数理解析研究所講究録
第 1716 巻 2010 年 37-42

MULTIPLICITY DISTANCE OF KNOTS

KOUKI TANIYAMA

ABSTRACT. We define multiplicity and multiplicity distance in a category. We define neighbourhood category of knots and discuss its multiplicity distance.

§1. Multiplicity in a category

Let $\mathcal{C} = (\mathcal{O}, \{\mathrm{Hom}(X,Y)\}_{X,Y\in\mathcal{O}}, \circ)$ be a category. Namely $\mathcal{O}$ is a set of objects and $\mathrm{Hom}(X,Y)$ is the set of morphisms from an object X to an object Y.

${}^{\forall}\varphi \in \mathrm{Hom}(X,Y), {}^{\forall}\psi \in \mathrm{Hom}(Y,Z)$, $\psi\circ\varphi \in \mathrm{Hom}(X,Z)$ is defined such that the following (1) (2) and (3) hold.

(Notation: $\varphi \in \mathrm{Hom}(X,Y) \Leftrightarrow \varphi : X \to Y$)

(1) ${}^{\forall}\varphi : X \to Y$, ${}^{\forall}\psi : Y \to Z$, ${}^{\forall}\tau : Z \to W$, $(\varphi\circ\psi)\circ\tau = \varphi\circ(\psi\circ\tau)$

(2) ${}^{\forall}X \in \mathcal{O}$, ${}^{\exists}\mathrm{id}_X : X \to X$ s.t. $\forall\varphi : X \to Y$, $\varphi\circ\mathrm{id}_X = \varphi$, and $\forall\psi : Y \to X$, $\mathrm{id}_X\circ\psi = \psi$,

(3) $\mathrm{Hom}(X,Y)\cap\mathrm{Hom}(Z,W) \neq \emptyset$
$\Rightarrow X = Z$ and $Y = W$.

Definition 1.

$$m : \bigcup_{(X,Y)\in\mathcal{O}\times\mathcal{O}} \mathrm{Hom}(X,Y) \to \mathbb{R}_{\geq 1}\cup\{\infty\}$$

is a *multiplicity* on $\mathcal{C}$

$\overset{\mathrm{def}}{\Longleftrightarrow}$ (1) ${}^{\forall}X \in \mathcal{O}$, $m(\mathrm{id}_X) = 1$, and

(2) ${}^{\forall}\varphi : X \to Y$, ${}^{\forall}\psi : Y \to Z$, $m(\psi\circ\varphi) \leq m(\varphi)m(\psi)$.

Key words and phrases. multiplicity, category, knot.

Here we think $\forall x \in \mathbb{R}_{\geq 1}$, $x \leq \infty$, $x \cdot \infty = \infty \cdot x = \infty$, $\infty \leq \infty$ and $\infty \cdot \infty = \infty$.

Definition 2. $X, Y \in \mathcal{O}$

$$m(X : Y) = \inf\{m(\varphi) \mid \varphi \in \mathrm{Hom}(X, Y)\}$$

$m(X : Y)$: *multiplicity* of X over Y with respect to m.

Proposition 3. *m: multiplicity on $\mathcal{C}$*

(1) $^\forall X \in \mathcal{O}$, $m(X : X) = 1$,

(2) $^\forall X, Y, Z \in \mathcal{O}$, $m(X : Z) \leq m(X : Y)m(Y : Z)$.

m has *finite multiplicity property (FMP)*
$\overset{\text{def}}{\Longleftrightarrow} \forall X, Y \in \mathcal{O}$, $m(X : Y) < \infty$.

Definition 4.

$$d_m(X, Y) = \log_e(m(X : Y)m(Y : X))$$

$d_m(X, Y)$: *m-distance* of X and Y

Proposition 5. *m: multiplicity on $\mathcal{C}$ with FMP*

$(\mathcal{O}, d_m)$ is a pseudo metric space i.e.

(D1′) $d_m(X, Y) \geq 0$, $d_m(X, X) = 0$,

(D2) $d_m(X, Y) = d_m(Y, X)$,

(D3) $d_m(X, Z) \leq d_m(X, Y) + d_m(Y, Z)$.

Example 6. $\mathcal{C}$: subcategory of the category of sets and maps (SET)

$X, Y \in \mathcal{O}$, $f : X \to Y$

$$m(f) = \sup\{|f^{-1}(y)| | y \in Y\}$$

m: *map multiplicity*

Proposition 7. *map multiplicity is a multiplicity, i.e.*

(1) $^\forall X \in \mathcal{O}$, $m(\mathrm{id}_X) = 1$, and

(2) $^\forall f : X \to Y$, $^\forall g : Y \to Z$, $m(g \circ f) \leq m(f)m(g)$.

Example 8. R: PID, M, N: R-modules finitely generated over R.

$r(M)$: the minimal number of generators of M over R.

$f : M \to N$: R-linear map

$$m_{\text{ker}}(f) = e^{r(\text{ker} f)}$$

$$m_{\text{coker}}(f) = e^{r(\text{coker} f)}$$

$(\text{coker} f = N/f(M))$

Proposition 9. *(1)* m_{ker} *is a multiplicity.*

(2) m_{coker} *is a multiplicity.*

When R is a field, M and N are finite dimensional vector spaces over R, and

$d_{m_{\text{ker}}}(M, N) = d_{m_{\text{coker}}}(M, N) = |\text{dim} M - \text{dim} N|.$

§2. Neighbourhood category of knots

$\mathcal{K}$: unoriented knot types in $\mathbb{S}^3$, $K_1, K_2 \in \mathcal{K}$.

$f = (V_2, k_1)$ is a morphism from K_1 to K_2

$\overset{\text{def}}{\Longleftrightarrow}$ $V_2 \subset \mathbb{S}^3$: knotted solid torus with knot type K_2,

$k_1 \subset \text{int} V_2$: knot in $\mathbb{S}^3$ with knot type K_1.

$(V, k) = (V', k') \overset{\text{def}}{\Longleftrightarrow} {}^{\exists} h : \mathbb{S}^3 \to \mathbb{S}^3$ ori. preserving homeomorphism s.t.

$h(V) = V'$ and $h(k) = k'$

$\text{Hom}(K_1, K_2)$: the set of morphisms from K_1 to K_2.

$f : K_1 \to K_2$, $g : K_2 \to K_3$, $f = (V_2, k_1)$, $g = (V_3, k_2)$

V_2' : regular neighbourhood of k_2 with $V_2' \subset \mathrm{int} V_3$, k_1' : knot in V_2' s.t. $(V_2', k_1') = (V_2, k_1)$.

We define $g \circ f = (V_3, k_1')$.

Proposition 10. $\mathcal{C}_{\mathcal{K}} = (\mathcal{K}, \{\mathrm{Hom}(K, J)\}_{K,J\in\mathcal{K}}, \circ)$ *is a category.*

$\mathcal{C}_{\mathcal{K}}$: *neighbourhood category of Knots*

$\mathrm{id}_K : K \to K$ is given by $\mathrm{id}_K = (V, k)$ where V is a regular neighbourhood of a knot k with knot type K.

§3. Multiplicity distance of knots

Definition 11. $f : K_1 \to K_2$: morphism, $f = (V_2, k_1)$,
$h : V_2 \to \mathbb{S}^1 \times \mathbb{D}^2$: homeomorphism, $\pi : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1$: natural projection.

h is generic $\overset{\text{def}}{\Longleftrightarrow}$ $\pi \circ h|_{k_1} : k_1 \to \mathbb{S}^1$ is a Morse map,
i.e. it has only finitely many critical points
in different levels.

$$m(h) = \max\{|(\pi \circ h|_{k_1})^{-1}(y)| \mid y \in \mathcal{S}^1\}$$
$$m(f) = \min\{m(h) \mid h : V_2 \to \mathbb{S}^1 \times \mathbb{D}^2 \text{ generic homeomorphism}\}$$

Proposition 12. m *is a multiplicity on* $\mathcal{C}_{\mathcal{K}}$.

Proposition 13. *(1)* $f : K_1 \to K_2$, $m(f) = 1 \ \Rightarrow\ K_1 = K_2$.
(2) $m(K_1 : K_2) = 1 \Leftrightarrow K_1 = K_2$.

Proposition 14. $(\mathcal{K}, d_m)$ *is an unbounded metric space.*

d_m : *multiplicity distance of knots*

Definition 15. (Ozawa [1])
k : knot in $\mathbb{S}^3 - \{(0,0,0,1), (0,0,0,-1)\}$,

$\pi : \mathbb{S}^3 - \{(0,0,0,1),(0,0,0,-1)\} \cong \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$

: natural projection, $\pi|_k : k \to \mathbb{R}$: Morse function.

$\text{trunk}(k) = \max\{|\pi^{-1}(y)| | y \in \mathbb{R}\}$

$\text{trunk}(K) = \min\{\text{trunk}(k) | \text{knot type of } k \text{ is } K\}$

Proposition 16. (Ozawa [1]) ${}^\forall n \in \mathbb{N}$, ${}^\exists K \in \mathcal{K}$ *s.t.* $\text{trunk}(K) \geq n$.

Proposition 17. *(1)* $K, J \in \mathcal{K}$,

$$\frac{\text{trunk}(K)}{\text{trunk}(J)} \leq m(K : J) \leq \text{trunk}(K).$$

(2) $K, J \in \mathcal{K}$, *J is not a companion of* K,

$\Rightarrow m(K : J) = \text{trunk}(K)$.

Proposition 18. $K, J \in \mathcal{K}$,

(1) $m(K : 0_1) \leq \text{braid}(K)$,

(2) $m(K : 0_1) \leq 2\text{bridge}(K) - 1$,

(3) $m(K : J) = 2 \Rightarrow K$ *is a* $(2,p)$*- cable of* J, *or* $K = 0_1$.

Corollary 19. $K \in \mathcal{K}$,

$$m(K : 0_1) \geq \frac{\text{trunk}(K)}{\text{trunk}(0_1)} = \frac{\text{trunk}(K)}{2}.$$

$K \neq 0_1$,

$$d_m(K, 0_1) \geq \log_e \frac{\text{trunk}(K)}{2} \cdot 2 = \log_e \text{trunk}(K).$$

$\{d_m(K, 0_1) | K \in \mathcal{K}\}$ *is unbounded.*

Definition 20. $K \in \mathcal{K}$, $m(K) = m(K : 0_1)$.

$m(K)$: *multiplicity* of K, $m(K) \in \mathbb{N}$.

Remark 21. $m(K) \leq n \Leftrightarrow d_m(K, 0_1) \leq \log_e 2n$.

Proposition 22. $K \in \mathcal{K}$,

(1) $m(K) = 1 \Leftrightarrow K = 0_1$,

(2) $m(K) = 2 \Leftrightarrow K$ *is a* $(2,p)$*-torus knot,*

(3) $m(K) = 3 \Leftrightarrow K \neq 0_1$, *$K$ is not a $(2,p)$-torus knot, and*

(a) K is a closed 3-braid, or

(b) K is a connected sum of some 2-bridge knots.

Proposition 23. *K:Montesinos knot* $\Rightarrow m(K) \leq 4$.

The detail will appear in [2].

References

[1] M. Ozawa, Waist and trunk of knots, to appear in *Geometriae Dedicata.*

[2] K. Taniyama, Multiplicity of a space over another space, in preparation.

Department of Mathematics, School of Education, Waseda University, Nishi-Waseda 1-6-1, Shinjuku-ku, Tokyo, 169-8050, Japan

E-mail address: taniyama@waseda.jp