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A Nevanlinna-Pick Interpolation Problem and a solution

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Abstract

Model matching control is an important and useful method to solve the feedback control for electric devices, the vibration control for architecture and so on. In these days, a method of Robust model matching control is often used. In that method, usually appear the formula $\|T_1(s)-T_2(s)Q(s)\|_\infty \leq \gamma$, where $T_j(j=1,2)$ are known weighted functions and Q is an unknown free parameter. Engineers want to know infimum value of γ. The Nevanlinna-Pick Interpolation is useful and powerful method to obtain the infimum γ. In this paper, we show a solution to a interpolation between n-tuples which lies on circles.

1 Introduction

It is an important problem in engineering to obtain the infimum value of γ in the formula

$$\|T_1(s)-T_2(s)Q(s)\|_\infty \leq \gamma, \tag{1}$$

where $T_j(j=1,2)$ are known weighted functions and Q is an unknown free parameter. To solve this problem, let $(s_j)_{j=1}^{n}$ be an n-tuple of zeros of T_2 and let γ be considered as a fixed parameter. Then set

$$G(s) := \frac{1}{\gamma}\left(T_1(s)-T_2(s)Q(s)\right). \tag{2}$$

Since T_1 is known and s_j is a zero of T_2, the correspondences

$$G(s_j)=\frac{1}{\gamma}T_1(s_j) =: w_j, \quad j=1,\ldots,n, \tag{3}$$

induces the interpolation problem between n-tuples, (s_j) and (w_j). Engineers often suppose that functions are analytic. Thus the interpolation problem for engineers can be reduced to Nevanlinna-Pick Interpolation Problem for mathematicians. Therefore we can assume that both preimage and image regions are the unit disk $\mathbb{D}$ and G is an analytic map of $\mathbb{D}$ into $\overline{\mathbb{D}}$.

The Nevanlinna-Pick Interpolation Problem is that: Let $(z_j)_{j=0}^{n-1}$ be an n-tuple of distinct points in $\mathbb{D}$ and $(w_j)_{j=0}^{n-1}$ be an n-tuple of points in $\mathbb{D}$. Let $\mathscr{B}$ denotes the class of analytic maps of $\mathbb{D}$ into $\overline{\mathbb{D}}$. Then does there exist a solution $f \in \mathscr{B}$ to solve the interpolation

$$f(z_j) = w_j, \quad j = 0, \dots, n-1? \tag{4}$$

G. Pick([9]) and R. Nevanlinna([8]) showed the necessary and sufficient condition for the existence of such a solution f.

Theorem A (Nevanlinna-Pick, cf. Theorem 2.2 in [5]). *For some positive integer n, let $(z_1, \dots, z_n)$ be an n-tuple of distinct points in $\mathbb{D}$, and $(w_1, \dots, w_n)$ be an n-tuple of points in $\mathbb{D}$. Then there exists $f \in \mathscr{B}$ satisfying the interpolation*

$$f(z_j) = w_j, \qquad j = 1, 2, \dots, n \tag{5}$$

if and only if the quadratic form

$$Q_n(t_1, \dots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j\overline{w_k}}{1 - z_j\overline{z_k}} t_j\overline{t_k} \tag{6}$$

is non-negative, i.e., $Q_n \geq 0$. When $Q_n \geq 0$ there is a Blaschke product of degree at most n which solves (5).

The Hermitian matrix P_n for $(z_1, \dots, z_n)$ and $(w_1, \dots, w_n)$, defined as

$$P_n := \left[\frac{1 - w_j\overline{w_k}}{1 - z_j\overline{z_k}}\right]_{1 \leq j,k \leq n} \tag{7}$$

is called the Pick matrix. The condition $Q_n \geq 0$ is equivalent to that the Pick matrix for $(z_1, \dots, z_n)$ and $(w_1, \dots, w_n)$ is positive semi-definite, in other words, all of its eigenvalues are nonnegative real numbers.

In principle, Pick's theorem determines whether the given Nevanlinna-Pick interpolation problem has a solution or not. However both the quadratic form (6) and the Pick matrix (7) are not very easy to calculate. Therefore more explicit conditions are expected in many cases.

Baribeau, Rivard and Wegert showed in [2] an explicit necessary and sufficient condition for a problem in their example. The problem is the interpolation between $(r, ri, -r, -ri)$ and $(s, -si, -s, si)$ respectively, where $0 < r, s < 1$ and i denotes the imaginary unit. Their simple and beautiful condition is that $s \leq r^3$. But they obtained their result by direct calculation, not theoretically.

In this paper, we will shows that the images of a solution to interpolations between $(re^{2\pi ij/n})_{j=0}^{n-1}$ and $(se^{-2\pi ij/n})_{j=0}^{n-1}$, where $n \geq 2$ and where $0 < r, s < 1$. Before we shows a precise method and results, we introduce our theorem.

Theorem. *Let $n \geq 2$ be a given integer and let $0 < r, s < 1$ be given numbers. Set $z_j = re^{2\pi ij/n}$ and $w_j = se^{-2\pi ij/n}$ for $0 \leq j \leq n-1$. Then there exists an analytic map f of $\mathbb{D}$ into $\overline{\mathbb{D}}$ which maps z_j to w_j for $0 \leq j \leq n-1$ if and only if $s \leq r^{n-1}$.*

Proof of the theorem is shown in [10].

Remark 1.1. The solution can be given as a Blaschke product of degree at most n. In this case the boundary of $\mathbb{D}$ is mapped onto itself. Moreover it can also be given as a rational map f of degree at most $n-1$ such that $\overline{f(\mathbb{D})} \subset \mathbb{D}$.

Remark 1.2. If $s < r^{n-1}$ and the solution f for the interpolation between $(re^{2j\pi i/n})$ and $(r^{n-1}e^{-2j\pi i/n})$ has been known, then $\widetilde{f} = (s/r^{(n-1)})f$ is a solution for the interpolation between $(re^{2\pi ij/n})$ and $(se^{-2\pi ij/n})$ and $\overline{\widetilde{f}(\mathbb{D})} \subset \mathbb{D}$.

2 Construct a solution f_n

Let m_a denote the Möbius transformation with respect to $a \in \mathbb{D}$ as

$$m_a(z) := \frac{z-a}{1-\overline{a}z}. \tag{8}$$

2.1 reduction

Let $z'_j = m_{z_{n-1}}(z_j)$ and $w'_j = m_{w_{n-1}}(w_j)$ for $0 \leq j \leq n-2$. Since m_a is an automorphism of $\mathbb{D}$ and z_js are distinct each other, $z'_j \neq 0$ for $0 \leq j \leq n-2$. Then, suppose that we have already known a solution $\widehat{f}_{n-1}$ to the interpolation between $(z'_0, \ldots, z'_{n-2})$ and $(w'_0/z'_0, \ldots, w'_{n-2}/z'_{n-2})$. Obviously a function $\widetilde{f}_n(z) = z \cdot \widehat{f}_{n-1}(z)$ is a solution to the interpolation between $(z'_0, \ldots, z'_{n-2}, 0)$ and $(w'_0, \ldots, w'_{n-2}, 0)$. It follows that

$$f_n(z) = m^{-1}_{w_{n-1}} \circ \widetilde{f}_n \circ m_{z_{n-1}}(z) \tag{9}$$

solves an interpolation between $(z_0, \ldots, z_{n-1})$ and $(w_0, \ldots, w_{n-1})$.

Clearly, a solution to an interpolation between (z_0) and (w_0) is given as a constant map

$$f_1(z) = w_0, \tag{10}$$

as an automorphism

$$f_1(z) = m^{-1}_{w_0} \circ m_{z_0}(z), \tag{11}$$

or as some holomorphism which is not an automorphism, i.e., which is contracting with respect to hyperbolic metric.

Hence, we can construct a solution to an interpolation between any given n-tuples by reduction in theory, if and only if there exists a solution. This method is shown in [6].

In general a solution is not unique. See [3] when a solution becomes unique.

2.2 Some examples

We show some examples here. In these figures, the outer circle is the unit circle, the middle circle is $\{|z| = r\}$, where $r = 0.8$, and the inner circle is $\{|z| = s\}$, where $s = r^{n-1}$.

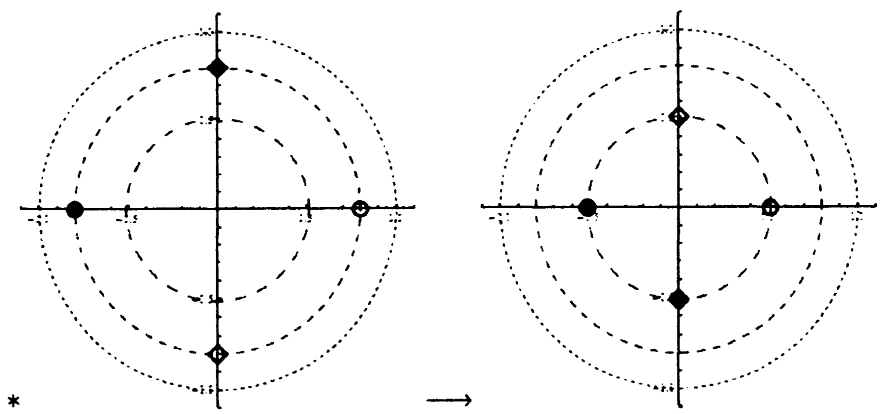
2.2.1 Line correspondency

....	$\lvert z\rvert = 1$
- - - - - - - - - - - - - - - - - - - -	$\lvert z\rvert = r$
— — — — — — — — — — — — — — —	$\lvert z\rvert = s$

2.2.2 4-tuples

This is an example of an interpolation between 4-tuples.

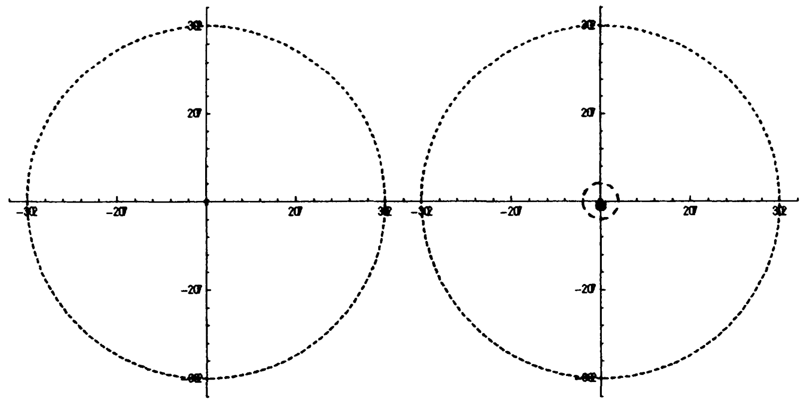
$$(r, ri, -r, -ri) \to (s, -si, -s, si)$$

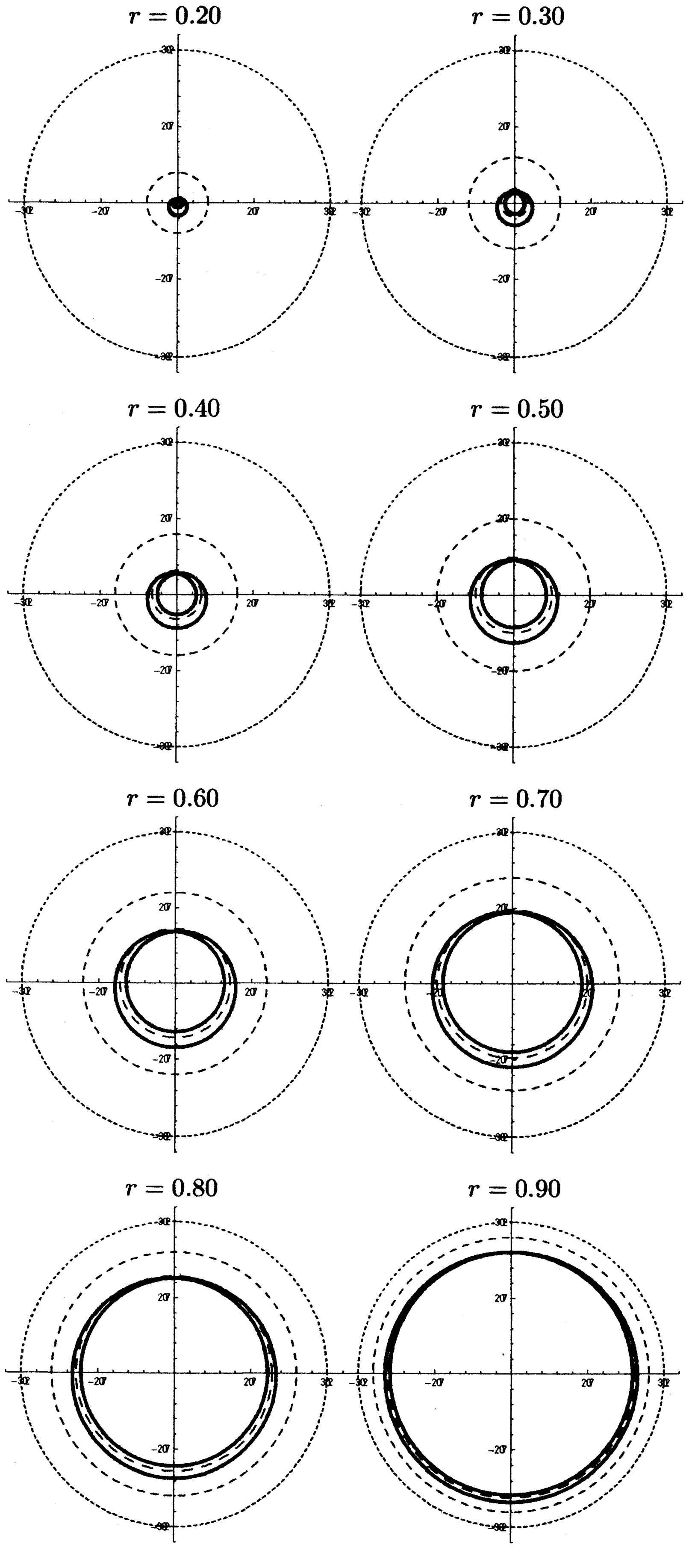


2.2.3 Images of $\{|z| = r\}$

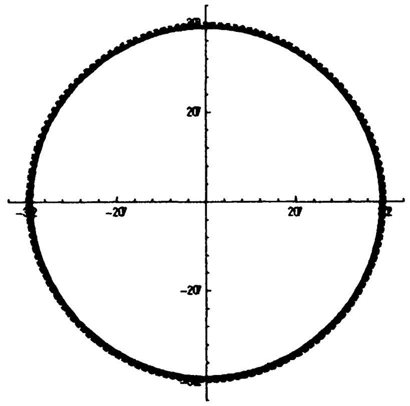
Next figures are images of $\{|z| = r\}$ by a solution to an interpolation between $\{r, ri, -r\}$ and $\{s, -si, -s\}$. Thease figures are symmetric with respect to the imaginary axis, but in general images are not symmetric.

$r = 0.01$ $\qquad\qquad\qquad$ $r = 0.10$



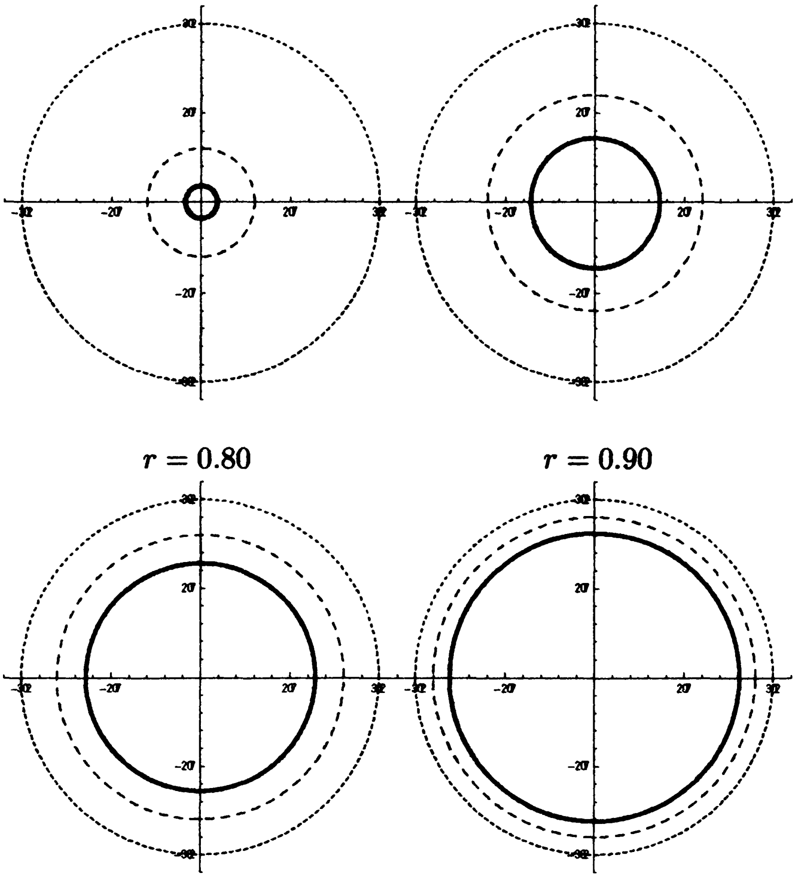

$r = 0.20$
$r = 0.30$
$r = 0.40$
$r = 0.50$
$r = 0.60$
$r = 0.70$
$r = 0.80$
$r = 0.90$

$r = 0.99$



Next figures are interpolation between $\{r, r\omega, r\omega^2\}$ and $\{s, s\omega^2, s\omega\}$, where ω is the primitive cubric root of 1.

$r = 0.30$ $r = 0.60$ $r = 0.80$ $r = 0.90$



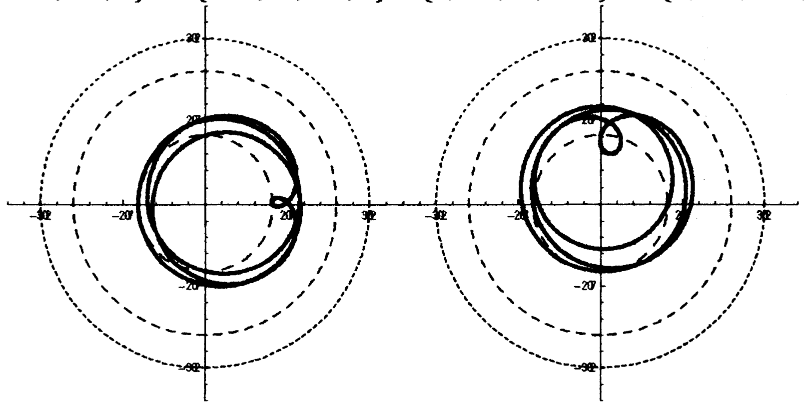
In this case, the image of a circle centered at origin is also a circle centered at origin.

These figures shows that the image of the circle $\{|z| = r\}$ tends to the unit circle as $r \to 1-$.

Remark 2.1. If one changes constructing order, then the solution will be changed. It shows that the solution is not unique even if the degree is same as n.

An example is shown below. This is the two images of 4-tuples cases. In both case $r = 0.80, s = r^4$. The differnce between each cases is only order of points in tuples. However, the solution we obtain becomes differ.

$\{ri, -ri, -r, r\} \to \{-si, si, -s, s\}$ $\{r, -r, ri, -ri\} \to \{s, -s, -si, si\}$



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