<table>
<thead>
<tr>
<th>Title</th>
<th>Coefficient estimates of functions in the class concerning with spirallike functions (Extensions of the historical calculus transforms in the geometric function theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Hamai, Kensei; Hayami, Toshio; Kuroki, Kazuo; Owa, Shigeyoshi</td>
</tr>
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</tr>
</tbody>
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Coefficient estimates of functions in the class concerning with spirallike functions

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Abstract

For analytic functions \( f(z) \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \) in the open unit disk \( U \), a new subclass \( S_\alpha \) of \( f(z) \) concerning with spirallike functions in \( U \) is introduced. The object of the present paper is to discuss an extremal function for the class \( S_\alpha \) and coefficient estimates of functions \( f(z) \) belonging to the class \( S_\alpha \).

1 Introduction

Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form

\[
(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C}; |z| < 1 \} \).

If \( f(z) \in \mathcal{A} \) satisfies the following inequality

\[
(1.2) \quad \text{Re} \left( \frac{1}{\alpha} \frac{zf'(z)}{f(z)} \right) > 1 \quad (z \in U)
\]

for some complex number \( \alpha (|\alpha - \frac{1}{2}| < \frac{1}{2}) \), then we say that \( f(z) \in S_\alpha \). If \( \alpha = |\alpha| e^{i\varphi} \), then the condition (1.2) is equivalent to

\[
\text{Re} \left( e^{-i\varphi} \frac{zf'(z)}{f(z)} \right) > |\alpha| \quad (z \in U).
\]

Therefore, we note that a function \( f(z) \in S_\alpha \) is spirallike in \( U \) which implies that \( f(z) \) is univalent in \( U \).

Further, if \( 0 < \alpha < 1 \), then \( f(z) \in S_\alpha \) is starlike of order \( \alpha \) (cf. Robertson[3]).

Let \( \mathcal{P} \) denote the class of functions \( p(z) \) of the form

\[
(1.3) \quad p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k
\]

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which are analytic in $U$ and satisfy
\[ \text{Re } p(z) > 0 \quad (z \in U). \]

Then we say that $p(z) \in \mathcal{P}$ is the Carathéodory function (cf. Carathéodory [1] or Duren [2]).

**Remark 1.1** Let us consider a function $f(z) \in \mathcal{A}$ which satisfies
\[ |\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}| < \frac{1}{2\alpha} \quad (z \in U) \]
for $|\alpha - \frac{1}{2}| < \frac{1}{2}$. If we write that $F(z) = \frac{zf'(z)}{f(z)}$, then the inequality (1.4) can be written by
\[ \left| \frac{2\alpha - F(z)}{F(z)} \right| < 1 \quad (z \in U). \]

This implies that
\[ \alpha \overline{F(z)} + \overline{\alpha} F(z) > 2|\alpha|^2 \quad (z \in U). \]
It follows that
\[ \left( \frac{F(z)}{\alpha} \right) + \overline{\left( \frac{F(z)}{\alpha} \right)} > 2 \quad (z \in U). \]
Therefore, the inequality (1.4) is equivalent to
\[ \text{Re} \left( \frac{1}{\alpha} \frac{zf'(z)}{f(z)} \right) > 1 \quad (z \in U). \]

## 2 Coefficient estimates

In this section, we discuss the coefficient estimates of $a_n$ for $f(z) \in S_\alpha$. To establish our results, we need the following lemma due to Carathéodory [1].

**Lemma 2.1** If a function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$ satisfies the following inequality
\[ \text{Re } p(z) > 0 \quad (z \in U), \]
then
\[ |c_k| \leq 2 \quad (k = 1, 2, 3, \cdots) \]
with equality for
\[ p(z) = \frac{1 + z}{1 - z}. \]

Now, we introduce the following theorem.

**Theorem 2.2** Extremal function for the class $S_\alpha$ is $f(z)$ defined by
\[ f(z) = \frac{z}{(1 - z)^{2\alpha(\text{Re} \frac{1}{2\alpha}) + 1}}. \]
Proof. From the definition of the class $S_\alpha$, we have that
\[ \text{Re}\left(\frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1\right) > 0. \]
Moreover, it is clear that
\[ \text{Re}\left(\frac{1}{\alpha}\right) > 1 \quad (|\alpha - \frac{1}{2}| < \frac{1}{2}). \]
Then, if the function $F(z)$ is defined by
\[ F(z) = \frac{\frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1 - i\text{Im}(\frac{1}{\alpha})}{\text{Re}(\frac{1}{\alpha}) - 1}, \]
we see that
\[ \text{Re}F(z) > 0 \text{ and } F(0) = 1, \]
so that, $F(z) \in \mathcal{P}$.
Therefore, Lemma 2.1 shows us that
\[ F(z) = \frac{\frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1 - i\text{Im}(\frac{1}{\alpha})}{\text{Re}(\frac{1}{\alpha}) - 1} = \frac{1+z}{1-z}. \]
It follows that,
\[ \frac{f'(z)}{f(z)} - \frac{1}{z} = 2\alpha \left(\text{Re}\left(\frac{1}{\alpha}\right) - 1\right) \frac{1}{1-z}. \]
Integrating both sides from 0 to $z$ on $t$, we have that
\[ \int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)dt = 2\alpha \left(\text{Re}\left(\frac{1}{\alpha}\right) - 1\right) \int_0^z \frac{1}{1-t}dt, \]
which implies that
\[ \log\frac{f(z)}{z} = \log\frac{1}{(1-z)^{2\alpha(\text{Re}(\frac{1}{\alpha}) - 1)}}. \]
Therefore, we obtain that
\[ f(z) = \frac{z}{(1-z)^{2\alpha(\text{Re}(\frac{1}{\alpha}) - 1)}}. \]
This is the extremal function of the class $S_\alpha$. \hfill \Box

Next, we discuss the coefficient estimates of $f(z)$ belonging to the class $S_\alpha$.

Theorem 2.3 If a function $f(z) \in S_\alpha$, then
\[ |a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha))) - |\alpha| + (k-1)) \quad (n = 2, 3, 4 \cdots). \]
Equality holds true for $f(z)$ given by (2.1).
Proof. By using same method with Theorem 2.2, if we set $F(z)$ that

\[(2.2)\]

\[F(z) = \frac{\frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1 - i\text{Im}\left(\frac{1}{\alpha}\right)}{\text{Re}\left(\frac{1}{\alpha}\right) - 1},\]

then it is clear that $F(z) \in P$.

Letting

\[F(z) = 1 + c_1 z + c_2 z^2 + \cdots,\]

Lemma 2.1 gives us that

\[|c_m| \leq 2 \quad (m = 1, 2, 3 \cdots).\]

Now, from (2.2),

\[
\left(\text{Re}\left(\frac{1}{\alpha}\right) - 1\right) F(z) = \frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1 - i\text{Im}\left(\frac{1}{\alpha}\right).
\]

Let $\text{Re}(\frac{1}{\alpha}) - 1 = s$ and $1 + i\text{Im}(\frac{1}{\alpha}) = A$.

This implies that

\[(\alpha s F(z) + \alpha A)f(z) = zf'(z).\]

Then, the coefficients of $z^n$ in both sides lead to

\[na_n = (\alpha s + \alpha A) a_n + \alpha s (a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}).\]

Therefore, we see that

\[a_n = \frac{\alpha s}{n - \alpha s - \alpha A} (a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}).\]

This shows that

\[|a_n| = \frac{|\alpha (\text{Re}\left(\frac{1}{\alpha}\right) - 1)|}{n - \alpha (\text{Re}\left(\frac{1}{\alpha}\right) - 1) - \alpha (1 + i\text{Im}(\frac{1}{\alpha}))} |a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}|\]

\[\leq \frac{\cos(\arg(\alpha)) - |\alpha|}{n - 1} |a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_{n-r} c_r + \cdots + a_2 c_{n-2} + c_{n-1}|\]

\[\leq \frac{\cos(\arg(\alpha)) - |\alpha|}{n - 1} (|a_{n-1}| |c_1| + |a_{n-2}| |c_2| + \cdots + |a_{n-r}| |c_r| + \cdots + |a_2| |c_{n-2}| + |c_{n-1}|)\]

\[\leq \frac{\cos(\arg(\alpha)) - |\alpha|}{n - 1} \left(2|a_{n-1}| + 2|a_{n-2}| + \cdots + 2|a_2| + 2\right)\]

\[\leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n - 1} \sum_{k=1}^{n-1} |a_k| \quad (|a_1| = 1).\]

To prove that

\[|a_n| \leq \frac{1}{(n - 1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k - 1)),\]
we need to show that

$$|a_n| \leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n-1} \sum_{k=1}^{n-1} |a_k| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)).$$

We use the mathematical induction for the proof.

When \( n = 2 \), this assertion is true.

We assume that the proposition is true for \( n = 2, 3, 4, \ldots, m-1 \).

For \( n = m \), we obtain that

$$|a_m| \leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \sum_{k=1}^{m-1} |a_k|$$

$$= \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \left( \sum_{k=1}^{m-2} |a_k| + |a_{m-1}| \right)$$

$$= \frac{m-2}{m-1} \left( \sum_{k=1}^{m-2} |a_k| + |a_{m-1}| \right)$$

$$\leq \frac{m-2}{(m-1)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k-1)$$

$$+ \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \frac{1}{(m-2)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k-1)$$

$$= \frac{1}{(m-1)!} \prod_{k=1}^{m-1} (2(\cos(\arg(\alpha)) - |\alpha|) + k-1) (m-2 + 2(\cos(\arg(\alpha)) - |\alpha|))$$

Thus the inequality (2.3) is true for \( n = m \). By the mathematical induction, we prove that

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)) \quad (n = 2, 3, 4 \ldots).$$

For the equality, we consider the extremal function \( f(z) \) given by Theorem 2.2. Since

$$f(z) = \frac{z}{(1-z)^{2\alpha(\text{Re}(\frac{1}{\alpha})-1)}},$$

if we let

$$2\alpha(\text{Re}(\frac{1}{\alpha}) - 1) = j,$$

then \( f(z) \) becomes that

$$f(z) = z(1-z)^{-j} = z \left( \sum_{n=0}^{\infty} \binom{-j}{n} (-z)^n \right) = z + \sum_{n=2}^{\infty} \frac{j(j+1) \cdots (j+n-2)}{(n-1)!} z^n.$$
From the above, we obtained

\[ a_n = \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2\alpha(\Re\left(\frac{1}{\alpha}\right) - 1) + k - 1). \]

For \( n = 2 \),

\[ |a_2| = 2|\alpha||\Re\left(\frac{1}{\alpha}\right) - 1| = 2(\cos(\arg(\alpha)) - |\alpha|). \]

Furthermore, for \( n \geq 3 \), we have that

\[
|a_n| = \left| \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2\alpha(\Re\left(\frac{1}{\alpha}\right) - 1) + k - 1) \right| \\
= \frac{1}{(n-1)!} \prod_{k=1}^{n-1} |2\alpha(\Re\left(\frac{1}{\alpha}\right) - 1) + k - 1| \\
\leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1).
\]

Equality holds true for some real \( \alpha (0 < \alpha < 1) \).

This completes the proof of Theorem 2.3.

Example 2.4 Let \( \alpha = \frac{1}{2} + \frac{1}{4}i \) in (2.1). Then we have that

\[ f(z) = \frac{z}{(1 - z)^{\frac{6+3i}{10}}}. \]

This function \( f(z) \) maps the unit disk \( U \) onto the following domain.
Example 2.5 If we take $\alpha = \frac{2}{3} + \frac{1}{4}i$ in (2.1), then we have that

$$f(z) = \frac{z}{(1-z)^{\frac{184+69i}{438}}}.$$ 

This function $f(z)$ maps the unit disk $U$ onto the following domain.

References


