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# Bounds of minimal dilatation for pseudo-Anosovs and the magic 3-manifold

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## 1 Minimal dilatation of pseudo-Anosovs

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures, and let $\mathrm{Mod}(\Sigma)$ be the mapping class group. Mapping classes $\phi \in \mathrm{Mod}(\Sigma)$ are classified into 3 types, periodic, reducible, pseudo-Anosov. There exist two numerical invariants of pseudo-Anosov mapping classes. One is the entropy $\mathrm{ent}(\phi)$ which is the logarithm of the dilatation $\lambda(\phi) > 1$. The other is the volume $\mathrm{vol}(\phi)$ which is the hyperbolic volume of the mapping torus of $\phi$

$$\mathbb{T}(\phi) = \Sigma \times [0,1]/\sim,$$

where $\sim$ identifies $(x,1)$ with $(f(x),0)$ for any representative $f \in \phi$.

We denote by $\delta_{g,n}$, the minimal dilatation for pseudo-Anosov elements $\phi \in \mathrm{Mod}(\Sigma_{g,n})$. We set $\delta_g = \delta_{g,0}$. A natural question arises.

**Question 1.1.** *What is the value of $\delta_{g,n}$? Find a pseudo-Anosov element of $\mathrm{Mod}(\Sigma_{g,n})$ whose dilatation is equal to $\delta_{g,n}$.*

The above question is hard in general. For instance, in the case of closed surfaces, it is open to determine the values $\delta_g$ for $g \geq 3$. On the other hand, one understands the asymptotic behavior of the minimal entropy $\log \delta_g$. Penner proved that $\log \delta_g \asymp \frac{1}{g}$ [12]. The following question posed by McMullen.

**Question 1.2** ([11]). *Does $\lim\limits_{g\to\infty} g \log \delta_g$ exist? What is its value?*

Note that $\lim\limits_{g\to\infty} g \log \delta_g$ exists if and only if $\lim\limits_{g\to\infty} |\chi(\Sigma_g)| \log \delta_g$ exists, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$.

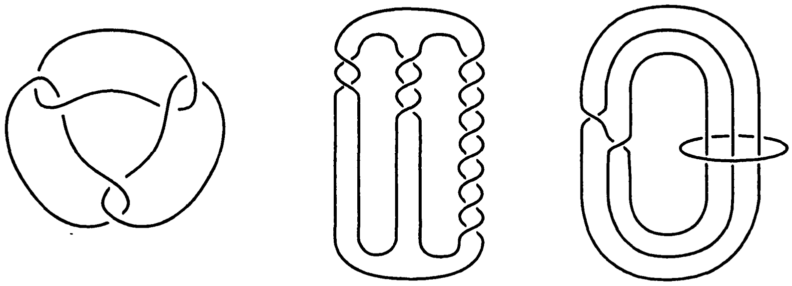

Figure 1: (left) 3 chain link $\mathcal{C}_3$. (center) $(-2,3,8)$-pretzel link or Whitehead sister link. (right) link $6_2^2$. ($N$ equals the exterior of $\mathcal{C}_3$. $N(\frac{-3}{2})$ is homeomorphic to the $(-2,3,8)$-pretzel link exterior. $N(\frac{-1}{2})$ is homeomorphic to the $6_2^2$ link exterior.)

Related questions on the minimal dilatation are ones for orientable pseudo-Anosovs. A pseudo-Anosov mapping class $\phi$ is said to be *orientable* if the invariant (un)stable foliation for a pseudo-Anosov homeomorphism $\Phi \in \phi$ is orientable. We denote by $\delta_g^+$, the minimal dilatation for orientable pseudo-Anosov elements of $\mathrm{Mod}(\Sigma_g)$ for a closed surface $\Sigma_g$ of genus $g$.

In this paper, we report our results in [7, 8] on the minimal dilatation by investigating the so called *magic manifold* $N$ which is the exterior of the 3 chain link $\mathcal{C}_3$, see Figure 1(left). In Section 2, we describe a motivation for the study of pseudo-Anosovs which occur as the monodromies on fibers for Dehn fillings of $N$. In Section 3, we state our results.

We would like to note that this paper only contains some results in [7, 8] and does not contain their proofs. The readers who are interested in the details should consult (the introduction of) [7, 8].

# 2 Why is the magic manifold an intriguing example?

Gordon and Wu named the exterior of the link $\mathcal{C}_3$ the *magic manifold* $N$, see [3]. The reason why this manifold is called "magic" is that many important examples for the study of the exceptional Dehn fillings can be obtained from the Dehn fillings of a single manifold $N$. The magic manifold is fibered and it has the smallest known volume among orientable hyperbolic 3-manifolds having 3 cusps. Many manifolds having at most 2 cusps with small volume are obtained from $N$ by Dehn fillings, see [10].

## 2.1 Entropy versus volume

Both invariants entropy $\mathrm{ent}(\phi)$ and volume $\mathrm{vol}(\phi)$ know some complexity of pseudo-Anosovs $\phi$. A natural question is how these are related.

**Theorem 2.1** ([6]). *There exists a constant $B = B(\Sigma)$ depending only on the topology of $\Sigma$ such that the inequality,*

$$B\,\mathrm{vol}(\phi) \leq \mathrm{ent}(\phi)$$

*holds for any pseudo-Anosov $\phi$ on $\Sigma$. Furthermore, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \Sigma) > 1$ depending only on $\varepsilon$ and the topology of $\Sigma$ such that the inequality*

$$\mathrm{ent}(\phi) \leq C\,\mathrm{vol}(\phi)$$

*holds for any pseudo-Anosov $\phi$ on $\Sigma$ whose mapping torus $\mathbb{T}(\phi)$ has no closed geodesics of length $< \varepsilon$.*

The first part of Theorem 2.1 says that if the entropy is small, the volume can not be large.

For a non-negative integer $c$, we set

$$\begin{aligned}
\lambda(\Sigma; c) &= \min\{\lambda(\phi) \mid \phi \in \mathrm{Mod}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\},\\
\mathrm{vol}(\Sigma; c) &= \min\{\mathrm{vol}(\phi) \mid \phi \in \mathrm{Mod}(\Sigma),\ \mathbb{T}(\phi) \text{ has } c \text{ cusps}\}.
\end{aligned}$$

A variation on the questions of the minimal dilatations is to determine $\lambda(\Sigma; c)$ and to find a mapping class realizing the minimum. In [6], the authors and S. Kojima obtain experimental results concerning the minimal dilatation. In the case the mapping class group $\mathrm{Mod}(D_n)$ of an $n$-punctured disk $D_n$, they observe that for many pairs $(n, c)$, there exists a pseudo-Anosov element simultaneously reaching both minima $\lambda(D_n; c)$ and $\mathrm{vol}(D_n; c)$. Experiments tell us that in case $c = 3$, the mapping tori reaching both minima are homeomorphic to $N$. Moreover when $c = 2$, it is observed that there exists a mapping class $\phi$

realizing both $\lambda(D_n;2)$ and $\mathrm{vol}(D_n;2)$ and its mapping torus $\mathbb{T}(\phi)$ is homeomorphic to a Dehn filling of $N$ along one cusp. This is a motivation for us for focusing on $N$.

## 2.2 Small dilatation pseudo-Anosovs

After we have finished our papers [6, 7], we leaned the small dilatation pseudo-Anosovs, introduced by Farb, Leininger and Margalit.

For any number $P > 1$, define the set of pseudo-Anosov homeomorphisms

$$\Psi_P = \{\text{pseudo-Anosov } \Phi : \Sigma \to \Sigma \mid \chi(\Sigma) < 0,\ |\chi(\Sigma)| \log \lambda(\Phi) \leq \log P\}.$$

Elements $\Phi \in \Psi_P$ are called *small dilatation pseudo-Anosov homeomorphisms.* If one takes $P$ sufficiently large (e.g. $P \geq 2 + \sqrt{3}$), then $\Psi_P$ contains a pseudo-Anosov homeomorphism $\Phi_g : \Sigma_g \to \Sigma_g$ for each $g \geq 2$. By a result in [5], $\Psi_P$ also contains pseudo-Anosov homeomorphism $\Phi_n : D_n \to D_n$ for each $n \geq 3$. Let $\Sigma^\circ \subset \Sigma$ be the surface obtained by removing the singularities of the (un)stable foliation for $\Phi$ and $\Phi|_{\Sigma^\circ} : \Sigma^\circ \to \Sigma^\circ$ denotes the restriction. Observe that $\lambda(\Phi) = \lambda(\Phi|_{\Sigma^\circ})$. The set

$$\Psi_P^\circ = \{\Phi|_{\Sigma^\circ} : \Sigma^\circ \to \Sigma^\circ \mid (\Phi : \Sigma \to \Sigma) \in \Psi_P\}$$

is infinite. Let $\mathcal{T}(\Psi_P^\circ)$ be the set of homeomorphism classes of mapping tori by elements of $\Psi_P^\circ$.

**Theorem 2.2** ([2]). *The set $\mathcal{T}(\Psi_P^\circ)$ is finite. Namely, for each $P > 1$, there exist finite many complete, non compact hyperbolic 3-manifolds $M_1, M_2, \cdots, M_r$ fibering over $S^1$ so that the following holds. Any pseudo-Anosov $\Phi \in \Psi_P$ occurs as the monodromy of a Dehn filling of one of the $M_k$. In particular, there exists a constant $V = V(P)$ such that* $\mathrm{vol}(\Phi) \leq V$ *holds for any $\Phi \in \Psi_P$.*

It is not known that how large the set of manifolds $\{M_1, \cdots, M_r\}$ is. By Theorem 2.2, one sees that the following set $\mathcal{V}$ is finite.

$$\mathcal{V} = \{\mathbb{T}(\Phi|_{\Sigma^\circ}) \mid n \geq 3,\ \Phi \text{ is pseudo-Anosov on } \Sigma = D_n,\ \lambda(\Phi) = \delta(D_n)\},$$

where $\delta(D_n)$ denotes the minimal dilatation for pseudo-Anosov elements of $\mathrm{Mod}(D_n)$ on $D_n$.

In [7], we show that for each $n \geq 9$ (resp. $n = 3, 4, 5, 7, 8$), there exists a pseudo-Anosov homeomorphism $\Phi_n : D_n \to D_n$ with the smallest known entropy (resp. the smallest entropy) which occurs as the monodromy on an $n$-punctured disk fiber for the Dehn filling of $N$. A pseudo-Anosov homeomorphism $\Phi_6 : D_6 \to D_6$ with the smallest entropy occurs as the monodromy on a 6-punctured disk fiber for $N$. In particular, $N \in \mathcal{V}$. See also work of Venzke [13]. This result suggests that one may have a chance to find pseudo-Anosov homeomorphisms with small dilatation on other surfaces which arise as the monodromies on fibers for Dehn fillings of $N$. This is another motivation for us.

# 3 Results

Let us introduce the following polynomial

$$f_{(k,\ell)}(t) = t^{2k} - t^{k+\ell} - t^k - t^{k-\ell} + 1 \text{ for } k > 0,\ -k < \ell < k,$$

having a unique real root $\lambda_{(k,\ell)}$ greater than 1 [8]. For the rational number $r$, let $N(r)$ be the Dehn filling of $N$ along the slope $r$.

**Theorem 3.1.** *Let $r \in \{\frac{-3}{2}, \frac{-1}{2}, 2\}$. For each $g \geq 3$, there exists a monodromy $\Phi_g = \Phi_g(r)$ on a closed fiber of genus $g$ for a Dehn filling of $N(r)$, where the filling is on the boundary slope of a fiber of $N(r)$, such that*

$$\lim_{g\to\infty} g \log \lambda(\Phi_g) = \log(\tfrac{3+\sqrt{5}}{2}).$$

*In particular*

$$\lim_{g\to\infty} \sup g \log \delta_g \leq \log(\tfrac{3+\sqrt{5}}{2}).$$

**Remark 3.2.** *Independently, Hironaka has obtained Theorem 3.1 in case $r = \frac{-1}{2}$ [4]. Independently, Aaber and Dunfield have obtained Theorem 3.1 in case $r = \frac{-3}{2}$ [1]. They have obtained similar results on the dilatation to those given in [8].*

By using monodromies on closed fibers coming from $N(\frac{-3}{2})$, we find an upper bound of $\delta_g$ for $g \equiv 0, 1, 5, 6, 7, 9 \pmod{10}$ and $g \geq 5$.

**Theorem 3.3.** **(1)** $\delta_g \leq \lambda_{(g+2,1)}$ *if* $g \equiv 0, 1, 5, 6 \pmod{10}$ *and* $g \geq 5$.

**(2)** $\delta_g \leq \lambda_{(g+2,2)}$ *if* $g \equiv 7, 9 \pmod{10}$ *and* $g \geq 7$.

For more details of an upper bound of $\delta_g$ for other $g$ (e.g. $g \equiv 2, 4 \pmod{10}$), see [8]. The bound in Theorem 3.3 improves the one by Hironaka [4].

We turn to the study on the minimal dilatations $\delta_g^+$ for orientable pseudo-Anosovs. The minima $\delta_g^+$ were determined for $g = 2$ by Zhirov [14], for $3 \leq g \leq 5$ by Lanneau-Thiffeault [9], and for $g = 8$ by Lanneau-Thiffeault and Hironaka [9, 4]. Those values are given by $\delta_2^+ = \lambda_{(2,1)}$, $\delta_3^+ = \lambda_{(3,1)} = \lambda_{(4,3)} \approx 1.40127$, $\delta_4^+ = \lambda_{(4,1)} \approx 1.28064$, $\delta_5^+ = \lambda_{(6,1)} = \lambda_{(7,4)} \approx 1.17628$ and $\delta_8^+ = \lambda_{(8,1)} \approx 1.12876$.

We recall the lower bounds of $\delta_6^+$ and $\delta_7^+$ and the question on $\delta_g^+$ for $g$ even by Lanneau-Thiffeault.

**Theorem 3.4** ([9]).

**(1)** $\delta_6^+ \geq \lambda_{(6,1)} \approx 1.17628$.

**(2)** $\delta_7^+ \geq \lambda_{(9,2)} \approx 1.11548$.

**Question 3.5** ([9]). *For $g$ even, is $\delta_g^+$ equal to $\lambda_{(g,1)}$?*

We give an upper bound of $\delta_g^+$ in case $g \equiv 1, 5, 7, 9 \pmod{10}$ using orientable pseudo-Anosov monodromies coming from $N(\frac{-3}{2})$.

**Theorem 3.6. (1)** $\delta_g^+ \leq \lambda_{(g+2,2)}$ *if* $g \equiv 7, 9 \pmod{10}$ *and* $g \geq 7$.

**(2)** $\delta_g^+ \leq \lambda_{(g+2,4)}$ *if* $g \equiv 1, 5 \pmod{10}$ *and* $g \geq 5$.

The bound in Theorem 3.6 improves the one by Hironaka [4]. Theorem 3.6(1) together with Theorem 3.4(2) gives:

**Corollary 3.7.** $\delta_7^+ = \lambda_{(9,2)}$.

Independently, Corollary 3.7 was established by Aaber and Dunfiled [1].

The following tells us that the sequence $\{\delta_g^+\}_{g \geq 2}$ is not monotone decreasing.

**Proposition 3.8.** *If Question 3.5 is true, then $\delta_g^+ < \delta_{g+1}^+$ whenever $g \equiv 1, 5, 7, 9 \pmod{10}$ and $g \geq 7$. In particular the inequality $\delta_7^+ < \delta_8^+$ holds.*

Our pseudo-Anosov homeomorphisms providing the upper bound of $\delta_g$ in Theorem 3.3(1) are not orientable. This together with the inequality $\lambda_{(7,1)} < \lambda_{(6,1)} = \delta_5^+$ implies:

**Corollary 3.9.** $\delta_5 < \delta_5^+$.

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