<table>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2010), 1717: 100-106</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/170322">http://hdl.handle.net/2433/170322</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On a remark of strongly convex functions of order $\beta$ and convex of order $\alpha$

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Abstract

For analytic functions $f(z)$ in the open unit disk $U$, two subclasses $C(\alpha, \beta)$ and $S^*(\alpha, \beta)$ are introduced. The object of the present paper is to investigate a strongly starlikeness of order $\delta$ and starlikeness of order $\beta(\alpha)$ of strongly convex functions of order $\gamma$ and convex of order $\alpha$.

1 Introduction

Let $S$ denote the set of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$.

Suppose that $f(z) \in S$ satisfies the following conditions

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in U) \quad (1)$$

or

$$\left| \arg \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in U), \quad (2)$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If $f(z) \in S$ satisfies (1), then we say that $f(z)$ is strongly convex of order $\beta$ and convex of order $\alpha$ in $U$ and we denote by $C(\alpha, \beta)$ the class of such functions $f(z)$. If $f(z) \in S$ satisfies (2), then $f(z)$ is said to be strongly starlike of order $\beta$ and starlike of order $\alpha$ in $U$ and we also denote by $S^*(\alpha, \beta)$ the class of such functions $f(z)$.

In view of the results by MacGregor [1] and Wilken and Feng [3], it is well known that $f(z) \in C(\alpha, 1)$ implies $f(z) \in S^*(\beta(\alpha), 1)$ where

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2010 Mathematics Subject Classification: 30C45

Keywords and phrases: Univalent, strongly convex, strongly starlike
\[ \beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & (0 \leq \alpha < 1; \alpha \neq \frac{1}{2}) \\ \frac{1}{2\log 2} & (\alpha = \frac{1}{2}). \end{cases} \]

In the present paper, we discuss some properties for \( f(z) \) concerning with the classes \( C(\alpha, \beta) \) and \( S^*(\alpha, \beta) \).

## 2 Main result

To consider our problems, we have to recall here the following lemma due to Nunokawa [2].

**Lemma 1.** Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( U \). Suppose that there exists a point \( z_0 \in U \) such that

\[ |\arg p(z)| < \frac{\pi}{2\alpha} \quad (|z| < |z_0|) \]

and

\[ |\arg p(z_0)| = \frac{\pi}{2\alpha} \]

where \( \alpha > 0 \). Then we have

\[ \frac{z_0p'(z_0)}{p(z_0)} = ik\alpha \]

where

\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{where} \quad \arg p(z_0) = \frac{\pi}{2\alpha} \]

and

\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{where} \quad \arg p(z_0) = -\frac{\pi}{2\alpha}, \]

where

\[ p(z_0)^{1/\alpha} = \pm ia \quad (a > 0). \]

Now, we derive

**Theorem 1.** If \( f(z) \in C(\gamma, \alpha) \) with \( 0 \leq \alpha < 1 \), then \( f(z) \in S^*(\delta, \beta(\alpha)) \), where \( 0 < \delta < 1 \) and

(i) if \( k = \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \) and

\[ \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} \geq \frac{\pi}{2} \delta, \]
then we put
\[ \gamma = \delta + \frac{2}{\pi} \tan^{-1} \frac{R_0 \sin(\Theta - \frac{\pi}{2} \delta)}{1 + R_0 \cos(\Theta - \frac{\pi}{2} \delta)}, \]
\[ \Theta = \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}, \]
\[ R_0 = \frac{\delta}{2(1 - \beta(\alpha)^2)} \left\{ \left( \frac{1 + \delta}{1 - \delta} \right)^{\frac{1 - \delta}{2}} + \left( \frac{1 - \delta}{1 + \delta} \right)^{\frac{1 + \delta}{2}} \right\}, \]

(ii) if \( k = \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \) and
\[ \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} < \frac{\pi}{2} \delta, \]
then we put
\[ \gamma = \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}, \]

(iii) if \( k = \frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \) and
\[ \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} \leq -\frac{\pi}{2} \delta, \]
then we put
\[ \gamma = -\delta - \frac{2}{\pi} \tan^{-1} \frac{R_0 \sin(\Theta + \frac{\pi}{2} \delta)}{1 + R_0 \cos(\Theta + \frac{\pi}{2} \delta)}, \]
\[ \Theta = \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}, \]
\[ R_0 = \frac{\delta}{2(1 - \beta(\alpha)^2)} \left\{ \left( \frac{1 + \delta}{1 - \delta} \right)^{\frac{1 - \delta}{2}} + \left( \frac{1 - \delta}{1 + \delta} \right)^{\frac{1 + \delta}{2}} \right\}, \]

(iv) if \( k = \frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \) and
\[ \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} > -\frac{\pi}{2} \delta, \]
then we put
\[ \gamma = \tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}. \]
Proof. Let us define the function \( p(z) \) by
\[
p(z) = \frac{zf'(z)}{f(z)}.
\]
Then we have that \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). It follows that
\[
p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)},
\]
and
\[
1 + \frac{zf''(z)}{f'(z)} - \alpha = p(z) - \beta(\alpha) + \frac{zp'(z)}{p(z)} + \beta(\alpha) - \alpha
\]
\[
= (p(z) - \beta(\alpha)) \left( 1 + \frac{zp(z) - \beta(\alpha)/p(z)}{p(z) - \beta(\alpha)} \frac{1}{p(z)} + \frac{\beta(\alpha) - \alpha}{p(z) - \beta(\alpha)} \right).
\]
Therefore, we see that
\[
\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)
\]
\[
= \arg(p(z) - \beta(\alpha)) + \arg\left(1 + \frac{zp(z) - \beta(\alpha)/p(z)}{p(z) - \beta(\alpha)} \frac{1}{p(z)} + \frac{\beta(\alpha) - \alpha}{p(z) - \beta(\alpha)} \right).
\]
If there exists a point \( z_0, |z_0| < 1 \) such that
\[
|\arg(p(z) - \beta(\alpha))| < \frac{\pi}{2} \delta \quad (|z| < |z_0|)
\]
and
\[
|\arg(p(z_0) - \beta(\alpha))| = \frac{\pi}{2} \delta,
\]
then, let us put
\[
q(z) = \frac{p(z) - \beta(\alpha)}{1 - \beta(\alpha)}, \quad q(0) = 1.
\]
Applying Lemma 1, we have that
\[
\frac{z_0q'(z_0)}{q(z_0)} = \frac{z_0q'(z_0)}{p(z_0) - \beta(\alpha)} = i\delta k
\]
where
\[
q(z_0) = \left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)}\right) = \pm ia \quad (a > 0)
\]
and
\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \quad \text{when} \quad \arg(p(z) - \beta(\alpha)) = \frac{\pi}{2} \delta
\]
and
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \quad \text{when} \quad \arg(p(z) - \beta(\alpha)) = -\frac{\pi}{2} \delta.
At first, let us consider the case \( \arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2} \delta \), then it follows that

\[
\arg \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) = \arg(p(z_0) - \beta(\alpha)) \left( 1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0) - \beta(\alpha)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)} \right) \\
= \frac{\pi}{2} \delta + \arg \left( 1 + \frac{i\delta k}{(\beta(\alpha) + (ia)^{\delta})(1 - \beta(\alpha))} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^{\delta}} \right) \\
\geq \frac{\pi}{2} \delta + \arg \left( 1 + \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^{\delta}} \right) \\
= \frac{\pi}{2} \delta + \arg \left( 1 + e^{-i\delta \frac{\pi}{2}} \left( \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^{\delta}} \right) \right).
\]

Next, let us consider

\[
\arg \left( \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^{\delta}} \right) = \tan^{-1} \left( \frac{\frac{\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^{\delta}}}{\frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^{\delta}}} \right) = \tan^{-1} \left( \frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right).
\]

On the other hand, applying the same method in the result by Nunokawa [2], we have

\[
\left| \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^{\delta}} + \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^{\delta}} \right| \geq \left| \frac{\delta k}{2(1 - \beta(\alpha)^2)a^{\delta}} \left( \alpha + \frac{1}{a} \right) \right| \\
= \frac{\delta}{2(1 - \beta(\alpha)^2)a^{\delta}} \left( a^{1-\delta} + \frac{1}{a^{1+\delta}} \right) \\
\geq \frac{\delta}{2(1 - \beta(\alpha)^2)} \left( \frac{1 + \delta}{1 - \delta} + \frac{1 - \delta}{1 + \delta} \right) \\
= R_0
\]
say.

Therefore, for the case \( \arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2} \delta \) and

\[
\tan^{-1} \left( \frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) \geq \frac{\pi}{2} \delta,
\]

we have

\[
\arg \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) = \frac{\pi}{2} \delta + \arg \left( 1 + \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^{\delta}} \right) \\
\geq \frac{\pi}{2} \delta + \arg \left( 1 + e^{-i\delta \frac{\pi}{2}} \left( \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^{\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^{\delta}} \right) \right) \\
\geq \frac{\pi}{2} \delta.
\]
we have
\[
\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right)
= \arg(p(z_0) - \beta(\alpha)) + \arg\left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0) - \beta(\alpha)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)}\right)
\geq \frac{\pi}{2} \delta + \arg\left(1 + R_0 e^{i(\Theta - \frac{\pi}{2} \delta)}\right)
= \frac{\pi}{2} \delta + \tan^{-1}\left(\frac{R_0}{1 + R_0 \cos(\Theta - \frac{\pi}{2} \delta)}\right),
\]

where
\[
\Theta = \tan^{-1}\left(\frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}\right).
\]
This is the contradiction for the condition of the theorem.
On the other hand, for the case
\[
\arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2} \delta
\]
and
\[
\Theta = \tan^{-1}\left(\frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}\right) < \frac{\pi}{2} \delta,
\]
putting $a \to +0$ in (3), we easily have
\[
\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right)
= \arg(p(z_0) - \beta(\alpha)) + \arg\left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0) - \beta(\alpha)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)}\right)
\geq \frac{\pi}{2} \delta + \arg e^{i(\Theta - \frac{\pi}{2} \delta)}
= \Theta
= \tan^{-1}\left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}\right).
\]
This is also the contradiction for the theorem.
For the case
\[
\arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2} \delta
\]
and
\[
\tan^{-1}\left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}\right) \leq -\frac{\pi}{2} \delta,
\]
where
\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right)
\]
and
\[
\left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)}\right)^{\frac{1}{2}} = -ia \quad (a > 0),
\]
or for the case
\[ \text{arg}(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2} \delta \]
and
\[ \tan^{-1} \left( \frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) < -\frac{\pi}{2} \delta, \]
where
\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \]
and
\[ \left( \frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)} \right)^{\frac{1}{\delta}} = -ia \quad (a > 0), \]
applying the same method as the above, we have the contradiction for the theorem. Therefore the proof of the theorem is completed.

\[ \square \]

**Remark 1.** We have to say that Theorem 1 is not sharp. To consider the sharp result, we need another method. Therefore we leave this problem for our discussion.

**References**


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