that $\delta(D_{n_i}) = \lambda(\Phi_i)$.*

10   The w-index of surface links

(Masahide Iwakiri)

A *surface link* is the image of a smooth embedding of a closed surface into $\mathbb{R}^4$. It is known [44, 88] that any surface link can be presented by the closure of some surface braid. We define the *w-index* $w(F)$ of a surface link F by

$$w(F) := \min \left\{ w(S) \ \middle| \ \begin{array}{l} S \text{ is a surface braid} \\ \text{whose closure is isotopic to } F \end{array} \right\}$$

where $w(S)$ is the number of triple points of S.

A *quandle* was defined by Joyce [39] (see also [59]). Let A be an abelian group, and let f be an A-valued 3-cocycle of a finite quandle. In [10], the quandle cocycle invariant $\Phi_f(F)$ of a surface link F was defined to be in $\mathbb{Z}[A]$. We can make an estimate of the triple point number of F by using the quandle cocycle invariant when $\Phi_f(F) \notin \mathbb{Z} \cdot 0_A$ for some 3-cocycle f, where 0_A denotes the unit of A.

We define the numbers $\omega_a^i(f)$, $\omega_a(f)$, $\omega^i(f)$ and $\omega(f)$ for any A-valued 3-cocycle f and any element $a \notin \mathbb{Z} \cdot 0_A$ of $\mathbb{Z}[A]$ by

$$\omega_a^i(f) := \min \left\{ w(S) \ \middle| \ \begin{array}{l} S \text{ is a surface braid of degree } i \\ \text{such that } \Phi_f(\text{closure of } S) = a \end{array} \right\},$$

$$\omega_a(f) := \min\{\omega_a^i(f) \mid i \in \mathbb{N}\},$$

$$\omega^i(f) := \min\{\omega_a^i(f) \mid a \notin \mathbb{Z} \cdot 0_A\},$$

$$\omega(f) := \min\{\omega_a(f) \mid a \notin \mathbb{Z} \cdot 0_A\} \ (= \min\{\omega^i(f) \mid i \in \mathbb{N}\}).$$

If there is no surface braid satisfying the condition in the definition of $\omega_a^i(f)$, then we define it to be ∞. The number $\omega(f)$ measures the sharpness of the estimation of $w(F)$ by $\Phi_f(F)$.

Problem 10.1 (M. Iwakiri). *Determine $\omega_a^i(f)$, $\omega_a(f)$, $\omega^i(f)$ and $\omega(f)$ for $i \in \mathbb{N}$, an A-valued 3-cocycle f and an element a of $\mathbb{Z}[A]$.*

By definition, $\omega(f) \le \omega_a(f) \le \omega_a^i(f)$ and $\omega(f) \le \omega^i(f) \le \omega_a^i(f)$. By considering stabilizations of surface braids ([44]), we have $\omega_a^{i+1}(f) \le \omega_a^i(f)$ and $\omega^{i+1}(f) \le \omega^i(f)$. Since the closure of any surface braid of degree ≤ 3 is a ribbon surface link ([43]), $\omega_a^i(f) = \infty$ for $i \le 3$.

Let θ_p be Mochizuki's $\mathbb{Z}_p$-valued 3-cocycle ([64]) of the dihedral quandle of order p, and identify $\mathbb{Z}[\mathbb{Z}_p]$ with $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$. It is shown in [36] that $\omega(\theta_3) = \omega^i(\theta_3) = \omega_a^i(\theta_3) = 6$ for any $i \ge 4$ and $a = 3 + 6t, \ 3 + 6t^2$. Further, it is shown in [37] that $\omega(\theta_5) \le \omega^4(\theta_5) = \omega_a^4(\theta_5) = 10$ for $a = 5 + 10t^2 + 10t^3$.

We can consider similar numbers for the triple point number, instead of the w-index, as follows. We define the *triple point number $t(F)$* of a surface link F by

$$t(F) := \min \big\{ t(D) \ \big| \ D \text{ is a link diagram of } F \big\},$$

where $t(D)$ is the number of triple points of D. Further, we define the numbers $\tau_a(f)$ and $\tau(f)$ for any A-valued 3-cocycle f and any element $a \notin \mathbb{Z} \cdot 0_A$ of $\mathbb{Z}[A]$ by

$$\tau_a(f) := \min \big\{ t(F) \mid F \text{ is a surface link such that } \Phi_f(F) = a \big\},$$
$$\tau(f) := \min\{\tau_a(f) \mid a \notin \mathbb{Z} \cdot 0_A\}.$$

If there is no surface link satisfying the condition in the definition of $\tau_a(f)$, then we define it to be ∞. The number $\tau(f)$ measures the sharpness of the estimation of $t(F)$ by $\Phi_f(F)$.

Problem 10.2 (M. Iwakiri). *Determine $\tau_a(f)$ and $\tau(f)$ for an A-valued 3-cocycle f and an element a of $\mathbb{Z}[A]$.*

By definition, $\tau(f) \leq \tau_a(f)$. It follows from results of [82] that $\tau(\theta_3) = \tau_a(\theta_3) = 4$ for $a = 3 + 6t, 3 + 6t^2$. Further, it follows from results of [27] that $\tau(\theta_5) \geq 6$. Furthermore, it follows from results of [81] that $\tau(\theta_5) = \tau_a(\theta_5) = 8$ for $a = 5 + 10t^2 + 10t^3$.

11 Invariants of 3-manifolds derived from their covering presentation

(Eri Hatakenaka, Takefumi Nosaka)

In this section, manifolds are connected and orientable. Consider a 4-fold branched cover of S^3 whose branch locus is a link $L \subset S^3$. Such a branched cover is called *simple* if its monodromy representation $\varphi : \pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ takes each meridian to a transposition in $\mathfrak{S}_4$. A *labeled link* is a link with such a representation. It is known that any closed 3-manifold is homeomorphic to a simple 4-fold branched cover of S^3. In this sense, a labeled link (L, φ) can be regarded as a presentation of a closed 3-manifold M. Further, it is known that two labeled links present homeomorphic 3-manifolds if and only if the labeled links are related by a finite sequence of the MI and MII moves. We call this presentation of closed 3-manifolds the *covering presentation*.

Problem 11.1 (E. Hatakenaka, T. Nosaka). *Construct invariants of a closed 3-manifold M from an invariant of a labeled link (L, φ).*

A simple representation $\pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ naturally induces a quandle homomorphism $Q(L) \to S_6$, where $Q(L)$ denotes a link quandle of L and S_6 denotes the conjugation quandle consisting of all transposition of $\mathfrak{S}_4$, i.e., $S_6 := \{(ij) \in \mathfrak{S}_4\}$. In [28], the *4-fold symmetric quandle homotopy invariant* of closed 3-manifolds was constructed from the quandle homotopy invariant of L for such a quandle X that there is a surjective quandle homomorphism $X \to S_6$ with appropriate properties. It is a problem to construct other invariants of closed 3-manifolds derived from the covering presentation.

Further, we pose another problem.

Problem 11.2 (E. Hatakenaka, T. Nosaka). *Extend the 4-fold symmetric quandle homotopy invariant to an invariant of 3-manifolds with boundaries. Further, construct a topological quantum field theory (TQFT) of the invariant.*

Quantum invariants can be formulated by TQFT, and it enables us to compute the invariants by cut-and-paste method of 3-manifolds: for example, Turaev-Viro and Dijkgraaf-Witten invariant. It is shown in [28] that Dijkgraaf-Witten invariant is derived from 4-fold symmetric quandle homotopy invariant of closed manifolds. So, we expect to construct a TQFT for the 4-fold symmetric quandle homotopy invariant.

12 Sutured manifolds and invariants of 3-manifolds

(Hiroshi Goda, Takuya Sakasai)

For a given knot K with a Seifert surface R, we can construct a sutured manifold (M_R, K) called the *complementary sutured manifold for R* by cutting open the knot exterior along R. The boundary of M_R is the union of two copies of R glued along their common boundary K, thus M_R can be seen as a cobordism over R. As an analogue of Heegaard genus, the *handle number* [21] is defined for a sutured manifold. Let $h(R)$ denote the handle number of the complementary sutured manifold M_R. The *Morse-Novikov number* $\mathcal{MN}(K)$ of K is also defined as the minimal number of the handle numbers among all Seifert surfaces of K. By considering many recent applications of Heegaard Floer homology and its variants, we pose:

Question 12.1 (H. Goda, T. Sakasai). *Can we estimate $h(R)$ and $\mathcal{MN}(K)$ by using the sutured Floer homology or the knot Floer homology?*

Next, we consider the following another approach for estimating handle numbers. A knot K is said to be *homologically fibered* if it has a (necessarily minimal genus) Seifert surface R whose complementary sutured manifold is a homology cobordism over R. This condition is known to be equivalent to that the Alexander polynomial $\Delta_K(t)$ of K is monic and has degree equal to the twice of the genus of K. As is well known, fibered knots satisfy this condition. The motivation for defining this class of knots comes from the study of *homology cylinders*, which was initiated by Goussarov [24] and Habiro [26] independently in their study of finite type invariants of 3-manifolds via *clover* or *clasper* surgery. A homology cylinder (M, m) is, roughly speaking, a pair of a homology cobordism M over a surface Σ and an identification m of ∂M with $\partial(\Sigma \times [0, 1])$. Homologically fibered knots provide a construction of homology cylinders.

Question 12.2 (H. Goda, T. Sakasai). *Can we estimate $h(R)$ and $\mathcal{MN}(K)$ for a homologically fibered knot K with a minimal genus Seifert surface R by using finite type invariants for homology cylinders?*

By definition, a knot K is fibered if and only if its minimal genus Seifert surface R satisfies $h(R) = 0$, in particular, $\mathcal{MN}(K) = 0$. In [22], we gave a lower estimate

of the handle number of a minimal genus Seifert surface of a homological fibered knot by using a certain (Reidemeister) torsion invariant τ for homology cylinders. It was shown by Friedl-Juhász-Rasmussen [16] that τ may be regarded as a special case of a *decategorification* of sutured Floer homology. While Ghiggini [20], Ni [71] and Juhász [40] showed that sutured Floer homology, a variant of Heegaard Floer homology, can detect the fiberedness of a knot, there are no examples so far where τ cannot detect the non-fiberedness of a non-fibered homological fibered knot (see the computations in [22, 23]).

Question 12.3 (H. Goda, T. Sakasai). *Does the torsion τ detect the fiberedness of a homologically fibered knot?*

This question is highly related to the question: How much the decategorification of sutured Floer homology loses information.

Finally, to make the relationship between homology cylinders and homologically fibered knots more closer, we pose:

Problem 12.4 (H. Goda, T. Sakasai). *Define (local) moves among homologically fibered knots. Then by using them, construct a theory of finite type invariants specialized to homologically fibered knots.*

13   Sutured Floer homology

(András Juhász)

Sutured manifolds were introduced by Gabai in [17] to study taut foliations on three-manifolds, and in [18] and [19] he used this theory to prove the so called property R conjecture. Loosely speaking, a sutured manifold is a pair (M, γ) such that M is a compact oriented three-manifold with boundary, and γ is a set of oriented simple closed curves in ∂M that divide ∂M into pieces $R_-(\gamma)$ and $R_+(\gamma)$. A sutured manifold is called balanced if $\chi(R_+(\gamma)) = \chi(R_-(\gamma)))$, it has no closed components, and each component of ∂M has at least one suture on it.

In [40], I introduced an invariant of balanced sutured manifolds, called sutured Floer homology, in short SFH. It assigns a finitely generated Abelian group to every balanced sutured manifold, and it is a common generalization of the hat versions of Heegaard Floer homology and knot Floer homology. It satisfies the following decomposition formula: if (M, γ) is a balanced sutured manifold and (M', γ') is obtained from it by a "nice" sutured manifold decomposition, then $SFH(M', \gamma')$ is a direct summand of $SFH(M, \gamma)$, see [41].

Question 13.1 (A. Juhász). *Is it possible to define SFH for sutured manifolds that have no sutures on some components of ∂M? Is there an analogue of $HF^\pm$ and HF^∞ in the sutured Floer setting? If the answer is yes, does an analogue of the decomposition formula hold for them?*

Given a Seifert surface R of a knot $K \subset S^3$, let $S^3(R)$ denote the sutured manifold complementary to R. Note that $SFH(S^3(R)) \cong \widehat{HFK}(K, g(R))$. However,

$SFH(M,\gamma)$ decomposes as a direct sum along relative Spin^c-structures on (M,γ), which form an affine space over $H_1(M)$.

Question 13.2 (A. Juhász). *Is there a knot K in S^3 that has two minimal genus Seifert surfaces R_1 and R_2 that can be distinguished using $SFH(S^3(R_i))$ together with the Spin^c-grading? Is there an example where the sutured Floer homology polytopes (see [42]) of $S^3(R_1)$ and $S^3(R_2)$ are different?*

Question 13.3 (A. Juhász). *Suppose that K is a knot in S^3, and*

$$rk\left(\widehat{HFK}(K,g(K))\right) < 2^{k+1}.$$

Then the sutured manifold $S^3(K)$ complementary to K has a length at most $2k+1$ sutured manifold hierarchy. Is it true that $S^3(K)$ has a depth at most $2k+1$ taut foliation?

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