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Critical points parameters for triply connected
Bell domains

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1 Introduction

The fundamental problem in the geometric function theory is to find a family of canonical domains. Recently, S. Bell proposed a new family of domains which admit canonically a simple proper holomorphic map to the unit disc $U$. Actually, they are enough.

**Theorem 1** ([1]). *Every non-degenerate d-ply connected planar domain $W$ with $d > 1$ is mapped biholomorphically (or, conformally) onto a domain $W_{a,b}$, defined by*

$$W_{a,b} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{d-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

*with suitable complex vectors

$$a = (a_1, a_2, \ldots, a_{d-1}), \quad b = (b_1, b_2, \ldots, b_{d-1}).$$

This theorem can be considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains.

We call such a domain $W_{a,b}$ as in Theorem 1.1 a *Bell representation* of $W$. The function $f_{a,b}$ defined by

$$f_{a,b}(z) = z + \sum_{k=1}^{d-1} \frac{a_k}{z - b_k}$$
is a proper holomorphic map from $W_{a,b}$ onto $U$. Set $B_d$ be the set of all vectors $(a, b)$ in $\mathbb{C}^{2d-2}$ such that $W_{a,b}$ is a Bell representation of $d$-ply connected planar domains, and we call $B_d$ the coefficient body of degree $d$. (Cf. [2].)

Now, from a well-known fact on the theory of moduli, we can conclude that $d$-ply connected non-degenerate planar domains have real $3d - 6$ moduli (or Teichmüller) parameters if $d \geq 3$. First we state this fact more precisely.

**Definition 1.** Let $d \geq 2$. We call a $d$-ply connected non-degenerate planar domain $W$ equipped with an order of boundary components of $W$ a boundary-marked planar domain of type $d$.

Two marked planar domains $W_1$ and $W_2$ of type $d$ are conformally equivalent if there is a conformal mapping $f : W_1 \to W_2$ which preserves the boundary-markings.

Let $D_d$ be the set of all equivalence classes of boundary-marked planar domains of type $d$. We call $D_d$ the deformation space of a boundary-marked planar domain of type $d$.

Then the following fact is classical.

**Proposition 2.** If $d \geq 3$, then $D_d$ can be considered as a domain in $\mathbb{R}^{3d-6}$.

**Proof.** By Koebe's theorem ([3]), every $d$-ply connected non-degenerate planar domain can be mapped conformally onto a Koebe circle domain.

On the other hand, it is easy to see that boundary-marked Koebe circle domains have real $3d - 6$ real global parameters up to Möbius transformations. □

In the case of triply connected planar domains, there always exists a canonical symmetry for every such one. Moreover, it is believed that the intersection of the coefficient body $B_3$ with each one of the following families gives an explicit model of $D_3$. We will discuss about it.

**Definition 2.** Set

$$B^+ = \{(a, b, d) \in \mathbb{R}^3 \mid a > 0, b > 0, d > 0\},$$

and

$$B^- = \{(a, b, d) \in \mathbb{R}^3 \mid a > 0, b < 0, d < 0\}.$$

We assume that $B^\pm$ are naturally embedded in $\mathbb{C}^3$. Also in the sequel, we write as

$$W_{a,b,d} = \{z \in \mathbb{C} \mid |f_{a,b,d}(z)| < 1\},$$

where

$$f_{a,b,d}(z) = z + \frac{b}{z-a} + \frac{d}{z+a}.$$
2 Main results

First, we clarify the correspondence of $(a, b, d)$ with the set of critical points and the phase transition of the covering structures of $f_{a,b,d}$ for the case of $B^+$.

First note the following

**Lemma 3.** For every $f = f_{a,b,d}$ with $(a, b, d) \in B^+$, either
1) $f$ has for real critical points $\{r, p, s, t\}$, or
2) $f$ has two real critical points $\{r, t\}$ and two others $\{p + si, p - si\}$. Here we may assume that

1) $r < p \leq s < t$, or
2) $r < t$, $s > 0$,

respectively.

For every $f = f_{a,b,d}$ with $(a, b, d) \in B^-$, $f$ has two pair of complex conjugates $\{r + it, r - it\}$ and $\{p + si, p - si\}$. Here we assume that

$r \leq p$, $t > 0$, $s > 0$.

In the case of $B^+$, the phase transition occurs at the locus $\text{Discr}(F) = 0$, where $\text{Discr}(F)$ is the constant times

$$bda^2 \left( (4a^2 - b - d)^3 - 108bda^2 \right)$$

$$F(z) = (z - a)^2(z + a)^2 - b(z + a)^2 - d(z - a)^2.$$

Here, we include the figures which show the typical manner of the phase transition.
Figure 1: $a = 0.05$, $b = 0.001$, $c = 0.00155$
Figure 2: \( a = 0.05, b = 0.001, c = 0.00153853756925731479 \)
Figure 3: $a = 0.05$, $b = 0.001$, $c = 0.0015$
Next, recall that $F(z)$ is represented also as
\[ F(z) = z^4 + \sigma_1 z^3 + \sigma_2 z^2 + \sigma_3 z + \sigma_4. \]
Clearly, $\sigma_1 = 0$ and the vectors $(\sigma_2, \sigma_3, \sigma_4)$ correspond to the sets \{r, s, t\} bijectively, which is called the relations between solutions and coefficients.

Also a direct computation gives

**Lemma 4.** The Jacobian
\[
\frac{\partial(\sigma_2, \sigma_3, \sigma_4)}{\partial(a, b, d)}
\]
is
\[-8a^2(4a^2 - b - d).\]

Now, the main theorems are the following

**Theorem 5.** In the case of $B^{-}$, the set of three real parameters
\[(r, s, t)\]
gives the set of global coordinates of $B^{-}$. In other words, the map $\Pi^{-}$ of $B^{-}$ to $(r, s, t) \in \mathbb{R}^3$ is a homeomorphism onto the image.

**Proof.** First, the map
\[
\phi : (a, b, d) \mapsto (\sigma_2, \sigma_3, \sigma_4)
\]
is locally homeomorphic by Lemma 4 and the assumptions that $b < 0$ and $d < 0$. Also $\phi$ is injective. Indeed, $a^2$ is a positive solution of
\[ 3x^2 + \sigma_2 x - \sigma_4 = 0. \]
And since $\sigma_4 > 0$, it has exactly one positive solution.

Next, we can show by a direct computation that the Jacobian
\[
\frac{\partial(\sigma_2, \sigma_3, \sigma_4)}{\partial(r, s, t)} = 4st \left(2(t^2 - s^2)^2 + 16r^2(2r^2 + s^2 + t^2)\right)
= 8st \left(4r^2 + (s - t)^2\right) \left(4r^2 + (s + t)^2\right),
\]
which is non-negative, and equals 0 if and only if $r = 0, s = t$. But these conditions imply that $a = b = d = 0$, and hence can not occur. Thus we conclude that
\[
\psi : (r, s, t) \mapsto (\sigma_2, \sigma_3, \sigma_4)
\]
is also locally homeomorphism and clearly $\psi^{-1}$ is injective.

Thus we can show that the map $\Pi^{-}$ of $B^{-}$ to $(r, s, t) \in \mathbb{R}^3$ is injective and locally homeomorphic, and hence is a homeomorphism onto the image. \(\square\)
Theorem 6. In the case $B^+$, the map $\Pi^+ : (a, b, d) \mapsto (r, s, t)$ is locally homeomorphic except for the degenerate locus

$$E_1 = \{(a, b, d) \mid 4a^2 - b - d = 0\},$$

The bifurcation locus is

$$E_2 = \{(a, b, d) \mid \text{Discr}(F) = (4a^2 - b - d)^3 - 108bda^2 = 0\}.$$

Proof. The first assertion follows from Lemma 4. And the second assertion is already stated before Lemma 4. \qed

Remark 1. On the subset of $B^+$ where $s^2 - b - d > 0$, $\Pi^+$ is injective.

Finally we include the figures of

$$(4a^2 - b - d)^3 - 108bda^2 = 0,$$

which are symmetric with respect to $\{a = 0\}$ and $\{b = c\}$. The planes in the figures are $a$-, $b$-, $c$-planes.
References

