HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGICALLY FIBERED KNOTS

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1. INTRODUCTION

This note is adapted from the talk at the 2010 Intelligence of Low-dimensional Topology at Research Institute for Mathematical Sciences, Kyoto University. For the detail, see the original papers [12], [13].

Let $\Sigma_{g,n}$ be a compact oriented surface of genus $g$ with $n \geq 1$ boundary components, and the triple $(M, i_+, i_-)$ be an oriented homology cobordism between $\Sigma_{g,n}$ and $\Sigma_{g,n}$ with two markings of $\partial M : i_+, i_- : \Sigma_{g,1} \leftrightarrow \partial M$. We call $(M, i_+, i_-)$ a homology cylinder over $\Sigma_{g,n}$. This object was introduced by Goussarov [14] and Habiro [16] since it is suitable for applying the theory of clovers and claspers, and then has been studied together with finite type invariants of 3-manifolds. The following have been known as methods for constructing homology cylinders:

- connected sums of the trivial cobordism with homology 3-spheres;
- Levine's method [19] using string links in the 3-ball;
- Habegger's method [15] giving homology cylinders as results of surgeries along string links in homology 3-balls; and
- clasper surgeries (see [14] and [16]).

In [12], the authors gave an explicit construction of homology cylinders, i.e. we introduced a notion of a homologically fibered knot and construct a homology cylinder using it. The family of the homologically fibered knots include that of the fibered knots. So, roughly speaking, the following relationships exist:

\[
\begin{array}{ccc}
\text{Pure Braid} & \leftrightarrow & \text{Mapping cylinder} & \leftrightarrow & \text{Fibered knot} \\
\cap & & \cap & & \cap \\
\text{Pure String link} & \overset{\text{Levine}}{\leftrightarrow} & \text{Homology cylinder} & \leftrightarrow & \text{Homologically fibered knot} \\
\overset{\text{(Habegger-Lin)}}{\text{Homology cylinder}} & \leftrightarrow & \text{Homologically fibered knot} & \overset{\text{(Goussarov, Habiro)}}{\text{Homology cylinder}} \\
\end{array}
\]

In [18], Kirk-Livingston-Wang introduced a Reidemeister torsion for string links, then the second author studied the corresponding Reidemeister torsion for homology cylinders in [23]. Note that this torsion may be regarded as a special case of a decategorification of sutured Floer homology [8]. In this note, we study the Reidemeister torsion for homologically fibered knots and show a factorization formula. Further, we give a MATHEMATICA program for explicit calculations of the invariants for homologically fibered knots.

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2. Homologically fibered knots

In this section, we introduce two main objects in this note: homology cylinders and sutured manifolds. First, we define homology cylinders over surfaces, which have their origin in Goussarov [14], Habiro [16], Garoufalidis-Levine [11] and Levine [19]. Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components.

Definition 2.1. A homology cylinder $(M, i_+, i_-)$ over $\Sigma_{g,n}$ consists of a compact oriented 3-manifold $M$ with two embeddings $i_+, i_- : \Sigma_{g,n} \to \partial M$ such that:

(i) $i_+$ is orientation-preserving and $i_-$ is orientation-reversing;
(ii) $\partial M = i_+(\Sigma_{g,n}) \cup i_-(\Sigma_{g,n})$ and $i_+(\Sigma_{g,n}) \cap i_-(\Sigma_{g,n}) = i_+(\partial \Sigma_{g,n}) = i_-(\partial \Sigma_{g,n})$;
(iii) $i_+|_{\partial \Sigma_{g,n}} = i_-|_{\partial \Sigma_{g,n}}$; and
(iv) $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ are isomorphisms.

If we replace (iv) with the condition that $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Q}) \to H_*(M; \mathbb{Q})$ are isomorphisms, then $(M, i_+, i_-)$ is called a rational homology cylinder.

We often write a (rational) homology cylinder $(M, i_+, i_-)$ briefly by $M$. Note that our definition is the same as that in [11] and [19] except that we may consider homology cylinders over surfaces with multiple boundaries.

Two (rational) homology cylinders $(M, i_+, i_-)$ and $(N, j_+, j_-)$ over $\Sigma_{g,n}$ are said to be isomorphic if there exists an orientation-preserving diffeomorphism $f : M \cong N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$. We denote the set of isomorphism classes of homology cylinders (resp. rational homology cylinders) over $\Sigma_{g,n}$ by $C_{g,n}$ (resp. $C_{g,n}^\mathbb{Q}$).

Example 2.2 (Mapping cylinder). For each diffeomorphism $\varphi$ of $\Sigma_{g,n}$ which fixes $\partial \Sigma_{g,n}$ pointwise (hence, $\varphi$ preserves the orientation of $\Sigma_{g,n}$), we can construct a homology cylinder by setting

$$(\Sigma_{g,n} \times [0, 1], \text{id} \times 1, \varphi \times 0),$$

where collars of $i_+(\Sigma_{g,n})$ and $i_-(\Sigma_{g,n})$ are stretched half-way along $(\partial \Sigma_{g,n}) \times [0, 1]$. It is easily checked that the isomorphism class of $(\Sigma_{g,n} \times [0, 1], \text{id} \times 1, \varphi \times 0)$ depends only on the (boundary fixing) isotopy class of $\varphi$. Therefore, this construction gives a map from the mapping class group $\mathcal{M}_{g,n}$ of $\Sigma_{g,n}$ to $C_{g,n}$. 

\[\text{FIGURE 1. Homology cylinder}\]
Next, we recall the definition of sutured manifolds given by Gabai [10]. We here use a special case of them.

A sutured manifold $(M, \gamma)$ is a compact oriented 3-manifold $M$ together with a subset $\gamma \subset \partial M$ which is a union of finitely many mutually disjoint annuli. For each component of $\gamma$, an oriented core circle called a suture is fixed, and we denote the set of sutures by $s(\gamma)$. Every component of $R(\gamma) = \partial M - \text{Int} \gamma$ is oriented so that the orientations on $R(\gamma)$ are coherent with respect to $s(\gamma)$, that is, the orientation of each component of $\partial R(\gamma)$ induced from that of $R(\gamma)$ is parallel to the orientation of the corresponding component of $s(\gamma)$. We denote by $R_+ (\gamma)$ (resp. $R_- (\gamma)$) the union of those components of $R(\gamma)$ whose normal vectors point out (resp. into) $M$.

**Example 2.3.** For a knot $K$ in $S^3$ and a Seifert surface $\overline{R}$ of $K$, we set $R := \overline{R} \cap E(K)$, called also a Seifert surface, where $E(K) = \overline{S^3 - N(K)}$ is the complement of a regular neighborhood $N(K)$ of $K$. Then $(M_R, \gamma) := (E(K) - N(R), \partial E(K) - N(\partial R))$ defines a sutured manifold. We call it the complementary sutured manifold for $R$. In this paper, we simply call it the sutured manifold for $R$.

![Diagram](image_url)

**Figure 2.** Complementary sutured manifold

Let $L$ be an oriented link in the 3-sphere $S^3$, and $\Delta_L(t)$ the normalized (one variable) Alexander polynomial of $L$, i.e. the lowest degree of $\Delta_L(t)$ is 0.

**Definition 2.4.** An $n$-component link $L$ in $S^3$ is said to be homologically fibered if $L$ satisfies the following two conditions:

(i) The degree of $\Delta_L(t)$ is $2g + n - 1$, where $g$ is the genus of a connected Seifert surface of $L$; and

(ii) $\Delta_L(0) = \pm 1$.

If an $n$-component link $L$ satisfies (i), then $L$ is said to be rationally homologically fibered.

The Alexander polynomial that satisfies the condition (ii) is said to be monic in this paper.

**Remark 2.5.** In general, if $L$ bounds a connected Seifert surface of genus $g$, then

$$2g + n - 1 \geq \text{(the degree of } \Delta_L(t)).$$

It is known ([5], [21]) that if $L$ has an alternating diagram that gives, by the Seifert algorithm, a connected Seifert surface of genus $g$, then the degree of $\Delta_L(t)$ is equal to $2g + n - 1$. 
Remark 2.6. Suppose \( L \) is an alternating link. Then, \( L \) is fibered if and only if \( \Delta_L(t) \) is monic, by Murasugi [22] (see also 13.26 (c) in [1]). Therefore, if a homologically fibered link \( L \) is not fibered, then \( L \) is non-alternating.

Let \( L \) be an \( n \)-component link and \( \Sigma_{g,n} \) the compact oriented surface that is diffeomorphic to a Seifert surface \( R \) of \( L \). We fix a diffeomorphism \( \vartheta : \Sigma_{g,n} \xrightarrow{\cong} R \) and denote by \((M_R, \gamma)\) the complementary sutured manifold for \( R \). Then we may see that there are an orientation-preserving embedding \( i_+ : \Sigma_{g,n} \to M_R \) and an orientation-reversing embedding \( i_- : \Sigma_{g,n} \to M_R \) with \( i_+(\Sigma_{g,n}) = R_+ (\gamma) \) and \( i_-(\Sigma_{g,n}) = R_- (\gamma) \), where two embeddings \( i_{\pm} \) are the composite mappings of \( \vartheta \) and embeddings \( i_{\pm} : R \leftarrow M_R \) such that \( i_{\pm} = i_{\pm} \circ \vartheta : \Sigma_{g,n} \to R_{\pm} (\gamma) \subset M_R \):

\[
\begin{array}{ccc}
\Sigma_{g,n} & \xrightarrow{\vartheta} & R \\
& \downarrow i_{\pm} & \\
M_R & & \\
\end{array}
\]

If \( i_+, i_- : H_1(\Sigma_{g,n}) \to H_1(M_R) \) are isomorphisms, we may regard \((M_R, \gamma)\) as a homology cylinder. The next proposition was essentially mentioned in [6]. A proof is given in [12].

Proposition 2.7. Let \( R \) be a Seifert surface of a link \( L \). If the complementary sutured manifold for \( R \) is a homology cylinder, then \( L \) is homologically fibered. Conversely, if \( L \) is homologically fibered, then the complementary sutured manifold for each minimal genus Seifert surface of \( L \) is a homology cylinder.

It is known that all homologically fibered knots are fibered among prime knots with at most 11 crossings. On the other hand, Friedl-Kim [9] (see also [2]) showed that there are 13 non-fibered homologically fibered knots with 12-crossings. See Figure 7.

3. FACTORIZATION FORMULAS OF ALEXANDER INVARANTS

Let \( R \) be a minimal genus Seifert surface of a rationally homologically fibered knot \( K \) in \( S^3 \), and \( M_R \) be the sutured manifold for \( R \cong \Sigma_{g,1} \). We fix a basis of \( H_1(R; \mathbb{Q}) \), which yields an isomorphism \( H_1(R; \mathbb{Q}) \cong \mathbb{Q}^{2g} \). Then we can rewrite the definition \( \Delta_K(t) = \det(S-tS^T) \) of the Alexander polynomial of \( K \) by using the invertibility (over \( \mathbb{Q} \)) of the Seifert matrix \( S \), and obtain a factorization

\[
\Delta_K(t) = \det(S) \det(I_{2g} - t\sigma(M_R))
\]

of \( \Delta_K(t) \). Note that \( \sigma(M_R) := S^{-1}S^T \) represents the composite of isomorphisms

\[
\mathbb{Q}^{2g} \cong H_1(R; \mathbb{Q}) \xrightarrow{i_-} H_1(M_R; \mathbb{Q}) \xrightarrow{\cong} H_1(R; \mathbb{Q}) \cong \mathbb{Q}^{2g}.
\]

The matrix \( \sigma(M_R) \) can be interpreted as a monodromy of \( M_R \) from a view point of the rational homology. Regarding the formula (3.1) as a basic case, we constructed in [12] its generalization under the framework of higher-order Alexander invariants due to Cochran [3], Harvey [17] and Friedl [7]. In this procedure, the Seifert matrix \( S \), the monodromy \( \sigma(M_R) \) and \( \Delta_K(t) \) are generalized to a certain Reidemeister torsion \( \tau^+_\rho(M_R) \), the Magnus
matrix $\tau_\rho(M_R)$ and some higher-order (non-commutative) Reidemeister torsion $\tau_\rho(E(K))$ associated with a representation $\rho$ of the fundamental group of $M_R$.

Here, we review higher-order Alexander invariants quickly. For a matrix $A$ with entries in a group ring $\mathbb{Z}G$ (or its quotient field) for a group $G$, we denote by $\widetilde{A}$ the matrix obtained from $A$ by applying the involution induced from $(x \mapsto x^{-1}, x \in G)$ to each entry. For a module $M$, we write $M^n$ for the module of column vectors with $n$ entries. For a finite cell complex $X$, we denote by $\tilde{X}$ its universal covering. We take a base point $p$ of $X$ and a lift $\tilde{p}$ of $p$ as a base point of $\tilde{X}$. $\pi := \pi_1(X, p)$ acts on $\tilde{X}$ from the right through its deck transformation group, so that the lift of a loop $l \in \pi$ starting from $\tilde{p}$ reaches $\tilde{p}l^{-1}$. Then the cellular chain complex $C_*(\tilde{X})$ of $\tilde{X}$ becomes a right $\mathbb{Z}\pi$-module. For each left $\mathbb{Z}\pi$-algebra $\mathcal{R}$, the twisted chain complex $C_*(X; \mathcal{R})$ is given by the tensor product of the right $\mathbb{Z}\pi$-module $C_*(\tilde{X})$ and the left $\mathbb{Z}\pi$-module $\mathcal{R}$, so that $C_*(X; \mathcal{R})$ and $H_*(X; \mathcal{R})$ are right $\mathcal{R}$-modules.

In the definition of higher-order Alexander invariants, PTFA groups play important roles, where a group $\Gamma$ is said to be poly-torsion-free abelian (PTFA) if it has a sequence
\[
\Gamma = \Gamma_0 \triangleright \Gamma_1 \triangleright \cdots \triangleright \Gamma_n = \{1\}
\]
whose successive quotients $\Gamma_i/\Gamma_{i+1}$ ($i \geq 0$) are all torsion-free abelian. An advantage of using PTFA groups is that the group ring $\mathbb{Z}\Gamma$ (or $\mathbb{Q}\Gamma$) of $\Gamma$ is known to be an Ore domain so that it can be embed into the field (skew field in general)
\[
\mathcal{K}_\Gamma := \mathbb{Z}\Gamma/(\mathbb{Z}\Gamma - \{0\})^{-1} = \mathbb{Q}\Gamma/(\mathbb{Q}\Gamma - \{0\})^{-1}
\]
called the right field of fractions. A typical example of PTFA groups is $\mathbb{Z}^n$, where $\mathcal{K}_{\mathbb{Z}}$ is isomorphic to the field of rational functions with $n$ variables.

For a rationally homologically fibered knot $K$, we take a homomorphism $\rho : G(K) := \pi_1(E(K)) \to \Gamma$ whose target $\Gamma$ is PTFA. We suppose that $\rho$ is non-trivial. We regard $\mathcal{K}_\Gamma$ as a local coefficient system on $E(K)$ through $\rho$.

**Lemma 3.1** (Cochran [3, Lemma 3.9]). For any non-trivial homomorphism $\rho : G(K) \to \Gamma$ to a PTFA group $\Gamma$, we have $H_*(E(K); \mathcal{K}_\Gamma) = 0$.

By this lemma, we can define the Reidemeister torsion
\[
\tau_\rho(E(K)) := \tau(C_*(E(K); \mathcal{K}_\Gamma)) \in K_1(\mathcal{K}_\Gamma)/ \pm \rho(G(K))
\]
for the acyclic complex $C_*(E(K); \mathcal{K}_\Gamma)$. We refer to Milnor [20] for generalities of torsions. By higher-order Alexander invariants for $K$, we here mean this torsion $\tau_\rho(E(K))$.

We now describe a factorization of $\tau_\rho(E(K))$ generalizing (3.1). Let $(M_R, i_+, i_-) \in C^0_{g, 1}$ be the rational homology cylinder obtained as the sutured manifold for a minimal genus Seifert surface $R$ of $K$. We use the same notation $\rho : \pi_1(M_R) \to \Gamma$ for the composition $\pi_1(M_R) \to G(K) \xrightarrow{\rho} \Gamma$. Applying Cochran-Orr-Teichner [4, Proposition 2.10], we have the following:

**Lemma 3.2.** $i_+, i_- : H_*(\Sigma_{g, 1}; p; \mathcal{K}_\Gamma) \to H_*(M_R, p; \mathcal{K}_\Gamma)$ are isomorphisms as right $\mathcal{K}_\Gamma$-vector spaces. Equivalently, $H_*(M_R, i_+\Sigma_{g, 1}; \mathcal{K}_\Gamma) = 0$. 
This lemma provides the following two kinds of invariants for $M_R$.

**The Magnus matrix** Let $X \subset \Sigma_{g,1}$ be the bouquet of $2g$ circles $\gamma_1, \ldots, \gamma_{2g}$ tied at $p$ (see Figure 3). $X$ is a deformation retract of $\Sigma_{g,1}$ relative to $p$. Therefore, for $\pm \in \{+, -\}$, we have

$$H_1(\Sigma_{g,1}, p; i_{\pm}^* \mathcal{K}_\Gamma) \cong H_1(X, p; i_{\pm}^* \mathcal{K}_\Gamma) = C_1(\tilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_{\pm}^* \mathcal{K}_\Gamma \cong \mathcal{K}_\Gamma^{2g}$$

with a basis

$$\{\tilde{\gamma}_1 \otimes 1, \ldots, \tilde{\gamma}_{2g} \otimes 1\} \subset C_1(\tilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_{\pm}^* \mathcal{K}_\Gamma$$

as a right $\mathcal{K}_\Gamma$-vector space. Here we fix a lift $\bar{p}$ of $p$ as a base point of $\tilde{X}$, and denote by $\tilde{\gamma}_i$ the lift of the oriented loop $\gamma_i$ starting from $\bar{p}$.

**Definition 3.3.** For $M_R = (M_R, i_+, i_-) \in C_{g,1}^\mathbb{Q}$, the *Magnus matrix*

$$r_\rho(M_R) \in GL(2g, \mathcal{K}_\Gamma)$$

of $M_R$ is defined as the representation matrix of the right $\mathcal{K}_\Gamma$-isomorphism

$$\mathcal{K}_\Gamma^{2g} \cong H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \xrightarrow{i_-} H_1(M_R, p; \mathcal{K}_\Gamma) \xrightarrow{i_+} H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \cong \mathcal{K}_\Gamma^{2g},$$

where the first and the last isomorphisms use the bases mentioned above.

The matrix $r_\rho(M_R)$ can be interpreted as a monodromy of $M_R$ from a view point of the twisted homology with coefficients in $\mathcal{K}_\Gamma$.

**$\Gamma$-torsion** Since the relative complex $C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)$ obtained from any cell decomposition of $(M_R, i_+(\Sigma_{g,1}))$ is acyclic by Lemma 3.2, we can define the following:

**Definition 3.4.** For $M_R = (M_R, i_+, i_-) \in C_{g,1}^\mathbb{Q}$, the $\Gamma$-torsion $\tau^{+}_\rho(M_R)$ of $M_R$ is defined by

$$\tau^{+}_\rho(M_R) := \tau(C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)) \in K_1(\mathcal{K}_\Gamma)/\pm \rho(\pi_1(M_R)).$$

A method for computing $r_\rho(M_R)$ and $\tau^{+}_\rho(M_R)$ is given in [12, Section 4], which is based on Kirk-Livingston-Wang's method [18] for invariants of string links, and we now recall it briefly. An *admissible presentation* of $\pi_1(M_R)$ is defined to be the one of the form

$$\langle i_-(\gamma_1), \ldots, i_- (\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) | \tau_1, \ldots, \tau_{2g+l} \rangle$$

**Figure 3.** Cell decomposition of $\Sigma_{g,1}$
for some integer $l$. That is, it is a finite presentation with deficiency $2g$ whose generating set contains $i_{-}(\gamma_{1}),\ldots,i_{-}(\gamma_{2g}),i_{+}(\gamma_{1}),\ldots,i_{+}(\gamma_{2g})$ and is ordered as above. Such a presentation always exists. For any admissible presentation, define $2g \times (2g + l)$, $l \times (2g + l)$ and $2g \times (2g + l)$ matrices $A, B, C$ over $\mathbb{Z}$ by

$$A = \left(\frac{\partial r_{j}}{\partial i_{-}(\gamma_{i})}\right)_{1 \leq i \leq 2g, 1 \leq j \leq 2g + l}, \quad B = \left(\frac{\partial r_{j}}{\partial z_{i}}\right)_{1 \leq i \leq 2g, 1 \leq j \leq 2g + l}, \quad C = \left(\frac{\partial r_{j}}{\partial i_{+}(\gamma_{i})}\right)_{1 \leq i \leq 2g, 1 \leq j \leq 2g + l}$$

Proposition 3.5 ([12, Propositions 4.5, 4.6]). As matrices with entries in $\mathcal{K}_{\Gamma}$, we have:

1. The square matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is invertible and $\tau_{\rho}^{+}(M_{R}) = \begin{pmatrix} A \\ B \end{pmatrix}$; and

2. $r_{\rho}(M_{R}) = -C \left( A^{-1} \begin{pmatrix} I_{2g} \\ 0_{(l, 2g)} \end{pmatrix} \right)$

Using the above invariants, the factorization formula for $\tau_{\rho}(E(K))$ is given as follows:

Theorem 3.6. Let $K$ be a rationally homologically fibered knot of genus $g$. For any non-trivial homomorphism $\rho : G(K) \to \Gamma$ to a PTFA group $\Gamma$, a loop $\mu$ representing the meridian of $K$ satisfies $\rho(\mu) \neq 1 \in \Gamma \subset \mathcal{K}_{\Gamma}$ and we have a factorization

$$\tau_{\rho}(E(K)) = \frac{\tau^{+}_{\rho}(M_{R}) \cdot (I_{2g} - \rho(\mu)r_{\rho}(M_{R}))}{1 - \rho(\mu)} \in K_{1}(\mathcal{K}_{\Gamma})/\pm \rho(G(K))$$

of the torsion $\tau_{\rho}(E(K))$.

To compare (3.3) with (3.1), recall Milnor's formula [20] that $\frac{\Delta_{K}(t)}{1 - t}$ represents the Reidemeister torsion associated with the abelianization map $\rho_{1} : G(K) \to \langle t \rangle \subset \mathbb{Q}(t)$. Taking $\rho_{1}$ as $\rho$, we recover the formula (3.1).

4. Computations

Although all the ingredients in the formula (3.3) are theoretically determined by information on fundamental groups, it is difficult to compute them explicitly because of the non-commutativity of $\mathcal{K}_{\Gamma}$ except in some special cases including the following.

Let $K$ be a homologically fibered knot with a minimal genus Seifert surface $R$ and let $M_{R}$ be the sutured manifold for $R$. Consider the group extension

$$1 \longrightarrow G(K)'/G(K)^{\prime} \longrightarrow D_{2}(K) \longrightarrow G(K)/G(K)' = H_{1}(E(K)) \cong \mathbb{Z} \longrightarrow 1$$

relating to the metabelian quotient $D_{2}(K) := G(K)/G(K)^{\prime}$ of $G(K)$. We have

$$G(K)'/G(K)^{\prime} \cong H_{1}(R) \cong H_{1}(M_{R})$$

since it coincides with the first homology of the infinite cyclic covering of $E(K)$, which can be seen as the product of infinitely many copies of $M_{R}$. In particular, we may regard $H_{1}(M_{R})$ as a natural (namely, independent of choices of minimal genus Seifert surfaces) subgroup of $D_{2}(K)$. We take $\rho$ to be the natural projection

$$\rho_{2} : G(K) \longrightarrow D_{2}(K).$$
It is known that $D_2(K)$ is PTFA, so that $\mathcal{K}_{D_2(K)}$ is defined. Then, Proposition 3.5 shows that $\tau_{\rho_2}^+(M_R)$ and $r_{\rho_2}(M_R)$ can be computed by calculations on a commutative subfield $\mathcal{K}_{H_1(M_R)}$ of $\mathcal{K}_{D_2(K)}$.

Let us see an example of calculations of our invariants. Let $K$ be the knot as the boundary of the Seifert surface $R$ illustrated in Figure 4. This is the knot 0057 in Figure 7. We can easily compute that $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and the genus of $R$ is 2. Hence $K$ is a homologically fibered knot and $R$ is of minimal genus. The graph $G$ in the right hand side of Figure 4 is obtained from $R$ by a deformation retract. Thus $\pi_1(M_R) \cong \pi_1(S^3 - \mathring{N}(G))$. Then $\pi_1(M_R)$ has a presentation:

\[
\langle z_1, z_2, \ldots, z_{10} | z_1 z_5 z_6^{-1}, z_2 z_3 z_4 z_1, z_3 z_9^{-1} z_5^{-1}, z_7 z_4 z_8^{-1}, z_8 z_1 z_6, z_2 z_5 z_7^{-1} z_5^{-1}, z_9 z_4 z_10^{-1} z_4^{-1} \rangle.
\]

The first 5 relations come from the vertices of $G$ and the last 2 relations come from the crossings of $G$. We can drop the last relation $z_9 z_4 z_10^{-1} z_4^{-1}$ because it is derived from the others.

![Figure 4](image4.png)

**Figure 4**

We take a spine of $R$ as in Figure 5, by which we can fix an identification of $\Sigma_{g,1}$ and $R$. A direct computation shows that

![Figure 5](image5.png)

**Figure 5**

\[
i_-(\gamma_1) = z_5 z_1, \quad i_-(\gamma_2) = z_2^{-1}, \quad i_-(\gamma_3) = z_5 z_7^{-1} z_8^{-1} z_4^{-1}, \quad i_-(\gamma_4) = z_4^{-1}
\]
\[
i_+(\gamma_1) = z_5, \quad i_+(\gamma_2) = z_6 z_9, \quad i_+(\gamma_3) = z_6 z_5^{-1} z_3 z_5 z_7^{-1} z_4^{-1} z_6^{-1}, \quad i_+(\gamma_4) = z_6 z_7 z_6^{-1}.
\]

Here the darker color in $R$ is the $+$-side. Then, we obtain an admissible presentation of $\pi_1(M_R)$:
Generators  
\[ i_-(\gamma_1), \ldots, i_-(\gamma_4), z_1, \ldots, z_{10}, i_+(\gamma_1), \ldots, i_+(\gamma_4) \]

Relations  
\[ z_1z_5^{-1}, z_2z_2z_3z_1, z_3z_9^{-1}z_5^{-1}, z_7z_4z_8^{-1}, z_8z_9z_{10}^{-1}z_6^{-1}z_2z_5z_7^{-1}z_5^{-1}, \]
\[ i_-(\gamma_1)z_1^{-1}z_5^{-1}, i_-i_-(\gamma_2)z_2, i_-i_-(\gamma_3)z_4z_8z_7z_5^{-1}, i_-i_-(\gamma_4)z_4, \]
\[ i_+(\gamma_1)z_5^{-1}, i_+(\gamma_2)z_9z_5^{-1}, i_+(\gamma_3)z_6z_4z_7z_5^{-1}z_3^{-1}z_5z_6^{-1}, i_+(\gamma_4)z_6z_7^{-1}z_6^{-1} \]

If we have an admissible presentation, we can use the program shown in Section 5. However, we here demonstrate a calculation by hand.

By sliding the edges \( v_1 \) and \( v_2 \) of \( G \) as in Figure 6, we obtain a graph whose complement is clearly a genus 4 handlebody. This means that the complement of \( G \) (and hence \( M_R \)) is homeomorphic to a genus 4 handlebody. Let \( D_1, \ldots, D_4 \) be the meridian disks of the handlebody as illustrated in the figure.

![Figure 6](image-url)

Then, \( H_1(M_R) \) is the free abelian group generated by \( t_i \) (\( i = 1, \ldots, 4 \)) where \( t_i \) corresponding to an oriented loop which intersects \( D_i \) transversely in one point from the above to the down side in Figure 6 and is disjoint from \( D_j \) (\( i \neq j \)).

We have the natural homomorphism \( \pi_1(M_R) \rightarrow H_1(M_R) \) which maps
\[
\begin{align*}
&z_1 \mapsto t_1^{-1} & z_2 \mapsto t_2t_3^{-1} & z_3 \mapsto t_1t_2^{-1}t_3t_4^{-1} & z_4 \mapsto t_4 & z_5 \mapsto t_1t_2^{-1} \\
z_6 \mapsto t_2^{-1} & z_7 \mapsto t_2t_3^{-1} & z_8 \mapsto t_2t_3^{-1}t_4 & z_9 \mapsto t_3t_4^{-1} & z_{10} \mapsto t_3t_4^{-1} \\
i_-(\gamma_1) \mapsto t_2^{-1} & i_-(\gamma_2) \mapsto t_2^{-1}t_3 & i_-(\gamma_3) \mapsto t_1t_2^{-1}t_3t_4^{-2} & i_-(\gamma_4) \mapsto t_4^{-1} \\
i_+(\gamma_1) \mapsto t_1t_2^{-1} & i_+(\gamma_2) \mapsto t_1t_3^{-1}t_4 & i_+(\gamma_3) \mapsto t_1t_2^{-1}t_3t_4^{-2} & i_+(\gamma_4) \mapsto t_2t_3^{-1}^{-1}
\end{align*}
\]

Under the bases \( \langle [\gamma_1], [\gamma_2], [\gamma_3], [\gamma_4] \rangle \) of \( H_1(\Sigma_{2,1}) \) and \( \langle t_1, t_2, t_3, t_4 \rangle \) of \( H_1(M_R) \), the induced maps \( i_-, i_+ \) are represented by
\[
S_- = \begin{pmatrix}
0 & 0 & 1 & 0 \\
-1 & -1 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & -2 & -1
\end{pmatrix}, \quad S_+ = \begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & -1 & -2 & 1 \\
0 & 1 & 2 & -1 \\
0 & -1 & -2 & 0
\end{pmatrix}
\]

respectively. Note that \( \det(I - t(S_+^{-1}S_-)) = 1 - 2t + 3t^2 - 2t^3 + t^4 \) is the Alexander polynomial of \( K \).

Since \( M_R \) is homeomorphic to a handlebody, we have the following admissible presentation of \( \pi_1(M_R) \) by setting \( x_1 := z_1^{-1}, x_2 := z_6^{-1}, x_3 := (z_6z_7)^{-1}, x_4 := z_4, \) which are mapped to \( t_1, t_2, t_3 \) and \( t_4 \) by the homomorphism \( \pi_1(M_R) \rightarrow H_1(M_R) \).

Generators  
\[ i_-(\gamma_1), \ldots, i_-(\gamma_4), x_1, x_2, x_3, x_4, i_+(\gamma_1), \ldots, i_+(\gamma_4) \]

Relations  
\[ i_-(\gamma_1)x_1x_2x_1^{-1}, i_-(\gamma_2)x_1x_3^{-1}x_2x_1^{-1}, i_-(\gamma_3)x_4x_2x_3^{-1}x_2x_1^{-1}, i_-(\gamma_4)x_4, \]
\[ i_+(\gamma_1)x_2x_1^{-1}, i_+(\gamma_2)x_4x_3^{-1}x_2, i_+(\gamma_3)x_2^{-1}x_4x_2x_3^{-1}x_2x_1^{-1}x_4x_3^{-1}x_2, i_+(\gamma_4)x_2^{-1}x_3 \]
We write $r_1, \ldots, r_8$ for these relations in order. Note that $\mathcal{K}_{\mathcal{H}_1(M_R)}$ is isomorphic to the field of rational functions with variables $x_1, \ldots, x_4$. Then we have:

\[
\begin{pmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
x_1 & g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\
x_2 & g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\
x_3 & g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \\
x_4 & g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} & g_{48} \\
i_+(\gamma_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i_+(\gamma_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i_+(\gamma_3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i_+(\gamma_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

where $g_{ij} = \frac{\partial r_j}{\partial x_i}$. Thus $\tau_{\rho_2}^+(M_R) = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\
g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\
g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \\
g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} & g_{48} \\
\end{pmatrix}$. As a torsion, it is equivalent to

\[
\begin{pmatrix} g_{15} & g_{16} & g_{17} & g_{18} \\
g_{25} & g_{26} & g_{27} & g_{28} \\
g_{35} & g_{36} & g_{37} & g_{38} \\
g_{45} & g_{46} & g_{47} & g_{48} \\
\end{pmatrix},
\]

where

\[
g_{15} = -1, \quad g_{16} = 0, \quad g_{18} = 0, \\
g_{25} = x_1^{-1}x_2, \quad g_{26} = x_2, \quad g_{28} = -x_3, \\
g_{35} = 0, \quad g_{36} = -x_2, \quad g_{38} = x_3, \\
g_{45} = 0, \quad g_{46} = x_2x_3^{-1}x_4, \quad g_{48} = 0,
\]

Then we have:

\[
\det(\tau_{\rho_2}^+(M_R)) = \det \begin{pmatrix} g_{15} & g_{16} & g_{17} & g_{18} \\
g_{25} & g_{26} & g_{27} & g_{28} \\
g_{35} & g_{36} & g_{37} & g_{38} \\
g_{45} & g_{46} & g_{47} & g_{48} \\
\end{pmatrix} = -\frac{x_2x_4^2}{x_1x_3^2}(x_2 - x_3 + x_2x_4).
\]
The Magnus matrix \( r_{p_{2}}(M_{R}) \) can be computed by the formula in Proposition 3.5 (2). However, we omit here.

**Remark 4.1.** If we change bases of \( H_{1}(\Sigma_{2,1}) \cong H_{1}(M_{R}) \) by

\[
x_{1} = \gamma_{2}^{-2}\gamma_{3}, \quad x_{2} = \gamma_{1}^{-1}\gamma_{2}^{-2}\gamma_{3}, \quad x_{3} = \gamma_{1}^{-1}\gamma_{2}^{-2}\gamma_{3}\gamma_{4}^{-1}, \quad x_{4} = \gamma_{2}^{-1}\gamma_{4}^{-1},
\]

where \( \gamma_{j} \) denotes \( i_{+}(\gamma_{j}) \), we have

\[
\det(\tau_{p_{2}}^{+}(M_{R})) = \frac{\gamma_{3}}{\gamma_{1}^{2}\gamma_{2}^{5}\gamma_{4}}(1 + \gamma_{2} - \gamma_{2}\gamma_{4}).
\]

This expression is used in the program in Section 5.

5. MATHEMATICA PROGRAM

The following is a MATHEMATICA program which calculates the invariants discussed in the previous section.

```mathematica
hlClass = {}; hlMonodromy = {}; torsionMatrix = {}; magnusMatrix = {};

invariants[g_, z_, RELATIONS_] := Module[{reindexedRel, hlMatrix, i, alex},
  GENUS = g;
  Ztotal = z;

  reindexedRel = Map[reindexing, RELATIONS, {2}];
  hlMatrix = Map[Take[#] & , homologyComputation[reindexedRel]];
  hlClass = Join[Map[monomialExpression, hlMatrix],
    Table[ToExpression[ToString[SequenceForm["\[Gamma]", i]]], {i, 2 GENUS}]];
  Print["Homology classes of generators = ", hlClass // DisplayForm];

  hlMonodromy = Transpose[Take[hlMatrix, 2 GENUS]];
  Print["Homological monodromy = ", hlMonodromy // MatrixForm]

  alex = Transpose[makeAlexanderMatrix[reindexedRel]];
  torsionMatrix = Take[alex, 2 GENUS + Ztotal];
  Print["torsion matrix = ", torsionMatrix // MatrixForm];
  Print["det(torsion) = ", Expand[Det[torsionMatrix]]];

  magnusMatrix = Simplify[Transpose[
    Take[Transpose[Drop[alex, 2 GENUS + Ztotal]].Inverse[
      torsionMatrix]], 2 GENUS]];
  Print["Magnus matrix = ", magnusMatrix // MatrixForm]
];

reindexing[num_] := Module[{numString, sg},
  If[NumberQ[num], num + 2 GENUS*Sign[num],
    numString = ToString[num];
    sg = If[StringTake[numString, 1] == "-", 1, 0];
    If[StringTake[numString, {1 + sg}] == "m",
      \((-1)^sg\)*ToExpression[StringDrop[numString, 1 + sg]],
      \((-1)^sg\)*ToExpression[StringDrop[numString, 1 + sg]] + 2 GENUS + Ztotal]]
```
A computation by this program goes as follows. Let \((M, i_+, i_-) \in C_{g,1}\) with an admissible presentation
\[
\left\langle i_-(\gamma_1), \ldots, i_-(\gamma_g), z_1, \ldots, z_l, i_+^{(\gamma_1)}, \ldots, i_+^{(\gamma_g)} \right| r_1, \ldots, r_{2g+l}\rightangle
\]
of \(\pi_1(M)\). The main function in the program is \texttt{invariants} having three slots as the input. These slots correspond to the genus \(g\), the number \(l\) of \(z\)-generators and the list of relations. For each word in the relations, we make a list by replacing \(i_-(\gamma_j)^{\pm 1}\), \(z_j^{\pm 1}\) and \(i_+^{(\gamma_j)^{\pm 1}}\) by \(\pm mj\), \(\pm j\) and \(\pm pj\). By lining up them, we obtain the list of relations.

For example, the knot 0815 in Figure 7 has a minimal genus Seifert surface giving a sutured manifold whose fundamental group has the following admissible presentation:

\begin{tabular}{ll}
\textbf{Generators} & \(i_-(\gamma_1), \ldots, i_-(\gamma_4), z_1, \ldots, z_1, i_+^{(\gamma_1)}, \ldots, i_+^{(\gamma_4)}\) \\
\textbf{Relations} & \(z_1z_2z_6, z_1z_2^{-1}z_4^{-1}, z_4z_1^{-1}z_5, z_10z_5^{-1}z_6z_7z_8, z_8z_6^{-1}z_9z_6, \\
& \quad z_7^{-1}z_6^{-1}z_3z_6, z_4z_3^{-1}z_4^{-1}z_{10}, \\
& \quad i_-(\gamma_1)z_4z_3^{-1}z_4^{-1}, i_-(\gamma_2)z_4z_111, i_-(\gamma_3)z_9, i_-(\gamma_4)z_2^{-1}z_9^{-1}, \\
& \quad i_+(\gamma_1)z_2^{-1}z_3^{-1}z_4^{-1}, i_+(\gamma_2)z_11z_1, i_+(\gamma_3)z_9z_3^{-1}z_1, i_+(\gamma_4)z_2z_2^{-1}z_9^{-1}\)
\end{tabular}

Then, the input is:

\begin{verbatim}
invariants[2, 11, \{\{1, 9, 6\}, \{1, -2, -4\}, \{4, -11, 5\}, \\
{-10, -5, 6, 7, 8\}, \{-8, -6, 9, 6\}, \{-7, -6, 3, 6\},]
\end{verbatim}
\{4, -3, -4, 10\}, \{m1, 4, -3, -4\}, \{m2, 4, 11\},
\{m3, 9\}, \{m4, -2, -9\}, \{p1, -2, -3, -4\}, \{p2, 11, 1\},
\{p3, 9, -3, 1\}, \{p4, 9, -2, -9\}\)

Then the function returns homology classes of generators in terms of $\gamma_j := \iota_+(\gamma_j) \in H_1(M_R)$, the homological monodromy matrix $\sigma(M_R)$, the torsion matrix $\tau^{+}_{\rho_2}(M_R)$ and the Magnus matrix $r_{\rho_2}(M_R)$. These data can be referred as the variables $h1Class$, $h1Monodromy$, $torsionMatrix$ and $magnusMatrix$.

Using this program, we can easily check the calculations presented in [13] for 13 non-fibered homologically fibered knots with 12-crossings (Figure 7).

**References**


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Figure 7. Non-fibered homologically fibered knots with 12-crossings