<table>
<thead>
<tr>
<th>Title</th>
<th>$(\alpha, \delta)$-neighborhood defining by a new operator for certain analytic functions (Extensions of the historical calculus transforms in the geometric function theory)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Kugita, Kazuyuki; Kuroki, Kazuo; Owa, Shigeyosi</td>
</tr>
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(α, δ)-neighborhood defining by a new operator for certain analytic functions

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Abstract

For analytic functions $f(z)$ in the open unit disk $U$, a new operator $D^j f(z)$ for any integer $j$ which is the generalization of Salagean differential operator and Alexander integral operator is introduced. The object of the present paper is to discuss some properties for $(α, δ)$-neighborhood defining by a new operator $D^j f(z)$ and to apply Miller-Mocanu lemma (J. Math. Anal. Appl. 65(1978)) for $(α, δ)$-neighborhood.

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}$, Salagean [3] has introduced the following operator $D^j f(z)$ which is called Salagean differential operator.

$$D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$$

and

$$D^j f(z) = D(D^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 1, 2, 3, \cdots).$$

Also, Alexander [1] has defined the following Alexander integral operator

$$D^{-1} f(z) = \int_{0}^{z} \frac{f(\zeta)}{\zeta} d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n.$$
Further, we introduce
\[ D^{-j} f(z) = D^{-1}(D^{-(j-1)} f(z)) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n \quad (j = 1, 2, 3, \cdots) \]
which is the generalization integral operator of Alexander integral operator. Therefore, combining Sălăgean differential operator and Alexander integral operator, we introduce the operator \( D^j f(z) \) by
\[ D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n \]
for any integer \( j \). Applying the above operator, we consider the subclass \((\alpha_1, \alpha_2, \cdots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \cdots, g_p)\) of \( A \) as follows. A function \( f(z) \in A \) is said to be in the class \((\alpha_1, \alpha_2, \cdots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \cdots, g_p)\) if it satisfies
\[ \left| \frac{D^{j+1} f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1} g_k(z)}{z} \right| < \delta \quad (z \in U) \]
for some \( \delta > \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2} \), where \( \beta = \arg \alpha_k \) for all \( k \) with \(-\pi \leq \beta \leq \pi\), and for some \( g_k(z) \in A \) \((k = 1, 2, \cdots, p)\). Let us define \((\alpha_1, \alpha_2, \cdots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \cdots, g_p)\) by
\[ (\alpha, \delta) - N_{m+1}^{j+1}(g) \equiv (\alpha_1, \alpha_2, \cdots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \cdots, g_p) \]
through this paper.

2 Main theorem

Let us define \( g_k(z) \in A \) \((k = 1, 2, \cdots, p)\) by
\[ g_k(z) = z + \sum_{n=2}^{\infty} b_{n,k} z^n \]
through this paper. Our first result of \( f(z) \) for \((\alpha, \delta) - N_{m+1}^{j+1}(g)\) is contained in

Theorem 2.1 If \( f(z) \in A \) satisfies
\[ \sum_{n=2}^{\infty} n |n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k}| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2} \]
for some \( \delta > \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2} \), where \( \beta = \arg \alpha_k \) for all \( k \) with \(-\pi \leq \beta \leq \pi\), and for some \( g_k(z) \in A \) \((k = 1, 2, \cdots, p)\), then \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \).
Proof. Note that
\[
\left| \frac{D^{j+1} f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1} g_k(z)}{z} \right| = \left| 1 + \sum_{n=2}^{\infty} n^{j+1} a_n z^{n-1} - \sum_{k=1}^{p} \alpha_k \left( 1 + \sum_{n=2}^{\infty} n^{m+1} b_{n,k} z^{n-1} \right) \right|
\]
\[
= \left| 1 - \sum_{k=1}^{p} \alpha_k + \sum_{n=2}^{\infty} n \left( n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right) z^{n-1} \right|
\]
\[
\leq \left| 1 - \sum_{k=1}^{p} \alpha_k \right| + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| z^{n-1}
\]
\[
< \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2} + \sum_{n=2}^{\infty} n |n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k}|.
\]

If
\[
\sum_{n=2}^{\infty} n |n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k}| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2},
\]
then we see that
\[
\left| \frac{D^{j+1} f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1} g_k(z)}{z} \right| < \delta \quad (z \in U).
\]

This gives us that \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \). \( \square \)

Example 2.2 For given \( g_k(z) = z + \sum_{n=2}^{\infty} b_{n,k} z^n \in \mathcal{A} \), we consider \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A} \) with
\[
a_n = \frac{\delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2}}{n^{j+2}(n-1)} e^{i \gamma} + n^{m-j} \sum_{k=1}^{p} \alpha_k b_{n,k} \quad (n = 2, 3, 4, \cdots).
\]

Then, we have that
\[
\sum_{n=2}^{\infty} n |n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k}| = \sum_{n=2}^{\infty} n \left| \frac{\delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2}}{n^{j+2}(n-1)} e^{i \gamma} \right|
\]
\[
= \left( \delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2} \right) \left( \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \right)
\]
\[
= \left( \delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2} \right) \left( \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) \right)
\]
\[
= \delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2}.
\]

Therefore, \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \).
In view of Theorem 2.1, we have the following corollary.

**Corollary 2.3** Let \( f(z) \in A \) satisfy

\[
\sum_{n=2}^{\infty} n \left| n^j a_n \right| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2}
\]

for some \( \delta > \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2} \), where \( \beta = \arg \alpha_k \) for all \( k \) with \( -\pi \leq \beta \leq \pi \), and for some \( g_k(z) \in A \) \((k = 1, 2, \ldots, p)\) with \( \arg a_n - \arg b_{n,k} = \beta \) \((n = 2, 3, 4, \ldots)\) for all \( k \), then \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \).

**Proof.** By Theorem 2.1, we have that if \( f(z) \in A \) satisfies

\[
\sum_{n=2}^{\infty} n \left| n^j a_n \right| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2},
\]

then \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \). Since \( \arg a_n - \arg b_{n,k} = \beta \), if \( \arg a_n = \varphi_n \), then \( \arg b_{n,k} = \varphi_n - \beta \). Therefore, we see that

\[
n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} = n^j |a_n| e^{i\varphi_n} - n^m \sum_{k=1}^{p} |\alpha_k| e^{i\beta} |b_{n,k}| e^{i(\varphi_n - \beta)}
\]

\[
= \left( n^j |a_n| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \right) e^{i\varphi_n},
\]

that is, that

\[
\left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| = \left| n^j |a_n| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \right|.
\]

This completes the proof of the corollary. \( \square \)

Next, we discuss the necessary conditions for neighborhoods.

**Theorem 2.4** If \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \) with

\[
\arg \left( n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right) = (n - 1)\varphi \quad (\varphi \in \mathbb{R}),
\]

for \( n = 2, 3, 4, \ldots, \) then,

\[
\sum_{n=2}^{\infty} n \left| n^j a_n \right| - n^m \sum_{k=1}^{p} |\alpha_k| |b_{n,k}| \leq -1 + \sum_{k=1}^{p} |\alpha_k| \cos \beta + \sqrt{\delta^2 - \left( \sum_{k=1}^{p} |\alpha_k| \sin \beta \right)^2}.
\]
Proof. For \( f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g) \), if we consider a point \( z \in U \) such that \( \arg z = -\varphi \), then
\[
z^{n-1} = |z|^{n-1} e^{-i(n-1)\varphi},
\]
and hence we have
\[
\left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| = \left| 1 - \sum_{k=1}^{p} \alpha_k + \sum_{n=2}^{\infty} n \left( n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right) z^{n-1} \right|
\]
\[
= \left| 1 - \sum_{k=1}^{p} \alpha_k + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| |z|^{n-1} \right| < \delta.
\]
Letting \( |z| \rightarrow 1^- \) we have
\[
\left| 1 - \sum_{k=1}^{p} \alpha_k + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| \right|
\]
\[
= \left\{ \left( 1 - \sum_{k=1}^{p} |\alpha_k| \cos \beta + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| \right)^2 + \left( \sum_{k=1}^{p} |\alpha_k| \sin \beta \right)^2 \right\}^{\frac{1}{2}} \leq \delta,
\]
which implies that
\[
\left( \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| \right)^2 + 2 \left( 1 - \sum_{k=1}^{p} |\alpha_k| \cos \beta \right) \left( \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| \right)
\]
\[
+ 1 + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta - \delta^2 \leq 0.
\]
Therefore, it is easy to see that
\[
\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^{p} \alpha_k b_{n,k} \right| \leq -1 + \sum_{k=1}^{p} |\alpha_k| \cos \beta + \sqrt{\delta^2 - \left( \sum_{k=1}^{p} |\alpha_k| \sin \beta \right)^2}.
\]

\[\square\]

3 Applications of Miller-Mocanu lemma

In this section, we will give a certain implication for the class \((\alpha, \delta) - N_{m+1}^{j+1}(g)\). To considering our problem, we need the following lemma given by Miller and Mocanu [2].

Lemma 3.1 Let \( n \) be a positive integer, and let \( F(z) \) be analytic in \( U \) with \( F^{(k)}(0) = 0 \) \((k = 1, 2, \cdots, n - 1)\), \( F(0) = a \) and \( F(z) \neq a \) for a complex number \( a \). If there exists a point \( z_0 \in U \) such that
\[
\max_{|z| \leq |z_0|} |F(z)| = |F(z_0)|,
\]
then
\[
\frac{z_0 F'(z_0)}{F(z_0)} = m,
\]
where $m$ is real and
\[
m \geq n \frac{|F(z_0) - a|^2}{|F(z_0)|^2 - |a|^2} \geq n \frac{|F(z_0)| - |a|}{|F(z_0)| + |a|}.
\]

Applying Lemma 3.1, we derive

**Theorem 3.2** If $f(z) \in A$ satisfies
\[
\left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| < \frac{2\delta^2}{\delta + \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2}} \quad (z \in \mathbb{U})
\]
for some $\delta \geq \sqrt{1 - 2 \sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2}$, where $\beta = \arg \alpha_k$ for all $k$ with $-\pi \leq \beta \leq \pi$, and for some $g_k(z) \in A$ $(k = 1, 2, \ldots, p)$, then
\[
\left| \frac{D^j f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^m g_k(z)}{z} \right| < \delta \quad (z \in \mathbb{U}),
\]
which implies that $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

**Proof.** We define the function $F(z)$ by
\[
F(z) = \frac{D^j f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^m g_k(z)}{z} \quad (z \in \mathbb{U}).
\]
Then,
\[
z \frac{F'(z)}{F(z)} = \frac{D^{j+1} f(z)}{z} - \frac{D^j f(z)}{z} - \sum_{k=1}^{p} \alpha_k \left( \frac{D^{m+1} g_k(z)}{z} - \frac{D^m g_k(z)}{z} \right)
= \frac{1}{F(z)} \left( \frac{D^{j+1} f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1} g_k(z)}{z} \right) - 1.
\]
Therefore,
\[
\left| \frac{D^{j+1} f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1} g_k(z)}{z} \right| = \left( 1 + \frac{z F'(z)}{F(z)} \right) F(z).
\]
Then $F(z)$ is analytic in $\mathbb{U}$ with $F(0) = 1 - \sum_{k=1}^{p} \alpha_k$ and $|F(0)| < \delta$. In view of the condition, let us suppose that there is a point $z_0 \in \mathbb{U}$ such that $\max_{|z| \leq |z_0|} |F(z)| = |F(z_0)| = \delta$. Then, by Lemma 3.1, we can write that
\[
F(z_0) = \delta e^{i\theta}, \quad \frac{z_0 F'(z_0)}{F(z_0)} = m \quad \text{and} \quad m \geq \left| \frac{\delta e^{i\theta} - \left( 1 - \sum_{k=1}^{p} \alpha_k \right)}{\delta^2 - \left( 1 - \sum_{k=1}^{p} \alpha_k \right)^2} \right|^2.
\]
Therefore, we see that
\[
\left| \frac{D^{j+1}f(z_0)}{z_0} - \sum_{k=1}^{p} \alpha_k \frac{D^{m+1}g_k(z_0)}{z_0} \right| = |1 + m| |F(z_0)|
\]
\[
= \delta (1 + m)
\]
\[
\geq \delta + \delta \frac{\left| e^{i\theta} - \left( 1 - \sum_{k=1}^{p} \alpha_k \right) \right|^2}{\delta^2 - \left| 1 - \sum_{k=1}^{p} \alpha_k \right|^2}
\]
\[
\geq \delta + \delta \frac{\delta - \left| 1 - \sum_{k=1}^{p} \alpha_k \right|}{\delta + \left| 1 - \sum_{k=1}^{p} \alpha_k \right|}
\]
\[
= \frac{2\tilde{\delta}^2}{\delta + \sqrt{1 - 2\sum_{k=1}^{p} |\alpha_k| \cos \beta + \left( \sum_{k=1}^{p} |\alpha_k| \right)^2}}
\]

This contradicts our condition in Theorem 3.2. Thus, there is no point \( z_0 \in U \) such that \( |F(z_0)| = \delta \). This means that \( |F(z)| < \delta \) for all \( z \in U \). Therefore, we have that
\[
\left| \frac{D^j f(z)}{z} - \sum_{k=1}^{p} \alpha_k \frac{D^m g_k(z)}{z} \right| < \delta \quad (z \in U).
\]

\( \square \)

Taking \( p = 1 \) in Theorem 3.2, and letting
\[ \alpha_1 = e^{i\alpha} \text{ and } g_1(z) = g(z), \]
we find the following corollary.

**Corollary 3.3** If \( f(z) \in A \) satisfies
\[
\left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| < \frac{2\tilde{\delta}^2}{\delta + \sqrt{2(1 - \cos \alpha)}} \quad (z \in U)
\]
for some \( -\pi \leq \alpha \leq \pi \), \( \delta > \sqrt{2(1 - \cos \alpha)} \) and for some \( g(z) \in A \), then
\[
\left| \frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} \right| < \delta \quad (z \in U).
\]

In particular, by putting \( \delta = \tilde{\delta} + \sqrt{2(1 - \cos \alpha)} \) for some \( -\pi \leq \alpha \leq \pi \) and \( \tilde{\delta} > 0 \), we can obtain the assertion as follows.

**Corollary 3.4** If \( f(z) \in A \) satisfies
\[
(3.1) \quad \left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| < 2\tilde{\delta} + \frac{4(1 - \cos \alpha)}{\delta + 2\sqrt{2(1 - \cos \alpha)}} \quad (z \in U)
\]
for some $-\pi \leq \alpha \leq \pi$, $\tilde{\delta} > 0$ and for some $g(z) \in \mathcal{A}$, then

\begin{equation}
\left| \frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} \right| < \tilde{\delta} + \sqrt{2(1 - \cos \alpha)} \quad (z \in \mathbb{U}).
\end{equation}

**Remark 3.5** Recently, in the paper by Kugita, Kuroki and Owa [4], we obtained the implication that

\begin{equation}
\left| \frac{D^{j+1} f(z)}{z} - e^{i\alpha} \frac{D^{m+1} g(z)}{z} \right| < 2\tilde{\delta} - \sqrt{2(1 - \cos \alpha)} \quad (z \in \mathbb{U})
\end{equation}

implies the inequality (3.2), where $\tilde{\delta} > \sqrt{2(1 - \cos \alpha)}$. Here, a simple check gives us that if $f(z) \in \mathcal{A}$ satisfies the inequality (3.3), then $f(z)$ satisfies the inequality (3.1). Hence, it follows this fact that if $f(z) \in \mathcal{A}$ satisfies the assertion of Corollary 3.4, then the implication which were proven by Kugita, Kuroki and Owa [4] holds.

**References**


