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AN EXTENSION OF BURAU REPRESENTATION OF THE BRAID GROUPS

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Artin [3] introduced the braid group

$$B_n = \left\{ \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \ \text{ if } \ |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\}.$$

Alexander [1] found a connection between the braid group B_n and links in S^3. Markov [10] introduced (three) "Markov moves" on closed braids, and announced "Markov Theorem." (Weinberg [13] showed that one of three Markov moves was unneccesary.) This theorem says that studying closed braids modulo Markov moves, "Markov equivalence classes", is equivalent to studying ambient isotopy classes of links in S^3. Birman [6] gave a first complete proof of Markov Theorem.

Around the same time as the announcement of Markov theorem, Burau investigated a connection between a representation of B_n and Alexander polynomial of links. We introduce Burau representation of B_n in §1.

1. Burau representation

Burau [7] defined a representation, Burau representation, $\varphi_n \colon B_n \to M(n; \mathbb{Z}[t, t^{-1}])$ of the braid group B_n. The image $\varphi_n(\sigma_i)$ of a generator σ_i of B_n is represented by the matrix

$$\left(\begin{array}{c|cc|c} I_{i-1} & O & O & O \\ \hline O & 1-t & t & O \\ O & 1 & 0 & O \\ \hline O & O & O & I_{n-(i+1)} \end{array}\right).$$

This representation φ_n is reducible, and is reduced to an irreducible representation $\varphi'_n \colon B_n \to M(n-1; \mathbb{Z}[t, t^{-1}])$. The images of generators σ_1, σ_{n-1} and σ_i $(2 \leq i \leq n-2)$ of B_n by φ'_n are represented by the following matrices:

$$\varphi_n'(\sigma_1) = \left(\begin{array}{cc|c} -t & 1 & O \\ 0 & 1 & O \\ \hline O & O & I_{n-3} \end{array}\right) \qquad \varphi_n'(\sigma_{n-1}) = \left(\begin{array}{c|cc} I_{i-3} & O & O \\ \hline O & 1 & 0 \\ O & t & -t \end{array}\right)$$

$$\varphi_n'(\sigma_i) = \left(\begin{array}{c|ccc|c} I_{i-2} & O & O & O & O \\ \hline O & 1 & 0 & 0 & O \\ O & t & -t & 1 & O \\ O & 0 & 0 & 1 & O \\ \hline O & O & O & O & I_{n-(i+2)} \end{array}\right).$$

Remark 1. The representation φ_n is faithful when $n \leq 3$ [9], and $n \geq 5$ [12], [8], [5]. It is not known whether $\varphi_4 \colon B_4 \longrightarrow M(4; \mathbb{Z}[t, t^{-1}])$ is faithful.

Burau obtained a knot invariant, Alexander polynomial, by measuring how far "1" departs from being an eigenvalue of $\varphi_n'(\beta)$.

Theorem 2. [7] *Let β denote a word in B_n. Let K denote a link in S^3 that is a closed n-braid corresponding to β. Then* $\dfrac{\det(\varphi_n'(\beta) - I_{n-1})}{\det(\varphi_n'(\sigma_1\sigma_2\cdots\sigma_{n-1}) - I_{n-1})}$ *is equal to Alexander polynomial of K, $\Delta_K(t)$, up to multiplications by t.*

2. Extension of Burau representation

In this section, we use 2×2 matrices instead of $\mathbb{Z}[t, t^{-1}]$ in Burau representation, that is, we study a mapping $\psi_n \colon B_n \longrightarrow M(n; 2 \times 2 \text{ matrices})$.

Let Λ denote a set of elementary functions with variables $a, b, c, d, e, f, g, h, p, q, r, s, t, u, v, w$. Let

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, L = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, N = \begin{pmatrix} t & u \\ v & w \end{pmatrix}$$

denote elements in $GL(2;\Lambda)$. We assume that the 4×4 matrix $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ is invertible in $M(4;\Lambda)$. We define a mapping $\psi_n\colon B_n \to M(n;GL(2;\Lambda))$ by

$$\psi_n(\sigma_i) = \left(\begin{array}{c|cc|c} I_{2(i-1)} & O & O & O \\ \hline O & K & L & O \\ O & M & N & O \\ \hline O & O & O & I_{2(n-(i+1))} \end{array}\right).$$

This mapping ψ_n is a homomorphism when the following conditions are satisfied;

$M = K^{-1}L^{-1}K(I_2 - K)$,

$N = I_2 - K^{-1}L^{-1}KL$,

$h = \dfrac{1}{acf + b(bc+d-ad)g}\{cf(a(e-de)+c(be+f))$
$+ (bde + (a-d)(-a-bc-d+ad)f)g - b^2g^2\}$, and

$e = \dfrac{1}{2bc(-bc+ad)}\{a^2cf + 2bc^2f - acdf + 2b^2cg - ab^2cg - abdg + a^2bdg + b^2cdg$
$+ bd^2g - abd^2g + (acf + b(bc+d-ad)g)\sqrt{4bc+(a-d)^2}\}$.

Therefore we obtain a representation of B_n, $\psi_n\colon B_n \to M(n;GL(2;\Lambda))$. This representation ψ_n is reducible, and is reduced to a representation $\psi'_n\colon B_n \to M(n-1;GL(2;\Lambda))$.

Remark 3. The representation ψ_n has something to do with a "biquandle". See [4], for example.

In a similar method as Burau obtained Alexander polynomial from Burau representation, we obtain a knot invariant from the representation ψ'_n.

Theorem 4. [11] *Let β denote a word in B_n. Let K denote a link in S^3 that is a closed n-braid corresponding to β. Then* $\dfrac{\det(\psi'_n(\beta) - I_{2(n-1)})}{\det(\psi'_n(\sigma_1\sigma_2\cdots\sigma_{n-1}) - I_{2(n-1)})}$ *is a knot invariant, up to multiplications by $(a-1)(d-1)-bc$.*

3. Example and Problem

In this section, we denote $\Delta_2(\beta) = \dfrac{\det(\psi'_n(\beta) - I_{2(n-1)})}{\det(\psi'_n(\sigma_1\sigma_2\cdots\sigma_{n-1}) - I_{2(n-1)})}$, and $D = (a-1)(d-1) - bc$, $T = (a-1) + (d-1)$.

Example 5. We calculate $\Delta_2(\beta)$ for some numbers of β.

(1) Suppose $\beta = \sigma_1 \in B_2$. Then K is a trivial knot, and $\Delta_2(\sigma_1) = 1$.

(2) Suppose $\beta = 1 \in B_2$. Then K is a 2-component trivial link, and $\Delta_2(1) = 0$.

(3) Suppose $\beta = \sigma_1^2 \in B_2$. Then K is a positive Hopf link, and $\Delta_2(\sigma_1^2) = D + T + 1$.

(4) Suppose $\beta = \sigma_1^{-2} \in B_2$. Then K is a negative Hopf link, and
$\Delta_2(\sigma_1^{-2}) = \dfrac{1}{D^2}\Delta_2(\sigma_1^2)$. This is equal to $\Delta_2(\sigma_1^2)$, up to multiplications by D^2.

(5) Suppose $\beta = \sigma_1^3 \in B_2$. Then K is a right-handed trefoil knot, and
$\Delta_2(\sigma_1^3) = D^2 + DT + T^2 - D + T + 1$.

(6) Suppose $\beta = \sigma_1^{-3} \in B_2$. Then K is a left-handed trefoil knot, and
$\Delta_2(\sigma_1^{-3}) = \dfrac{1}{D^3}\Delta_2(\sigma_1^3)$. This is equal to $\Delta_2(\sigma_1^3)$, up to multiplications by D^3.

Observing calculations in Example 5, we pose the following conjecture.

Conjecture 6. *The knot invariant $\Delta_2(\beta)$ is an element in $\mathbb{Z}[D, D^{-1}, T, T^{-1}]$, up to multiplications by D, for every $\beta \in B_n$.*

In order to extend our extension of Burau representation, we pose the following problem.

Problem 7. *Choose your favorite algebra Ω with unit 1. (For example, Ω might be a "quandle algebra" or $GL(n;\Lambda)$ for some algebra Λ.) Let $\kappa, \lambda, \mu, \nu$ denote elements in Ω such that $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ is invertible in $M(2;\Omega)$. We define a mapping*

$$\phi_n\colon B_n \longrightarrow M(n;\Omega) \text{ by } \phi_n(\sigma_i) = \left(\begin{array}{c|cc|c} I_{i-1} & O & O & O \\ \hline O & \kappa & \lambda & O \\ O & \mu & \nu & O \\ \hline O & O & O & I_{n-(i+1)} \end{array}\right).$$

(1) *Select $\kappa, \lambda, \mu, \nu \in \Omega$ so that $\phi_n\colon B_n \longrightarrow M(n;\Omega)$ is a homomorphism.*
(We refer to [2] for one answer.)

(2) *Let β denote a word in B_n. Let K denote a link in S^3 that is a closed n-braid corresponding to β. Construct an invariant of "Markov equivalence classes" of β, that is, an invariant of K by evaluating the determinant, the trace, etc., of a matrix obtained from $\phi_n(\beta)$.*

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