An invitation to V.F.R. Jones' planar algebras

Nobuya SATO
Rikkyo University

Abstract
This article is aimed to explain the basic ideas of planar algebras for topological applications in mind. We will start with some familiar diagrammatic algebras and will give some known examples of subfactor planar algebras without any knowledge of subfactor theory. We will briefly see some applications of (subfactor) planar algebras to low dimensional topology.

1 Introduction – diagrammatic algebras –
In the last 25 years, we have witnessed that pictorial expression of some algebras is useful to construct link invariants and 3-manifold invariants. For instance, pictorial expressions of the elements in Temperley-Lieb algebras induce the Jones polynomials for links and those of so-called BMW algebras introduced by J. Murakami in [Mu] and Birman-Wenzl in [BW] independently induce the Kauffman polynomials.

As the introduction of this article, let us recall the definitions of the Temperley-Lieb algebras and the BMW algebras and we will describe the diagrams for the generating elements. In this article, our ground field is always assumed to be the field of complex numbers \( \mathbb{C} \).

Example 1 Temperley-Lieb algebras.

Definition 1 Let \( n \) be a natural number and \( \delta \) be a non-zero complex number. The \( n \)-th Temperley-Lieb algebra \( TL_n(\delta) \) is an algebra over \( \mathbb{C} \) generated by the unit 1 and \( E_1, \ldots, E_{n-1} \) obeying the following relations:
for $i, j = 1, \ldots, n - 1$,

\[ E_i^2 = \delta E_i, \quad (1.1) \]
\[ E_i E_j = E_j E_i \ (|i - j| \geq 2), \quad (1.2) \]
\[ E_i E_{i\pm 1} E_i = E_{i\pm 1} E_i E_{i\pm 1}. \quad (1.3) \]

We list a few known facts about $TL_n(\delta)$ for later use:

- It is finite dimensional. The dimension is given by the $n$-th Catalan number $\dim TL_n(\delta) = \frac{1}{n+1} \binom{2n}{n}$.
- It has a normalized trace $\tau_n$ which is compatible with the inclusion $TL_n(\delta) \subset TL_{n+1}(\delta)$. Namely, $\tau_n$ is a linear function on $TL_n(\delta)$ with $\tau_n(1) = 1$, $\tau(xy) = \tau(yx)$ for any $x, y \in TL_n(\delta)$ and $\tau_{n+1}(x) = \tau_n(x)$ for any $x \in TL_n(\delta)$.

The generators $E_i$'s have a diagrammatic presentation as in Fig 1.

![Fig. 1 Diagrams of the unit 1 and the generators $E_1, E_2$ of $TL_n(\delta)$ ($n = 3$)](image)

In general, we will depict each element of $TL_n(\delta)$ as a diagram in the fixed rectangle which has the $n$-marked points on both the top and the bottom of the rectangle.

Then, the multiplication $x \cdot y$ of two diagrams $x, y$ in the rectangles corresponding to $x, y \in TL_n(\delta)$ is defined in a similar manner to the way to define the multiplication of the braids. Namely, 'stack $y$ on $x$'.

However, this is not enough to give an algebra of the diagrammatic presentation of Temperley-Lieb algebras and we need the following additional rule.

**Rule 1:** If we have a loop in the consequence of a multiplication, then we remove a loop and multiply the picture by $\delta$.

We denote this diagrammatically defined algebra obtained from $TL_n(\delta)$ by $\mathcal{TL}_n(\delta)$. 
When $n = 3$, by easy computations, we see that basis elements of $\mathcal{T}\mathcal{L}_3(\delta)$ have the diagrams shown in Fig. 2.

![Diagrams of the basis elements of $\mathcal{T}\mathcal{L}_3(\delta)$](image)

Fig. 2 Diagrams of the basis elements of $\mathcal{T}\mathcal{L}_3(\delta)$ ($n = 3$)

Thus, we have an increasing sequence of finite dimensional algebras $\{\mathcal{T}\mathcal{L}_n(\delta)\}_{n=1}^{\infty}$. And we will adopt the diagram of the embedded element in $\mathcal{T}\mathcal{L}_{n+1}(\delta)$ for $x \in \mathcal{T}\mathcal{L}_n(\delta)$ by the following rule.

**Rule 2:** We use the diagram of a rectangle with one through strand in the right of $x$ for the embedded $x$. See Fig. 3.

![Diagrams for an embedded element](image)

Fig. 3 Diagrams for an embedded element

Then, for each $n$, we have an obvious isomorphism from $\mathcal{T}\mathcal{L}_n(\delta)$ onto $TL_n(\delta)$ by sending the diagrams of generators to corresponding $E_i$’s, which is compatible with the inclusions.

**Example 2** BMW algebras.

We will follow the exposition by Birman-Wenzl in [BW].

**Definition 2** Let $n$ be a natural number and let $l, m \in \mathbb{C}^\times$. We denote by $\mathcal{C}_n(l, m)$ the algebra generated by the unit 1 and $G_i, G_i^{-1}$ $(1 \leq i \leq n - 1)$ with the property $G_iG_i^{-1} = G_i^{-1}G_i = 1$ for all $i$ and with the following relations (1)–(8). Firstly, we introduce an element $E_i \in \mathcal{C}_n(l, m)$ in such
a way that $G_i + G_i^{-1} = m(1 + E_i)$ for all $i$. This identity is often called Kauffman's skein relation. The relations are:

(1) $G_i G_j = G_j G_i$, $E_i E_j = E_j E_i$ ($|i - j| > 1$)
(2) $G_{i\pm 1} G_i = G_i G_{i\pm 1}$, $E_{i\pm 1} E_i = E_i$
(3) $G_i G_{i\pm 1} = G_{i\pm 1} G_i = E_i E_{i\pm 1}$
(4) $G_i G_{i\pm 1} = G_{i\pm 1} G_i = G_i^{-1} E_i G_i^{-1}$
(5) $G_i G_{i\pm 1} = G_i^{-1} E_i G_i^{-1}$
(6) $E_i G_{i\pm 1} = E_{i\pm 1} G_i$
(7) $E_i G_{i\pm 1} = G_{i\pm 1} G_i = E_i E_{i\pm 1}$
(8) $E_i G_{i\pm 1} = G_{i\pm 1} G_i = l^{-1} E_i$

We call the algebra $C_n(l, m)$ the $n$-th BMW algebra.

We list a few known facts about $C_n(l, m)$ for later use:

- It is finite dimensional and the dimension is given by
  $$\dim C_n(l, m) = (2n - 1)!! = \prod_{j=1}^{n} (2j - 1).$$

- It has a normalized trace $\tau_n$ which is compatible with the inclusion $C_n(l, m) \subset C_{n+1}(l, m)$.
- After certain renormalization, the trace $\tau_n$ gives rise to the Kauffman polynomial for links with parameters $l, m$. The links are obtained from the closure of the 'braids' in $C_n(l, m)$.

It is well known that the generators $G_i$'s and the elements $E_i$'s of the algebra $C_n(l, m)$ has diagrammatic presentations. The diagrams of $E_i$'s are same as the ones used for $\mathcal{TL}_n(\delta)$ and we also apply Rule 1 to the diagrams obtained from $E_i$'s. The diagrams of $G_i$ and $G_i^{-1}$ have one crossing for each. See Fig. 4. This diagrammatic presentation of the elements of $C_n(l, m)$ defines an algebra over $\mathbb{C}$, in a similar way to $\mathcal{TL}_n(\delta)$

$G_1 = \begin{array}{c} \hline \hline \end{array}$, $G_1^{-1} = \begin{array}{c} \hline \hline \end{array}$, $G_2 = \begin{array}{c} \hline \hline \end{array}$, $G_2^{-1} = \begin{array}{c} \hline \hline \end{array}$

Fig. 4 Diagrams of the generators $G_1, G_1^{-1}, G_2$ and $G_2^{-1}$ ($n = 3$)
and we denote this diagrammatic algebra by $BMW_n(l, m)$. With Rule 2, we will see that $\{BMW_n(l, m)\}_{n=1}^{\infty}$ is an increasing sequence of finite dimensional algebras. Then, it is known that for each $n$, $BMW_n(l, m)$ is isomorphic to $C_n(l, m)$ as algebras [MT].

2 Planar tangles

Before we define planar algebras, we will introduce the notion of planar tangles, which plays a crucial role in planar algebras.

In the previous section, we depicted elements of $T\mathcal{L}_n(\delta)$ and $BMW_n(l, m)$ as $n$-boxes. Namely, a diagram of each element is presented by a $2n$-marked rectangular box; half of the marked points are located on the top of the rectangle and the rests are located on the bottom.

From now on, we shall change the way to present elements of $T\mathcal{L}_n(\delta)$ and $BMW_n(l, m)$ from $n$-boxes to “$n$-disks”.

In the sequel of this article, $D_0$ is always meant to be the unit disk centered at the origin in the complex plane:

$$D_0 = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

We will define planar $k$-tangles generalizing $n$-disks following [J2].

Definition 3  A planar $k$-tangle $T$ constitutes of:

- the unit disk $D_0$ with $2k(\geq 0)$ marked points on the boundary and a finite set of disjoint disks $D_i$ ($1 \leq i \leq b(T)$) inside $D_0$
- each inner disk $D_i$ has the even number (possibly zero) of marked points on the boundary of the disk, say, $2k_i(\geq 0)$.
- a finite set of disjoint smoothly embedded curves, which are either closed loops or whose boundaries are marked points of $D_i$ ($0 \leq i \leq b(T)$)
- each marked point is the boundary of some curve which meets the boundary of the corresponding disk transversally
- each connected component of the complement of the curves in $D_0 - \bigcup_{i=1}^{b(T)} D_i$ is called a region and it is either shaded or unshaded by the following rule:
  - we shade and unshade the regions inside $D_0$ so that regions whose
boundaries meet are shaded differently.

* each disk $D_i$ ($0 \leq i \leq b(T)$) has a distinguished interval. We shall mark ‘*’ for the distinguished interval.

We think that two planar $k$-tangles are equal if they are isotopic.

![Diagram of a planar 3-tangle]

**Fig. 5** An example of a planar 3-tangle

**Definition 4** We denote the set of planar $k$-tangles by $\mathcal{T}_k$. If an element $T$ in $\mathcal{T}_k$ has the distinguished interval (in the boundary of $D_0$) attached with unshaded (shaded) region, then $T$ is called positive (resp. negative). We denote the subset of positive (negative) planar $k$-tangles by $\mathcal{T}_{k,+}$ (resp. $\mathcal{T}_{k,-}$). Hence, $\mathcal{T}_k = \mathcal{T}_{k,+} \coprod \mathcal{T}_{k,-}$.

The above Fig. 5 is an example of positive 3-tangles.

**Remark 5** Instead of shading regions, we often use arrows for curves to indicate whether a region is unshaded or shaded. If a region is unshaded, curves consisting of the boundary of it are oriented positively, i.e., in the counter-clockwise direction. If a region is shaded, then the other way around. See [KS1] for more detailed accounts.

We introduce some important planar tangles for later use.

* Multiplication tangle $M_k$.
- Identity tangle $I_k$

- Inclusion tangle $I_{k}^{k+1}$

- Right (Left) trace tangle $\text{Tr}_R$ (resp. $\text{Tr}_L$), where $k$ stands for $k$ parallel strands

- Partial trace tangle $\mathcal{E}_k^{k-1}$
- Rotation tangle $\rho$ (* for the inner disk $D_1$ is rotated to the next unshaded region in the clockwise direction relative to the * for the outer disk $D_0$.)

We will explain a natural operation of the composition for two planar tangles. Suppose $T$ is a planar $k$-tangle with $b = b(T)$ internal disks. If one of these internal disks, say $D_i$, has $2k_i$ marked points, and if $S$ is a planar $k_i$-tangle, then we have the $k$-tangle obtained by isotoping $S$ so that the boundary and the distinguished interval with information of the shading of $S$ matches those of $D_i$. We denote the resulting planar $k$-tangle by $T \circ_i S$.

Fig. 6 An example of $T \circ_i S$
In Fig. 6, we give an example of the composition $T \circ_{2} S$ with the planar 3-tangle $T$ in Fig. 5. Here $S$ is depicted as the disk surrounded by the dashed circle. (We have used isotopy for $S$ to fit the disk $D_2$ of $T$.) Note that the numbers of the disk inside $S$ has been changed from $D_1, D_2$ of $S$ to $D_2, D_3$ of $T \circ_{2} S$ and that the * indicating the distinguished intervals on the outer disk of $S$ and * for $D_2$ of $T$ are removed in $T \circ_{2} S$.

**Example 3** Planar tangles of $\mathcal{T}\mathcal{L}_n(\delta)$.

We will use the isotopy from an element of $\mathcal{T}\mathcal{L}_n(\delta)$, which is a rectangle with $2n$-marked points (on the top and the bottom of it) connected by non-intersecting strands, to the unit disk $D_0$ with $2n$-marked points connected by non-intersecting smooth curves. Then, there are $2n$ intervals on the boundary circle of $D_0$. We shall mark ‘*’ for the distinguished interval which is isotoped from the left edge of the rectangle. And we will define the shading in such a way that the region which has the distinguished interval is positive. We will depict basis elements of $\mathcal{T}\mathcal{L}_n(\delta)$ in the case $n = 3$ as in Fig. 7.

![Fig. 7 Basis elements of $\mathcal{T}\mathcal{L}_n(\delta)$ as positive planar $n$-tangles ($n = 3$)](image)

**Example 4** Planar tangles of $BMW_n(l, m)$

In a similar manner to define the planar tangles of $\mathcal{T}\mathcal{L}_n(\delta)$, we will use the isotopy from a diagram of $n$-box in $BMW_n(l, m)$ to the $n$-disk. Then, we have the same pictures of $E_i$'s as those of $\mathcal{T}\mathcal{L}_n(\delta)$. However, say, $G_1$ and $G_1^{-1}$ will be depicted as in Fig. 8 in this procedure. So, these are not planar diagrams.

![Fig. 8 $G_1$ and $G_1^{-1}$ as 2-disks](image)
Since we know by \((1.4)\) that \(\dim BMW_2(l, m) = 3\), we have a basis \(\{1, G_1, G_1^{-1}\}\) of \(BMW_2(l, m)\). So, there should be the corresponding elements for planar diagrams if there would be such planar presentations. For this sake, we introduce two planar 2-tangles \(Q\) and \(R\) as in Fig. 9. They are substitutes of the crossings corresponding to \(G_1\) and \(G_1^{-1}\), respectively. Identifying the elements of planar 2-tangles and the labels for them, we call \(Q, R\) the planar 2-tangles labeled by \(Q\) and \(R\).

![Planar 2-tangles Q and R](image)

**Fig. 9** Planar 2-tangles \(Q\) and \(R\)

In the case that a planar tangle contains **disks with 4 marked points**, we allow some of these disks to be **labeled** by an element of \(L = \{Q, R\}\). Then, we consider the vector space \(V_n\) over \(\mathbb{C}\) which is the set of linear combinations of **fully labeled** \(n\)-tangles. For such a vector space to be isomorphic to \(BMW_n(l, m)\), we need to consider the algebra structure on \(V_n\). We will come back and discuss it in the next section.

We will end this section by giving the general definition of **labeled planar tangles**.

**Definition 6** Let \(L = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} L_k\) be a disjoint union of an arbitrary collection of a set \(L_k\) (possibly empty). A planar \(k\)-tangle \(T\) is said to be **labeled** by \(L\) if \(T\) is a planar \(k\)-tangle and is allowed to label an element of \(L_j\) for each internal disk \(D\) with \(2j\)-marked points only if \(L_j\) is not empty. If this is the case, we call \(L\) the **label set of planar tangles**.

### 3 Definitions of planar algebras

#### 3.1 Definition of planar algebras

**Definition 7** A (shaded) planar algebra \(P\) is a family of \(\mathbb{Z}/2\mathbb{Z}\)-graded vector spaces \(\{P_k, \pm\}_{k \in \mathbb{Z}_{\geq 0}}\) indexed by the set of non-negative integers \(\mathbb{Z}_{\geq 0}\), where \(P_k, \pm\) denotes the \(\pm\) graded space indexed by \(k\). To each
planar $k$-tangle $T$ for $k \geq 0$, if the set of inner disks of the unit disk $D_0$ of $T$ is not empty, there will be a linear map

$$Z_T : P_{k_1,\epsilon_1} \otimes \cdots \otimes P_{k_b,\epsilon_b} \to P_{k,\epsilon_0}$$  \hspace{1cm} (3.1)

where $P_{k_i,\epsilon_i}$ is the vector space indexed by half the number of marked boundary points of $D_i$ and $\epsilon_i = +$ if the distinguished interval of $D_i$ is unshaded and $\epsilon_i = -$ if it is shaded.

The map $Z_T$ is subject to the following requirements:

(i) (Naturality)

$$Z_{T \circ_i S} = Z_T \circ (\text{id} \otimes \cdots \otimes \text{id} \otimes Z_S \otimes \text{id} \cdots \otimes \text{id}),$$  \hspace{1cm} (3.2)

where $Z_S$ sits at the $i$-th place.

(ii) (Isotopy invariance) If $\varphi$ is an orientation preserving diffeomorphism of $\mathbb{C}$ then

$$Z_T = Z_{\varphi(T)},$$  \hspace{1cm} (3.3)

where the sets of internal disks of $T$ and $\varphi(T)$ are identified by $\varphi$.

(iii) (Unit preserving) $Z_{I_k} = \text{id}_{P_{k,\epsilon}}$, where $I_k$ is the identity tangle.

Remark 8  (i) We often work with either $\{P_{n,+}\}_n$ or $\{P_{n,-}\}_n$ for $n > 0$. We agree on the convention that $P_n$ will mean $P_{n,+}$.

(ii) The correspondence $Z : T \mapsto Z_T$ may be seen as a "representation of an algebra of planar tangles".

One notices that the multiplication tangle $M_n$ implies a linear map $Z_{M_n} : P_{n,\epsilon} \otimes P_{n,\epsilon} \to P_{n,\epsilon}$. Thus, by using $Z_{M_n}$, $P_{n,\epsilon}$ has multiplication and becomes an associative algebra over $\mathbb{C}$ with the unit.

Example 5  Temperley-Lieb planar algebras.

Let $T$ be a planar $k$-tangle which has $b = b(T)$ disks $D_1, \ldots, D_b$ inside the unit disk $D_0$ and each inner disk $D_i$ has $2k_i$ marked points. Put $P_j = TL_j(\delta) \ (j \in \{k_1, \ldots, k_b, k\})$ to simplify the notations. Then, $T$ implies a linear map

$$Z_T : P_{k_1} \otimes \cdots \otimes P_{k_b} \to P_k$$  \hspace{1cm} (3.4)
It would be obvious for $Z_T$ to satisfy the conditions (i) - (iii) in the
definition of planar algebras for $k > 0$.

When $k = 0$, we need to elaborate a little bit more. The 0-th
Temperley-Lieb planar algebra consists of planar 0-tangles by definition
and 0-tangles consist of non-intersecting loops. However, by Rule 1,
each loop is replaced by $\delta$. After all the loops are removed, the rest is
an empty diagram inside $D_0$. According to the shading, we have the
unshaded empty diagram and the shaded one denoted by $1_+$ and $1_-$,
respectively. Then, we see that $\mathcal{TL}_{0,\pm}(\delta) = \mathbb{C}1_{\pm}$. Sending $1_{\pm}$ to $1 \in \mathbb{C}$
gives an isomorphism from $\mathcal{TL}_{0,\pm}(\delta)$ to $\mathbb{C}$.

Thus, we now know that $\mathcal{TL}(\delta) = \bigcup_{n=0}^{\infty} \mathcal{TL}_n(\delta)$ is a planar algebra.

Definition 9 Let $\mathcal{TL}(\delta) = \bigcup_{n=0}^{\infty} \mathcal{TL}_{n,\epsilon_n}(\delta)$, where $\epsilon_n = +$ for $n > 0$.
Then, $\mathcal{TL}(\delta)$ with the above correspondence $Z$ is a planar algebra and is
called the Temperley-Lieb planar algebra. We call an element of $\mathcal{TL}(\delta)$
Temperley-Lieb element, or TL-element for brevity.

3.2 A planar algebra with generators and relations

Definition 10 Let $L$ be a label set. Define $P_k(L)$ to be the vector space
over $\mathbb{C}$ with the basis given by the collection of planar $k$-tangles all of
which inner disks are labeled by $L$. Each planar $k$-tangle $T$ induces a lin-
ear map $Z_T$ satisfying both (3.1) for $P_j = P_j(L)$ and (3.2). Thus, $P_k(L)$
has an algebra structure given by $Z_{M_k}$, where $M_k$ is the multiplication
tangle. $P = \bigcup_{n=0}^{\infty} P_n(L)$ is called the universal planar algebra with the
label set $L$.

Definition 11 For a universal planar algebra $P = \bigcup_{n=0}^{\infty} P_n(L)$ with the
label set $L$. Let $R$ be a subset of $L$. We define $J_k(R)$ to be the smallest
vector subspace of $P_k(L)$ containing the set $R$. Then, $P(L; R) := \bigcup_{n=0}^{\infty} P_n(L)/J_n(R)$ is called the planar algebra generated by $L$ with rela-
tions $R$.

Example 6 BMW planar algebras.

Here is a recipe for $V_n$ in Example 4 to become a planar algebra due
to V.F.R. Jones [J1].

We use the multiplication tangle $M_n$ to define the multiplication on
$V_n$. Hence, $V_n$ in Example 4 is an associative algebra with the unit.
Let $L = L_2 = \{Q, R\}$. Put $P_n(L) = V_n$ in Definition 10, it defines a universal planar algebra $P(L) = \bigcup_{k=0}^{\infty} P_k(L)$ with the label set $L$. Then, we define the algebra $BMW_n(l, m)$ to be $P_k(L)$ modulo the following relations on $Q, R$:

(i) $\emptyset Q = \emptyset \circ$, $\emptyset R = \emptyset \copyright$

(ii) $\emptyset R \circ Q = \emptyset Q \circ R$

(iii) $l = l^{-1} = Q$

(iv) $Q = l^{-1} Q = l^{-1} Q$

(v) $Q + R = m (\emptyset \circ + \emptyset \circ)$

Here, we omit the shading since it will be clear from the positions of *’s.

We will see that the correspondence

$$R \mapsto \bigcirc \quad \text{and} \quad Q \mapsto \bigcirc$$

extends to a homomorphism from $BMW_n(l, m)$ to $BMW_n(l, m)$ and that it is in fact an isomorphism. See [J1] for more detailed accounts.

When $n = 0$, the 0-th BMW planar algebra consists of planar 0-tangles, which are linear combinations of finitely many links. Since we have assumed the Kauffman skein relation, we see that planar 0-tangles are replaced by the Kauffman polynomials for fixed $l, m \in \mathbb{C}^\times$. As in a similar manner to the case of $\mathcal{T}L(\delta)$, we see that $BMW_{0, \pm}(l, m) = \mathbb{C}1_{\pm}$. 
Definition 12 We call $\mathcal{BMW}(l, m) = \cup_{n=0}^{\infty} \mathcal{BMW}_{n, \epsilon_n}(l, m)$, where $\epsilon_n = +$ for $n > 0$, with the correspondence $Z$ the BMW planar algebra.

Definition 13 Let $P$ be a planar algebra.

(i) $P$ is called connected if $\dim P_{0, \pm} = 1$. If this is the case, we define the isomorphism $Z_{\pm} : P_{0, \pm} \to \mathbb{C}$ by sending positive/negative empty diagrams to $1 \in \mathbb{C}$.

(ii) If $P$ is connected and if $Z_{Tr_R} = Z_{Tr_L}$ holds for every planar tangles, then $P$ is called spherical and we denote $Z_{Tr_R} = Z_{Tr_L}$ by simply $\text{Tr}$. If $P$ is spherical, the map $Z(T) := Z_{\pm} \circ Z_{\text{Tr}}(T)$ gives an isotopy invariant of the closure of a planar tangle $T$ by the trace tangle. We call $Z$ the partition function of $P$.

4 Definitions of subfactor planar algebras

Definition 14 We define the adjoint tangle $T^*$ of $T$ by the complex conjugate of $T$. (Recall $T$ is defined on the complex plane.)

Definition 15 A planar algebra $P$ is a subfactor planar algebra if:

(i) $P$ is a connected, spherical planar algebra with $Z(1) = \delta > 0$, where the unit 1 is considered as planar 1-tangle,

(ii) each $P_{k, \pm}$ is a finite dimensional $*$-algebra satisfying the following condition:

$$Z_T(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1^* \otimes \cdots \otimes x_b^*),$$

(4.1)

for $x_i \in P_{k_i, \epsilon_i}$ ($1 \leq i \leq b$),

(iii) for any $x \in P_{k, \pm}$, $\text{tr}_k(x) := \delta^{-k} Z_{\text{Tr}}(x)$ is a positive definite trace on $P_{k, \pm}$ for all $k \geq 1$.

Remark 16 (i) If $P$ is a subfactor planar algebra, then each $P_{k, \pm}$ is a semi-simple $*$-algebra. Namely, $P_{k, \pm}$ is isomorphic to the $*$-algebra of finite direct sums of full matrix algebra over $\mathbb{C}$.

(ii) In the case of a subfactor planar algebra, we will have the canonical vector space isomorphism $P_{n, +} \simeq P_{n, -}$ for positive $n$ [J2, KS1]. For this reason, we sometimes treat only $P_n = P_{n, +}$.

(iii) When we treat the diagrammatic element $\delta^{-1} E_n$ as an element in
$P_{n+1}$, we set $e_n := \delta^{-1}Z_{E_n}(1) \in P_{n+1}$, and we call it $n$-th Jones projection in $P_{n+1}$. In fact, this plays a role of the $n$-th Jones projection in subfactor theory.

**Proposition 17** Let $P = \bigcup_{n=0}^{\infty}P_n$ be a subfactor planar algebra. Then, for any element $x, y \in P_{n+1}$, there exist $x_0, y_0 \in P_n$ such that $xe_ny = x_0e_ny_0$ holds, where $e_n$ is the $n$-th Jones projection as in the previous remark (iii).

**Proof.** It is enough to prove that for any $x \in P_{n+1}$, there exists $x_0$ such that $xe_n = x_0e_n$ holds. Put $x_0 := Z_{E_{n+1}^{n}}(xe_n)$, then this does the job. (Consider $x, x_0, e_n$ as planar tangles, and apply $E_{n+1}^{n}$ to the multiplication of $xe_n$. Readers are encouraged to draw pictures.) \(\square\)

As a consequence of this proposition, we conclude that $P_n e_n P_n$ is a two sided-ideal in $P_{n+1}$.

**Definition 18** Let $P = \bigcup_{n=0}^{\infty}P_n$ be a subfactor planar algebra. $P$ is said to have **finite depth** if there exists a positive integer $k$ such that $P_k e_k P_k = P_{k+1}$. We call the smallest such $k$ the **depth** of $P$.

We have the following very striking result for finite depth subfactor planar algebras ([KT1, KT2]).

**Theorem 19** (Kodiyalam-Tupurani) Any finite depth subfactor planar algebra is singly generated with finite relations.

**Remark 20** The proof of this result is surprisingly easy based on a simple trick of single generation property in linear algebras. See [KT1, KT2] for details.

## 5 Examples of finite depth subfactor planar algebras

Let $[n]_q$ be the $n$-th $q$-integer defined by $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. For each non-negative integer $n$, we have the distinguished projection $f^{(n)}$ in $TL_n(\delta)$ defined by the following recursive formula:

$$
\begin{cases} 
  f^{(0)} = 1, \\
  [n+1]_q f^{(n+1)} = [n+1]_q f^{(n)} + [n]_q f^{(n)} e_n f^{(n)},
\end{cases}
$$
where \( e_n = \delta^{-1}E_n \in TL_{n+1}(\delta) \). We call \( f^{(n)} \) Jones-Wenzl projections. In fact, \( f^{(n)} \) is the projection in \( TL_n(\delta) \) characterized by the following properties:

(i) \( f^{(n)} \neq 0 \),
(ii) \( e_i f^{(n)} = f^{(n)} e_i = 0 \) for \( i = 1, \ldots, n - 1 \).

As in Section 2, we have the diagrammatic presentation for \( f^{(n)} \) and the above identities [MPS1].

Let \( \delta = [2]_q = q + q^{-1} \).

### 5.1 When \( \delta < 2 \)

- **\( A_N \)-subfactor planar algebra**

  Let \( q = \exp(\pi i/(N + 1)) \). \( A_N \)-subfactor planar algebra \( P = \bigcup_{n=0}^{\infty} P_n \) is the quotient of \( TL(\delta) \) by the relation \( f^{(N)} = 0 \).

- **\( D_{2N} \)-subfactor planar algebra** [MPS1]

  Let \( q = \exp(\pi i/(4N - 2)) \). \( D_{2N} \)-subfactor planar algebra \( P = \bigcup_{n=0}^{\infty} P_n \) is generated by a single element \( S = S^* \in P_{2N-2} \) with the following relations:

  (i) \( \rho(S) = -S \),
  (ii) \( \mathcal{E}_{2N-2}^{2N-1}(S) = 0 \),
  (iii) \( S^2 = f^{(2N-2)} \),
  (iv) \( f^{(4N-3)} = 0 \).

Here, we have used the notation of planar tangles exhibited in Section 2, except (iii). In (iii), \( S^2 \) stands for \( S \) multiplied by itself. So, it should be written with the multiplication tangle \( M_{2N-2} \) with the input of two \( S \)'s.

- **\( E_6 \)- and \( E_8 \)-subfactor planar algebras**

  See [B] for these examples.
5.2 Other examples

- Asaeda-Haagerup subfactor planar algebra. [P]
- Extended Haagerup subfactor planar algebra. [BMPS]
- Finite dimensional \((C^*-)\) Hopf algebras. [KLS, KS2]

The first two examples are considered to be ‘exotic’. Namely, we expect that they would not be constructed from groups, quantum groups or conformal field theories.

6 Applications

Link invariants [MPS2]

We may define a crossing (or braiding) as the linear combinations of the two planar 2-tangles as in the following figure.

\[
\begin{align*}
\begin{array}{c}
\quad \\
\quad \\
\end{array}
\end{align*}
= \begin{array}{c}
iq^{1/2} \\
-iq^{-1/2}
\end{array}
\begin{array}{c}
\quad \\
\quad \\
\end{array}
\]

In the case of \(\mathcal{TL}(\delta)\) and the \(A_N\)-subfactor planar algebra, this gives rises to colored Jones polynomial formulated with Jones-Wenzl idempotents [MV]. Namely, each component \(L_i\) of a link \(L\) is replaced by \(n_i\)-parallel strands and we put the Jones-Wenzl idempotent \(f^{(n_i)}\) on \(n_i\)-cabled link component of \(L_i\).

In the case of the \(D_{2N}\)-subfactor planar algebra, by partial braiding property (Theorem 3.2 in [MPS1]), the even number of cabling is allowed to construct the colored Jones polynomial with respect to the new idempotents coming from the \(D_{2N}\)-subfactor planar algebra. Practically, on the \(2N - 2\) cabled link, we put \(P = (f^{(2N-2)} + S)/2\) (which is also a projection) instead of Jones-Wenzl projection, then we have a new link invariant \(J_{D_{2N}}\) constructed from the \(D_{2N}\)-subfactor planar algebra. The invariant \(J_{D_{2N}}\) induces unexpected relations to some known link polynomials with various specializations. See [MPS2] for the detailed accounts.
Topological Quantum Field Theories in (1+1)-dimension [KPS]

Let us put a planar $k$-tangle in $\mathbb{R}^3$. Consider the holes instead of the inner disks. Then, planar tangle with this modification gives rise to a 2-sphere with boundary circles, which come from the the boundaries of the inner disks and the outer disk. It is not hard to imagine that this picture will give a TQFT in (1+1)-dimensions. Namely, the objects are closed circles with the decorated arcs, and the morphism are planar tangles seen as decorated 2-spheres with holes which boundaries are the circles, i.e., objects. Thus, we obtain a cobordism category $\mathcal{D}$. In [KPS], the category $\mathcal{D}$ is essentially in one to one correspondence with a subfactor planar algebra. See [KPS] for the detailed accounts.

A topological operad of $\mathbb{P}^1$ [CM]

There is an interesting result on an operadic aspect of planar algebras from real algebraic geometry, although we have mentioned nothing about operads.

Ceyhan and Marcolli have established the following results. Planar algebras, which can be seen as algebras over the planar operad, are algebras over the topological operad of certain moduli spaces of stable maps to $\mathbb{P}^1$ with Lagrangian boundary conditions.

According to this result, we might expect some new results in planar algebras from the viewpoint of the algebras over the topological operad. In fact, they give two general frameworks to construct planar algebras. One is based on sting topology and the other is on Gromov-Witten theory.

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The author wishes that this article gives a help for the students and researchers in low dimensional topology interested in planar algebras to read further articles in this research area.

Finally, the author would like to thank the organizers to give him an opportunity to include this article in the proceedings of the workshop.
References


Department of Mathematics
Rikkyo University
3-34-1, Nishi-Ikebukuro, Toshima, Tokyo
171-8501, JAPAN
nobuya@rikkyo.ac.jp