HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGICALLY FIBERED KNOTS

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1. INTRODUCTION

This note is adapted from the talk at the 2010 Intelligence of Low-dimensional Topology at Research Institute for Mathematical Sciences, Kyoto University. For the detail, see the original papers [12], [13].

Let $\Sigma_{g,n}$ be a compact oriented surface of genus $g$ with $n \geq 1$ boundary components, and the triple $(M, i_+, i_-)$ be an oriented homology cobordism between $\Sigma_{g,n}$ and $\Sigma_{g,n}$ with two markings of $\partial M : i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$. We call $(M, i_+, i_-)$ a homology cylinder over $\Sigma_{g,n}$. This object was introduced by Goussarov [14] and Habiro [16] since it is suitable for applying the theory of clovers and claspers, and then has been studied together with finite type invariants of 3-manifolds. The following have been known as methods for constructing homology cylinders:

- connected sums of the trivial cobordism with homology 3-spheres;
- Levine's method [19] using string links in the 3-ball;
- Habegger's method [15] giving homology cylinders as results of surgeries along string links in homology 3-balls; and
- clasper surgeries (see [14] and [16]).

In [12], the authors gave an explicit construction of homology cylinders, i.e. we introduced a notion of a homologically fibered knot and construct a homology cylinder using it. The family of the homologically fibered knots include that of the fibered knots. So, roughly speaking, the following relationships exist:

\[
\begin{array}{ccc}
\text{Pure Braid} & \leftrightarrow & \text{Mapping cylinder} & \leftrightarrow & \text{Fibered knot} \\
\cap & \cap & \cap & \cap & \cap \\
\text{Pure String link} & \overset{\text{Levine}}{\leftrightarrow} & \text{Homology cylinder} & \leftrightarrow & \text{Homologically fibered knot} \\
(\text{Habegger-Lin}) & (\text{Goussarov, Habiro}) & & & 
\end{array}
\]

In [18], Kirk-Livingston-Wang introduced a Reidemeister torsion for string links, then the second author studied the corresponding Reidemeister torsion for homology cylinders in [23]. Note that this torsion may be regarded as a special case of a decategorification of sutured Floer homology [8]. In this note, we study the Reidemeister torsion for homologically fibered knots and show a factorization formula. Further, we give a MATHEMATICA program for explicit calculations of the invariants for homologically fibered knots.

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2. Homologically fibered knots

In this section, we introduce two main objects in this note: homology cylinders and sutured manifolds. First, we define homology cylinders over surfaces, which have their origin in Goussarov [14], Habiro [16], Garoufalidis-Levine [11] and Levine [19]. Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components.

**Definition 2.1.** A homology cylinder $(M, i_+, i_-)$ over $\Sigma_{g,n}$ consists of a compact oriented 3-manifold $M$ with two embeddings $i_+, i_- : \Sigma_{g,n} \to \partial M$ such that:

(i) $i_+$ is orientation-preserving and $i_-$ is orientation-reversing;
(ii) $\partial M = i_+(\Sigma_{g,n}) \cup i_-(\Sigma_{g,n})$ and $i_+(\Sigma_{g,n}) \cap i_-(\Sigma_{g,n}) = i_+(\partial \Sigma_{g,n}) = i_-(\partial \Sigma_{g,n})$;
(iii) $i_+|_{\partial \Sigma_{g,n}} = i_-|_{\partial \Sigma_{g,n}}$; and
(iv) $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ are isomorphisms.

If we replace (iv) with the condition that $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Q}) \to H_*(M; \mathbb{Q})$ are isomorphisms, then $(M, i_+, i_-)$ is called a rational homology cylinder.

![Figure 1. Homology cylinder](image)

We often write a (rational) homology cylinder $(M, i_+, i_-)$ briefly by $M$. Note that our definition is the same as that in [11] and [19] except that we may consider homology cylinders over surfaces with multiple boundaries.

Two (rational) homology cylinders $(M, i_+, i_-)$ and $(N, j_+, j_-)$ over $\Sigma_{g,n}$ are said to be isomorphic if there exists an orientation-preserving diffeomorphism $f : M \to N$ satisfying $j_+ = f o i_+$ and $j_- = f o i_-$. We denote the set of isomorphism classes of homology cylinders (resp. rational homology cylinders) over $\Sigma_{g,n}$ by $C_{g,n}$ (resp. $C_{g,n}^\mathbb{Q}$).

**Example 2.2** (Mapping cylinder). For each diffeomorphism $\varphi$ of $\Sigma_{g,n}$ which fixes $\partial \Sigma_{g,n}$ pointwise (hence, $\varphi$ preserves the orientation of $\Sigma_{g,n}$), we can construct a homology cylinder by setting

$$(\Sigma_{g,n} \times [0,1], \text{id} \times 1, \varphi \times 0),$$

where collars of $i_+(\Sigma_{g,n})$ and $i_-(\Sigma_{g,n})$ are stretched half-way along $(\partial \Sigma_{g,n}) \times [0,1]$. It is easily checked that the isomorphism class of $(\Sigma_{g,n} \times [0,1], \text{id} \times 1, \varphi \times 0)$ depends only on the (boundary fixing) isotopy class of $\varphi$. Therefore, this construction gives a map from the mapping class group $\mathcal{M}_{g,n}$ of $\Sigma_{g,n}$ to $C_{g,n}$. 
Next, we recall the definition of sutured manifolds given by Gabai [10]. We here use a special case of them.

A sutured manifold $(M, \gamma)$ is a compact oriented 3-manifold $M$ together with a subset $\gamma \subset \partial M$ which is a union of finitely many mutually disjoint annuli. For each component of $\gamma$, an oriented core circle called a suture is fixed, and we denote the set of sutures by $s(\gamma)$. Every component of $R(\gamma) = \partial M - \text{Int}\, \gamma$ is oriented so that the orientations on $R(\gamma)$ are coherent with respect to $s(\gamma)$, that is, the orientation of each component of $\partial R(\gamma)$ induced from that of $R(\gamma)$ is parallel to the orientation of the corresponding component of $s(\gamma)$. We denote by $R^+_\gamma$ (resp. $R^-_\gamma$) the union of those components of $R(\gamma)$ whose normal vectors point out of (resp. into) $M$.

**Example 2.3.** For a knot $K$ in $S^3$ and a Seifert surface $\overline{R}$ of $K$, we set $R := \overline{R} \cap E(K)$, called also a Seifert surface, where $E(K) = \overline{S^3 - N(K)}$ is the complement of a regular neighborhood $N(K)$ of $K$. Then $(M_R, \gamma) := (E(K) - N(R), \partial E(K) - N(\partial R))$ defines a sutured manifold. We call it the complementary sutured manifold for $R$. In this paper, we simply call it the sutured manifold for $R$.

![Figure 2. Complementary sutured manifold](image)

Let $L$ be an oriented link in the 3-sphere $S^3$, and $\Delta_L(t)$ the normalized (one variable) Alexander polynomial of $L$, i.e. the lowest degree of $\Delta_L(t)$ is 0.

**Definition 2.4.** An $n$-component link $L$ in $S^3$ is said to be homologically fibered if $L$ satisfies the following two conditions:

(i) The degree of $\Delta_L(t)$ is $2g + n - 1$, where $g$ is the genus of a connected Seifert surface of $L$; and

(ii) $\Delta_L(0) = \pm 1$.

If an $n$-component link $L$ satisfies (i), then $L$ is said to be rationally homologically fibered.

The Alexander polynomial that satisfies the condition (ii) is said to be monic in this paper.

**Remark 2.5.** In general, if $L$ bounds a connected Seifert surface of genus $g$, then

$$2g + n - 1 \geq \text{(the degree of } \Delta_L(t)).$$

It is known ([5], [21]) that if $L$ has an alternating diagram that gives, by the Seifert algorithm, a connected Seifert surface of genus $g$, then the degree of $\Delta_L(t)$ is equal to $2g + n - 1$. 
Remark 2.6. Suppose $L$ is an alternating link. Then, $L$ is fibered if and only if $\Delta_L(t)$ is monic, by Murasugi [22] (see also 13.26 (c) in [1]). Therefore, if a homologically fibered link $L$ is not fibered, then $L$ is non-alternating.

Let $L$ be an $n$-component link and $\Sigma_{g,n}$ the compact oriented surface that is diffeomorphic to a Seifert surface $R$ of $L$. We fix a diffeomorphism $\vartheta: \Sigma_{g,n} \cong R$ and denote by $(M_R, \gamma)$ the complementary sutured manifold for $R$. Then we may see that there are an orientation-preserving embedding $i_+ : \Sigma_{g,n} \to M_R$ and an orientation-reversing embedding $i_- : \Sigma_{g,n} \to M_R$ with $i_+(\Sigma_{g,n}) = R_+(\gamma)$ and $i_-(\Sigma_{g,n}) = R_-(\gamma)$, where two embeddings $i_\pm$ are the composite mappings of $\vartheta$ and embeddings $i_\pm : R \hookrightarrow M_R$ such that $i_\pm = i_\pm \circ \vartheta : \Sigma_{g,n} \to R_\pm(\gamma) \subset M_R$:

$$
\begin{array}{ccc}
\Sigma_{g,n} & \xrightarrow{\vartheta} & R \\
\downarrow i_\pm & & \downarrow i_\pm \\
M_R & & \\
\end{array}
$$

If $i_+, i_- : H_1(\Sigma_{g,n}) \to H_1(M_R)$ are isomorphisms, we may regard $(M_R, \gamma)$ as a homology cylinder. The next proposition was essentially mentioned in [6]. A proof is given in [12].

Proposition 2.7. Let $R$ be a Seifert surface of a link $L$. If the complementary sutured manifold for $R$ is a homology cylinder, then $L$ is homologically fibered. Conversely, if $L$ is homologically fibered, then the complementary sutured manifold for each minimal genus Seifert surface of $L$ is a homology cylinder.

It is known that all homologically fibered knots are fibered among prime knots with at most 11 crossings. On the other hand, Friedl-Kim [9] (see also [2]) showed that there are 13 non-fibered homologically fibered knots with 12-crossings. See Figure 7.

3. Factorization formulas of Alexander invariants

Let $R$ be a minimal genus Seifert surface of a rationally homologically fibered knot $K$ in $S^3$, and $M_R$ be the sutured manifold for $R \cong \Sigma_{g,1}$. We fix a basis of $H_1(R; \mathbb{Q})$, which yields an isomorphism $H_1(R; \mathbb{Q}) \cong \mathbb{Q}^{2g}$. Then we can rewrite the definition $\Delta_K(t) = \det(S - tS^T)$ of the Alexander polynomial of $K$ by using the invertibility (over $\mathbb{Q}$) of the Seifert matrix $S$, and obtain a factorization

\begin{equation}
\Delta_K(t) = \det(S) \det(I_{2g} - t\sigma(M_R))
\end{equation}

of $\Delta_K(t)$. Note that $\sigma(M_R) := S^{-1}S^T$ represents the composite of isomorphisms

$$
\mathbb{Q}^{2g} \cong H_1(R; \mathbb{Q}) \xrightarrow{\iota_-} H_1(M_R; \mathbb{Q}) \xrightarrow{\iota_+^{-1}} H_1(R; \mathbb{Q}) \cong \mathbb{Q}^{2g}.
$$

The matrix $\sigma(M_R)$ can be interpreted as a monodromy of $M_R$ from a view point of the rational homology. Regarding the formula (3.1) as a basic case, we constructed in [12] its generalization under the framework of higher-order Alexander invariants due to Cochran [3], Harvey [17] and Friedl [7]. In this procedure, the Seifert matrix $S$, the monodromy $\sigma(M_R)$ and $\Delta_K(t)$ are generalized to a certain Reidemeister torsion $\tau^+_\psi(M_R)$, the Magnus
matrix \( r_{\rho}(M_{R}) \) and some higher-order (non-commutative) Reidemeister torsion \( \tau_{\rho}(E(K)) \) associated with a representation \( \rho \) of the fundamental group of \( M_{R} \).

Here, we review higher-order Alexander invariants quickly. For a matrix \( A \) with entries in a group ring \( \mathbb{Z}G \) (or its quotient field) for a group \( G \), we denote by \( A \) the matrix obtained from \( A \) by applying the involution induced from \( (x \mapsto x^{-1}, x \in G) \) to each entry. For a module \( M \), we write \( M^{n} \) for the module of column vectors with \( n \) entries. For a finite cell complex \( X \), we denote by \( \tilde{X} \) its universal covering. We take a base point \( p \) of \( X \) and a lift \( \tilde{p} \) of \( p \) as a base point of \( \tilde{X} \). \( \pi := \pi_{1}(X, p) \) acts on \( \tilde{X} \) from the right through its deck transformation group, so that the lift of a loop \( l \in \pi \) starting from \( \tilde{p} \) reaches \( \tilde{p}l^{-1} \). Then the cellular chain complex \( C_{*}(\tilde{X}) \) of \( \tilde{X} \) becomes a right \( \mathbb{Z}\pi \)-module. For each left \( \mathbb{Z}\pi \)-algebra \( R \), the twisted chain complex \( C_{*}(X; R) \) is given by the tensor product of the right \( \mathbb{Z}\pi \)-module \( C_{*}(\tilde{X}) \) and the left \( \mathbb{Z}\pi \)-module \( R \), so that \( C_{*}(X; R) \) and \( H_{*}(X; R) \) are right \( R \)-modules.

In the definition of higher-order Alexander invariants, PTFA groups play important roles, where a group \( \Gamma \) is said to be poly-torsion-free abelian (PTFA) if it has a sequence

\[ \Gamma = \Gamma_{0} \triangleright \Gamma_{1} \triangleright \cdots \triangleright \Gamma_{n} = \{1\} \]

whose successive quotients \( \Gamma_{i}/\Gamma_{i+1} \) (\( i \geq 0 \)) are all torsion-free abelian. An advantage of using PTFA groups is that the group ring \( \mathbb{Z}\Gamma \) (or \( \mathbb{Q}\Gamma \)) of \( \Gamma \) is known to be an Ore domain so that it can be embed into the field (skew field in general)

\[ \mathcal{K}_{\Gamma} := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1} = \mathbb{Q}\Gamma(\mathbb{Q}\Gamma - \{0\})^{-1} \]

called the right field of fractions. A typical example of PTFA groups is \( \mathbb{Z}^{n} \), where \( \mathcal{K}_{\mathbb{Z}^{n}} \) is isomorphic to the field of rational functions with \( n \) variables.

For a rationally homologically fibered knot \( K \), we take a homomorphism \( \rho : G(K) := \pi_{1}(E(K)) \to \Gamma \) whose target \( \Gamma \) is PTFA. We suppose that \( \rho \) is non-trivial. We regard \( \mathcal{K}_{\Gamma} \) as a local coefficient system on \( E(K) \) through \( \rho \).

**Lemma 3.1** (Cochran [3, Lemma 3.9]). For any non-trivial homomorphism \( \rho : G(K) \to \Gamma \) to a PTFA group \( \Gamma \), we have \( H_{*}(E(K); \mathcal{K}_{\Gamma}) = 0 \).

By this lemma, we can define the Reidemeister torsion

\[ \tau_{\rho}(E(K)) := \tau(C_{*}(E(K); \mathcal{K}_{\Gamma}))/ \pm \rho(G(K)) \]

for the acyclic complex \( C_{*}(E(K); \mathcal{K}_{\Gamma}) \). We refer to Milnor [20] for generalities of torsions. By higher-order Alexander invariants for \( K \), we here mean this torsion \( \tau_{\rho}(E(K)) \).

We now describe a factorization of \( \tau_{\rho}(E(K)) \) generalizing (3.1). Let \( (M_{R}, i_{+}, i_{-}) \in C^{Q}_{g,1} \) be the rational homology cylinder obtained as the sutured manifold for a minimal genus Seifert surface \( R \) of \( K \). We use the same notation \( \rho : \pi_{1}(M_{R}) \to \Gamma \) for the composition \( \pi_{1}(M_{R}) \to G(K) \overset{\rho}{\to} \Gamma \). Applying Cochran-Orr-Teichner [4, Proposition 2.10], we have the following:

**Lemma 3.2.** \( i_{+}, i_{-} : H_{*}(\Sigma_{g,1}, p; i_{+}^{*} \mathcal{K}_{\Gamma}) \to H_{*}(M_{R}, p; \mathcal{K}_{\Gamma}) \) are isomorphisms as right \( \mathcal{K}_{\Gamma} \)-vector spaces. Equivalently, \( H_{*}(M_{R}, i_{\pm}(\Sigma_{g,1}); \mathcal{K}_{\Gamma}) = 0 \).
This lemma provides the following two kinds of invariants for $M_R$.

**The Magnus matrix** Let $X \subset \Sigma_{g,1}$ be the bouquet of $2g$ circles $\gamma_1, \ldots, \gamma_{2g}$ tied at $p$ (see Figure 3). $X$ is a deformation retract of $\Sigma_{g,1}$ relative to $p$. Therefore, for $\pm \in \{+,-\}$, we have

$$H_1(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_\Gamma) \cong H_1(X, p; i_\pm^* \mathcal{K}_\Gamma) = C_1(\tilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_\pm^* \mathcal{K}_\Gamma \cong \mathcal{K}_\Gamma^{2g}$$

with a basis

$$\{\tilde{\gamma}_1 \otimes 1, \ldots, \tilde{\gamma}_{2g} \otimes 1\} \subset C_1(\tilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_\pm^* \mathcal{K}_\Gamma$$

as a right $\mathcal{K}_\Gamma$-vector space. Here we fix a lift $\tilde{p}$ of $p$ as a base point of $\tilde{X}$, and denote by $\tilde{\gamma}_i$ the lift of the oriented loop $\gamma_i$ starting from $\tilde{p}$.

**Definition 3.3.** For $M_R = (M_R, i_+, i_-) \in C_{g,1}^\mathbb{Q}$, the **Magnus matrix**

$$r_\rho(M_R) \in GL(2g, \mathcal{K}_\Gamma)$$

of $M_R$ is defined as the representation matrix of the right $\mathcal{K}_\Gamma$-isomorphism

$$\mathcal{K}_\Gamma^{2g} \cong H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \xrightarrow{i_-^*} H_1(M_R, p; \mathcal{K}_\Gamma) \xrightarrow{i_+^*} H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \cong \mathcal{K}_\Gamma^{2g},$$

where the first and the last isomorphisms use the bases mentioned above.

The matrix $r_\rho(M_R)$ can be interpreted as a monodromy of $M_R$ from a view point of the twisted homology with coefficients in $\mathcal{K}_\Gamma$.

**\(\Gamma\)-torsion** Since the relative complex $C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)$ obtained from any cell decomposition of $(M_R, i_+(\Sigma_{g,1}))$ is acyclic by Lemma 3.2, we can define the following:

**Definition 3.4.** For $M_R = (M_R, i_+, i_-) \in C_{g,1}^\mathbb{Q}$, the **\(\Gamma\)-torsion** $\tau_\rho^+(M_R)$ of $M_R$ is defined by

$$\tau_\rho^+(M_R) := \tau(C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)) \in K_1(\mathcal{K}_\Gamma)/\pm \rho(\pi_1(M_R)).$$

A method for computing $r_\rho(M_R)$ and $\tau_\rho^+(M_R)$ is given in [12, Section 4], which is based on Kirk-Livingston-Wang's method [18] for invariants of string links, and we now recall it briefly. An **admissible presentation** of $\pi_1(M_R)$ is defined to be the one of the form

$$(3.2) \langle i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) | r_1, \ldots, r_{2g+l}\rangle$$
for some integer $l$. That is, it is a finite presentation with deficiency $2g$ whose generating set contains $i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), i_+(\gamma_1), \ldots, i_+(\gamma_{2g})$ and is ordered as above. Such a presentation always exists. For any admissible presentation, define $2g \times (2g + l), l \times (2g + l)$ and $2g \times (2g + l)$ matrices $A, B, C$ over $\mathbb{Z} \Gamma$ by

$$A = \left( \frac{\partial r_j}{\partial i_-(\gamma_i)} \right)_{1 \leq j \leq 2g, 1 \leq i \leq 2g + l}, \quad B = \left( \frac{\partial r_j}{\partial z_{i_1}} \right)_{1 \leq j \leq l, 1 \leq i \leq 2g + l}, \quad C = \left( \frac{\partial r_j}{\partial i_+(\gamma_i)} \right)_{1 \leq j \leq 2g, 1 \leq i \leq 2g + l}$$

**Proposition 3.5** ([12, Propositions 4.5, 4.6]). As matrices with entries in $\mathcal{K}_\Gamma$, we have:

1. The square matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is invertible and $\tau_{\rho}^+(M_R) = \begin{pmatrix} A \\ \Gamma \end{pmatrix}$; and
2. $r_\rho(M_R) = -C \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} I_{2g} \\ 0_{(l, 2g)} \end{pmatrix}$

Using the above invariants, the factorization formula for $\tau_{\rho}(E(K))$ is given as follows:

**Theorem 3.6.** Let $K$ be a rationally homologically fibered knot of genus $g$. For any non-trivial homomorphism $\rho : G(K) \to \Gamma$ to a PTFA group $\Gamma$, a loop $\mu$ representing the meridian of $K$ satisfies $\rho(\mu) \neq 1 \in \Gamma \subset \mathcal{K}_\Gamma$ and we have a factorization

$$\tau_{\rho}(E(K)) = \frac{\tau_{\rho}^+(M_R) \cdot (I_{2g} - \rho(\mu)r_\rho(M_R))}{1 - \rho(\mu)} \in K_1(\mathcal{K}_\Gamma)/ \pm \rho(G(K))$$

of the torsion $\tau_{\rho}(E(K))$.

To compare (3.3) with (3.1), recall Milnor's formula [20] that $\frac{\Delta_K(t)}{1 - t}$ represents the Reidemeister torsion associated with the abelianization map $\rho_1 : G(K) \to \langle t \rangle \subset \mathbb{Q}(t)$. Taking $\rho_1$ as $\rho$, we recover the formula (3.1).

4. **Computations**

Although all the ingredients in the formula (3.3) are theoretically determined by information on fundamental groups, it is difficult to compute them explicitly because of the non-commutativity of $\mathcal{K}_\Gamma$ except in some special cases including the following.

Let $K$ be a homologically fibered knot with a minimal genus Seifert surface $R$ and let $M_R$ be the sutured manifold for $R$. Consider the group extension

$$1 \longrightarrow G(K)'/G(K)'' \longrightarrow D_2(K) \longrightarrow G(K)/G(K)' = H_1(E(K)) \cong \mathbb{Z} \longrightarrow 1$$

relating to the metabelian quotient $D_2(K) := G(K)/G(K)''$ of $G(K)$. We have

$$G(K)'/G(K)'' \cong H_1(R) \cong H_1(M_R)$$

since it coincides with the first homology of the infinite cyclic covering of $E(K)$, which can be seen as the product of infinitely many copies of $M_R$. In particular, we may regard $H_1(M_R)$ as a natural (namely, independent of choices of minimal genus Seifert surfaces) subgroup of $D_2(K)$. We take $\rho$ to be the natural projection

$$\rho_2 : G(K) \longrightarrow D_2(K).$$
It is known that $D_2(K)$ is PTFA, so that $K_{D_2(K)}$ is defined. Then, Proposition 3.5 shows that $\tau_{\rho_2}(M_R)$ and $r_{\rho_2}(M_R)$ can be computed by calculations on a commutative subfield $K_{H_1(M_R)}$ of $K_{D_2(K)}$.

Let us see an example of calculations of our invariants. Let $K$ be the knot as the boundary of the Seifert surface $R$ illustrated in Figure 4. This is the knot 0057 in Figure 7. We can easily compute that $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and the genus of $R$ is 2. Hence $K$ is a homologically fibered knot and $R$ is of minimal genus. The graph $G$ in the right hand side of Figure 4 is obtained from $R$ by a deformation retract. Thus $\pi_1(M_R) \cong \pi_1(S^3 - \mathring{N}(G))$. Then $\pi_1(M_R)$ has a presentation:

$$\langle z_1, z_2, \ldots, z_{10} | z_1z_5z_6^{-1}, z_2z_3z_4z_1, z_3z_9^{-1}z_5^{-1}, z_7z_4z_8^{-1}, z_8z_10z_6, z_2z_6z_7^{-1}z_5^{-1}, z_9z_4z_10^{-1}z_4^{-1} \rangle.$$

The first 5 relations come from the vertices of $G$ and the last 2 relations come from the crossings of $G$. We can drop the last relation $z_9z_4z_10^{-1}z_4^{-1}$ because it is derived from the others.

![Figure 4](image_url)

We take a spine of $R$ as in Figure 5, by which we can fix an identification of $\Sigma_{g,1}$ and $R$. A direct computation shows that

![Figure 5](image_url)

Here the darker color in $R$ is the $+$-side. Then, we obtain an admissible presentation of $\pi_1(M_R)$:
Generators \( i_-(\gamma_1), \ldots, i_-(\gamma_4), i_+(\gamma_1), \ldots, i_+(\gamma_4) \)

Relations
\[
\begin{align*}
& z_1 z_6^{-1}, z_2 z_4 z_1, z_3 z_9^{-1} z_5^{-1}, z_7 z_4 z_8^{-1}, z_8 z_10 z_6, z_2 z_5 z_7^{-1} z_5^{-1}, \\
& i_-(\gamma_1) z_1^{-1} z_5^{-1}, i_-(\gamma_2) z_2, i_-(\gamma_3) z_4 z_6 z_7 z_5^{-1}, i_-(\gamma_4) z_4, \\
& i_+(\gamma_1) z_5^{-1}, i_+(\gamma_2) z_9^{-1} z_6^{-1}, i_+(\gamma_3) z_6 z_4 z_7 z_5^{-1} z_3^{-1} z_5 z_6^{-1}, i_+(\gamma_4) z_6 z_7^{-1} z_6^{-1}
\end{align*}
\]

If we have an admissible presentation, we can use the program shown in Section 5. However, we here demonstrate a calculation by hand.

By sliding the edges \( v_1 \) and \( v_2 \) of \( G \) as in Figure 6, we obtain a graph whose complement is clearly a genus 4 handlebody. This means that the complement of \( G \) (and hence \( M_R \)) is homeomorphic to a genus 4 handlebody. Let \( D_1, \ldots, D_4 \) be the meridian disks of the handlebody as illustrated in the figure.

![Figure 6](image)

Then, \( H_1(M_R) \) is the free abelian group generated by \( t_i \) (\( i = 1, \ldots, 4 \)) where \( t_i \) corresponding to an oriented loop which intersects \( D_i \) transversely in one point from the above to the down side in Figure 6 and is disjoint from \( D_j \) (\( i \neq j \)).

We have the natural homomorphism \( \pi_1(M_R) \to H_1(M_R) \) which maps
\[
\begin{align*}
& z_1 \mapsto t_1^{-1} \quad z_2 \mapsto t_2 t_3^{-1} \quad z_3 \mapsto t_1 t_2^{-1} t_3 t_4^{-1} \quad z_4 \mapsto t_4 \quad z_5 \mapsto t_1 t_2^{-1} \\
& z_6 \mapsto t_2^{-1} \quad z_7 \mapsto t_2 t_3^{-1} \quad z_8 \mapsto t_2 t_3^{-1} t_4 \quad z_9 \mapsto t_3 t_4^{-1} \quad z_{10} \mapsto t_3 t_4^{-1} \\
& i_-(\gamma_1) \mapsto t_2^{-1} \quad i_-(\gamma_2) \mapsto t_2 t_3^{-1} t_4 \quad i_-(\gamma_3) \mapsto t_1 t_2^{-3} t_3 t_4^{-2} \quad i_-(\gamma_4) \mapsto t_4 \\
& i_+(\gamma_1) \mapsto t_1 t_2^{-1} \quad i_+(\gamma_2) \mapsto t_1 t_2^{-1} t_3 t_4^{-1} \quad i_+(\gamma_3) \mapsto t_1 t_2^{-1} t_3 t_4^{-1} \quad i_+(\gamma_4) \mapsto t_1^{-1} t_2^{-1} \quad i_+(\gamma_4) \mapsto t_1^{-1} t_2^{-1}
\end{align*}
\]

Under the bases \(( [\gamma_1], [\gamma_2], [\gamma_3], [\gamma_4] )\) of \( H_1(\Sigma_{2,1}) \) and \( \langle t_1, t_2, t_3, t_4 \rangle \) of \( H_1(M_R) \), the induced maps \( i_-, i_+ \) are represented by
\[
S_\ = \ \begin{pmatrix}
0 & 0 & 1 & 0 \\
-1 & -1 & -3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & -2 & -1
\end{pmatrix}, \quad S_+ = \begin{pmatrix}
1 & 0 & 1 & 0 \\
-1 & -1 & -2 & 1 \\
0 & 1 & 2 & -1 \\
0 & -1 & -2 & 0
\end{pmatrix}
\]

respectively. Note that \( \det(I - t(S_+^{-1}S_-)) = 1 - 2t + 3t^2 - 2t^3 + t^4 \) is the Alexander polynomial of \( K \).

Since \( M_R \) is homeomorphic to a handlebody, we have the following admissible presentation of \( \pi_1(M_R) \) by setting \( x_1 := z_1^{-1}, x_2 := z_6^{-1}, x_3 := (z_6 z_7)^{-1} \) and \( x_4 := z_4 \), which are mapped to \( t_1, t_2, t_3 \) and \( t_4 \) by the homomorphism \( \pi_1(M_R) \to H_1(M_R) \).

Generators \( i_-(\gamma_1), \ldots, i_-(\gamma_4), i_+(\gamma_1), \ldots, i_+(\gamma_4) \)

Relations
\[
\begin{align*}
& i_-(\gamma_1) x_1 x_2 x_1^{-1}, i_-(\gamma_2) x_1 x_3^{-1} x_2 x_1^{-1}, i_-(\gamma_3) x_4 x_2 x_3^{-1} x_2 x_1^{-1}, i_-(\gamma_4) x_4, \\
& i_+(\gamma_1) x_2 x_1^{-1}, i_+(\gamma_2) x_4 x_3^{-1} x_2, i_+(\gamma_3) x_2 x_3^{-1} x_2 x_1^{-1} x_4 x_2 x_3^{-1} x_2, i_+(\gamma_4) x_2^{-1} x_3
\end{align*}
\]
We write $r_1, \ldots, r_8$ for these relations in order. Note that $\mathcal{K}_{H_1(M_R)}$ is isomorphic to the field of rational functions with variables $x_1, \ldots, x_4$. Then we have:

\[
(A) = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \\ i_-(\gamma_1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
i_-(\gamma_2) & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
i_-(\gamma_3) & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
i_-(\gamma_4) & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\]

\[
(B) = \begin{pmatrix}
x_1 & g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\
x_2 & g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\
x_3 & g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \\
x_4 & g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} & g_{48} \\
i_+(\gamma_1) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
i_+(\gamma_2) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
i_+(\gamma_3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
i_+(\gamma_4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\]

where $g_{ij} = \overline{\partial r_j / \partial x_i}$. Thus $\tau^+_{\rho_2}(M_R) = (A) = (B)$, as a torsion, it is equivalent to

\[
\begin{pmatrix}
g_{15} & g_{16} & g_{17} & g_{18} \\
g_{25} & g_{26} & g_{27} & g_{28} \\
g_{35} & g_{36} & g_{37} & g_{38} \\
g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix},
\]

where

\[
g_{15} = -1, \quad g_{16} = 0, \quad g_{18} = 0,
\]

\[
g_{25} = x_1^{-1}x_2, \quad g_{26} = x_2, \quad g_{28} = -x_3,
\]

\[
g_{35} = 0, \quad g_{36} = -x_2, \quad g_{38} = x_3,
\]

\[
g_{45} = 0, \quad g_{46} = x_2x_3^{-1}x_4, \quad g_{48} = 0,
\]

\[
g_{17} = -x_2x_3^{-1}x_4,
\]

\[
g_{27} = x_2 + x_1^{-1}x_2x_3^{-1}x_4 + x_1^{-1}x_2x_3^{-2}x_4 - x_1^{-1}x_2x_3^{-2}x_4^2,
\]

\[
g_{37} = -x_2 - x_1^{-1}x_2x_3^{-1}x_4,
\]

\[
g_{47} = x_2x_3^{-1}x_4 + x_1^{-1}x_2x_3^{-2}x_4^2.
\]

Then we have:

\[
\det(\tau^+_{\rho_2}(M_R)) = \det\begin{pmatrix} g_{15} & g_{16} & g_{17} & g_{18} \\
g_{25} & g_{26} & g_{27} & g_{28} \\
g_{35} & g_{36} & g_{37} & g_{38} \\
g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix} = \frac{x_2^3x_4^2}{x_1x_3}(x_2 - x_3 - x_2x_4).
\]
The Magnus matrix $r_{\rho_{2}}(M_{R})$ can be computed by the formula in Proposition 3.5 (2). However we omit here.

Remark 4.1. If we change bases of $H_{1}(\Sigma_{2,1}) \cong H_{1}(M_{R})$ by

$$x_{1} = \gamma_{2}^{-2}\gamma_{3}, \quad x_{2} = \gamma_{1}^{-1}\gamma_{2}^{-2}\gamma_{3}, \quad x_{3} = \gamma_{1}^{-1}\gamma_{2}^{-2}\gamma_{3}\gamma_{4}^{-1}, \quad x_{4} = \gamma_{2}^{-1}\gamma_{4}^{-1},$$

where $\gamma_{j}$ denotes $i_{+}(\gamma_{j})$, we have

$$\det(\tau_{\rho_{2}}^{+}(M_{R})) = \frac{\gamma_{3}}{\gamma_{1}^{2}\gamma_{2}^{5}\gamma_{4}}(1 + \gamma_{2} - \gamma_{2}\gamma_{4}).$$

This expression is used in the program in Section 5.

5. MATHEMATICA PROGRAM

The following is a MATHEMATICA program which calculates the invariants discussed in the previous section.

```mathematica
hlClass = {}; hlMonodromy = {}; torsionMatrix = {}; magnusMatrix = {};

invariants[g_, z_, RELATIONS_] := Module[{reindexedRel, hlMatrix, i, alex, GENUS = g, Ztotal = z;

  reindexedRel = Map[reindexing, RELATIONS, {2}];
  hlMatrix = Map[Transpose[Take[hlMatrix, 2 GENUS + Ztotal]], (i, 2 GENUS)];
  Print["Homology classes of generators = ", hlClass // DisplayForm];

  hlMonodromy = Transpose[Take[hlMatrix, 2 GENUS]]; Print["Homological monodromy = ", hlMonodromy // MatrixForm];

  alex = Transpose[makeAlexanderMatrix[reindexedRel]]; torsionMatrix = Take[alex, 2 GENUS + Ztotal];
  Print["torsion matrix = ", torsionMatrix // MatrixForm];
  Print["det(torsion) = ", Expand[Det[torsionMatrix]]];

  magnusMatrix = Simplify[Transpose[Take[Transpose[-Drop[alex, 2 GENUS + Ztotal].Inverse[torsionMatrix]], 2 GENUS]]];
  Print["Magnus matrix = ", magnusMatrix // MatrixForm];

  reindexing[num_] := Module[{numString, sg},

    If[NumberQ[num], num = 2 GENUS + Sign[num],
      numString = ToString[num]; sg = If[StringTake[numString, 1] == ":", 1, 0];
      If[StringTake[numString, 1 + sg] == "m", ((-1)^sg)ToExpression[StringDrop[numString, 1 + sg]],
        ((-1)^sg)*ToExpression[StringDrop[numString, 1 + sg] + 2 GENUS + Ztotal]]
    ];
}
```
Let \( s[2,11,] \) makeMonomial

\[
\text{foxDer}=0, i),
\]

\[
\text{entry} = entry - (\text{makeMonomial}[\text{take}[\text{word}, i])^(-1)\]};
\]

\[
\text{entry} = entry + (\text{makeMonomial}[\text{take}[\text{word}, i-1])^(-1)
\]

\[
\text{entry} = entry - (\text{makeMonomial}[\text{take}[\text{word}, i])^(-1)\]}

\[
\text{entry} = entry + (\text{makeMonomial}[\text{take}[\text{word}, i-1])^(-1)
\]

A computation by this program goes as follows. Let \( (M, i_{+}, i_{-}) \in \mathcal{C}_{g,1} \) with an admissible presentation

\[
\langle i_{-}(\gamma_{1}), \ldots, i_{-}(\gamma_{2g}), z_{1}, \ldots, z_{l}, i_{+}(\gamma_{1}), \ldots, i_{+}(\gamma_{2g}) \mid r_{1}, \ldots, r_{2g+l}\rangle
\]

of \( \pi_{1}(M) \). The main function in the program is \text{invariants} having three slots as the input. These slots correspond to the genus \( g \), the number \( l \) of \( z \)-generators and the list of relations. For each word in the relations, we make a list by replacing \( i_{-}(\gamma_{j})^{\pm 1} \), \( z_{j}^{\pm 1} \) and \( i_{+}(\gamma_{j})^{\pm 1} \) by \( \pm mj, \pm j \) and \( \pm pj \). By lining up them, we obtain the list of relations.

For example, the knot 0815 in Figure 7 has a minimal genus Seifert surface giving a sutured manifold whose fundamental group has the following admissible presentation:

\[
\text{Generators} \quad i_{-}(\gamma_{1}), \ldots, i_{-}(\gamma_{4}), z_{1}, \ldots, z_{11}, i_{+}(\gamma_{1}), \ldots, i_{+}(\gamma_{4})
\]

\[
\text{Relations} \quad z_{1}z_{2}z_{6}, z_{1}z_{2}^{-1}z_{4}^{-1}, z_{4}z_{11}z_{5}, z_{10}z_{5}^{-1}z_{6}z_{2}z_{8}, z_{8}z_{6}^{-1}z_{9}z_{6},
\]

\[
\quad z_{7}z_{6}^{-1}z_{3}z_{26}, z_{4}z_{3}^{-1}z_{4}^{-1}z_{10},
\]

\[
\quad i_{-}(\gamma_{1})z_{4}z_{3}^{-1}z_{4}^{-1}, i_{-}(\gamma_{2})z_{4}z_{11}, i_{-}(\gamma_{3})z_{9}, i_{-}(\gamma_{4})z_{2}^{-1}z_{9}^{-1},
\]

\[
\quad i_{+}(\gamma_{1})z_{2}^{-1}z_{3}z_{4}^{-1}, i_{+}(\gamma_{2})z_{11}z_{1}, i_{+}(\gamma_{3})z_{9}z_{3}^{-1}z_{1}, i_{+}(\gamma_{4})z_{9}z_{2}^{-1}z_{9}^{-1}
\]

Then, the input is:

\[
\text{invariants}[2, 11, \{1, 9, 6\}, \{1, -2, -4\}, \{4,-11, 5\},
\]

\[
\{-10, -5, 6, 7, 8\}, \{-8, -6, 9, 6\}, \{-7, -6, 3, 6\}
\]
\{4, -3, -4, 10\}, \{m1, 4, -3, -4\}, \{m2, 4, 11\},
\{m3, 9\}, \{m4, -2, -9\}, \{p1, -2, -3, -4\}, \{p2, 11, 1\},
\{p3, 9, -3, 1\}, \{p4, 9, -2, -9\}\)

Then the function returns homology classes of generators in terms of $\gamma_j := i_+ (\gamma_j) \in H_1 (M_R)$, the homological monodromy matrix $\sigma (M_R)$, the torsion matrix $\tau^+_{\rho_2} (M_R)$ and the Magnus matrix $r_{\rho_2} (M_R)$. These data can be referred as the variables h1Class, h1Monodromy, torsionMatrix and magnusMatrix.

Using this program, we can easily check the calculations presented in [13] for 13 non-fibered homologically fibered knots with 12-crossings (Figure 7).

REFERENCES


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Figure 7. Non-fibered homologically fibered knots with 12-crossings