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<td>Author(s)</td>
<td>Hamai, Kensei; Hayami, Toshio; Kuroki, Kazuo; Owa, Shigeyoshi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2010), 1717: 1-7</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/170332">http://hdl.handle.net/2433/170332</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Coefficient estimates of functions in the class concerning with spirallike functions

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Abstract

For analytic functions $f(z)$ normalized by $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $U$, a new subclass $S_\alpha$ of $f(z)$ concerning with spirallike functions in $U$ is introduced. The object of the present paper is to discuss an extremal function for the class $S_\alpha$ and coefficient estimates of functions $f(z)$ belonging to the class $S_\alpha$.

1 Introduction

Let $A$ be the class of functions $f(z)$ of the form

\begin{equation}
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}

which are analytic in the open unit disk $U = \{z \in \mathbb{C}; |z| < 1\}$. If $f(z) \in A$ satisfies the following inequality

\begin{equation}
    \text{Re} \left( \frac{1}{\alpha} \frac{zf''(z)}{f'(z)} \right) > 1 \quad (z \in U)
\end{equation}

for some complex number $\alpha (|\alpha - \frac{1}{2}| < \frac{1}{2})$, then we say that $f(z) \in S_\alpha$. If $\alpha = |\alpha| e^{i\varphi}$, then the condition (1.2) is equivalent to

\begin{equation}
    \text{Re} \left( e^{-i\varphi} \frac{zf'(z)}{f(z)} \right) > |\alpha| \quad (z \in U).
\end{equation}

Therefore, we note that a function $f(z) \in S_\alpha$ is spirallike in $U$ which implies that $f(z)$ is univalent in $U$.

Further, if $0 < \alpha < 1$, then $f(z) \in S_\alpha$ is starlike of order $\alpha$ (cf. Robertson[3]). Let $P$ denote the class of functions $p(z)$ of the form

\begin{equation}
    p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k
\end{equation}

2000 Mathematics Subject Classification: Primary 30C45.

Key Word and Phrases: Analytic, univalent, spirallike, extremal function.
which are analytic in $U$ and satisfy
\[ \text{Re } p(z) > 0 \quad (z \in U). \]
Then we say that $p(z) \in \mathcal{P}$ is the Carathéodory function (cf. Carathéodory [1] or Duren [2]).

**Remark 1.1** Let us consider a function $f(z) \in \mathcal{A}$ which satisfies
\[ |f(z) - zf'(z)| < \frac{1}{2\alpha} \quad (z \in U) \quad (1.4) \]
for $|\alpha - \frac{1}{2}| < \frac{1}{2}$. If we write that $F(z) = \frac{zf'(z)}{f(z)}$, then the inequality (1.4) can be written by
\[ |2\alpha - F(z)| < 1 \quad (z \in U). \]
This implies that
\[ \alpha F(z) + \bar{\alpha} F(z) > 2|\alpha|^2 \quad (z \in U). \]
It follows that
\[ \frac{F(z)}{\alpha} + \frac{F(z)}{\alpha} > 2 \quad (z \in U). \]
Therefore, the inequality (1.4) is equivalent to
\[ \text{Re} \left( \frac{1}{\alpha} \frac{zf'(z)}{f(z)} \right) > 1 \quad (z \in U). \]

### 2 Coefficient estimates

In this section, we discuss the coefficient estimates of $a_n$ for $f(z) \in S_\alpha$. To establish our results, we need the following lemma due to Carathéodory [1].

**Lemma 2.1** If a function $p(z) = 1 + \sum_{k=1}^\infty c_k z^k \in \mathcal{P}$ satisfies the following inequality
\[ \text{Re } p(z) > 0 \quad (z \in U), \]
then
\[ |c_k| \leq 2 \quad (k = 1, 2, 3, \cdots) \]
with equality for
\[ p(z) = \frac{1 + z}{1 - z}. \]

Now, we introduce the following theorem.

**Theorem 2.2** Extremal function for the class $S_\alpha$ is $f(z)$ defined by
\[ f(z) = \frac{z}{(1 - z)^{2\alpha(\text{Re}(\frac{1}{z}) - 1)}}. \]
Proof. From the definition of the class $S_\alpha$, we have that
\[
\text{Re}\left(\frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1\right) > 0.
\]
Moreover, it is clear that
\[
\text{Re}\left(\frac{1}{\alpha}\right) > 1 \quad (|\alpha - \frac{1}{2}| < \frac{1}{2}).
\]
Then, if the function $F(z)$ is defined by
\[
F(z) = \frac{\frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1 - i\text{Im}\left(\frac{1}{\alpha}\right)}{\text{Re}\left(\frac{1}{\alpha}\right) - 1},
\]
we see that
\[
\text{Re} F(z) > 0 \quad \text{and} \quad F(0) = 1,
\]
so that, $F(z) \in \mathcal{P}$.
Therefore, Lemma 2.1 shows us that
\[
F(z) = \frac{\frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1 - i\text{Im}\left(\frac{1}{\alpha}\right)}{\text{Re}\left(\frac{1}{\alpha}\right) - 1} = \frac{1+z}{1-z}.
\]
It follows that,
\[
\frac{f'(z)}{f(z)} - \frac{1}{z} = 2\alpha \left(\text{Re}\left(\frac{1}{\alpha}\right) - 1\right) \frac{1}{1-z}.
\]
Integrating both sides from 0 to $z$ on $t$, we have that
\[
\int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)dt = 2\alpha \left(\text{Re}\left(\frac{1}{\alpha}\right) - 1\right) \int_0^z \frac{1}{1-t}dt,
\]
which implies that
\[
\log\frac{f(z)}{z} = \log \frac{1}{(1-z)^{2\alpha(\text{Re}\left(\frac{1}{\alpha}\right)-1)}}.
\]
Therefore, we obtain that
\[
f(z) = \frac{z}{(1-z)^{2\alpha(\text{Re}\left(\frac{1}{\alpha}\right)-1)}}.
\]
This is the extremal function of the class $S_\alpha$.

\[\square\]

Next, we discuss the coefficient estimates of $f(z)$ belonging to the class $S_\alpha$.

\textbf{Theorem 2.3} If a function $f(z) \in S_\alpha$, then
\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\text{arg}(\alpha)) - |\alpha|) + (k-1)) \quad (n = 2, 3, 4 \cdots).
\]
Equality holds true for $f(z)$ given by (2.1).
Proof. By using same method with Theorem 2.2, if we set $F(z)$ that

$$F(z) = \frac{\frac{1}{2} z f'(z)}{\alpha f(z)} - 1 - i \text{Im} \left( \frac{1}{\alpha} \right)$$

then it is clear that $F(z) \in \mathcal{P}$.

Letting $F(z) = 1 + c_1 z + c_2 z^2 + \cdots$, Lemma 2.1 gives us that $|c_m| \leq 2$ $(m = 1, 2, 3 \cdots)$.

Now, from (2.2),

$$\left( \text{Re} \left( \frac{1}{\alpha} \right) - 1 \right) F(z) = \frac{1}{\alpha} \frac{zf'(z)}{f(z)} - 1 - i \text{Im} \left( \frac{1}{\alpha} \right).$$

Let $\text{Re} \left( \frac{1}{\alpha} \right) - 1 = s$ and $1 + i \text{Im} \left( \frac{1}{\alpha} \right) = A$.

This implies that

$$\left( \alpha s F(z) + \alpha A \right) f(z) = zf'(z).$$

Then, the coefficients of $z^n$ in both sides lead to

$$na_n = (\alpha s + \alpha A)a_n + \alpha s (a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_1 c_n + \cdots + a_{n-2} c_n + c_{n-1}).$$

Therefore, we see that

$$a_n = \frac{\alpha s}{n - \alpha s - \alpha A} (a_{n-1} c_1 + a_{n-2} c_2 + \cdots + a_1 c_n + \cdots + a_{n-2} c_n + c_{n-1}).$$

This shows that

$$|a_n| = \frac{\cos(\arg(\alpha)) - |\alpha|}{n - 1} (|a_{n-1} c_1| + |a_{n-2} c_2| + \cdots + |a_1 c_n| + \cdots + |a_{n-2} c_n| + |c_{n-1}|)$$

$$\leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n - 1} \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k - 1)).$$

To prove that

$$|a_n| \leq \frac{1}{(n - 1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k - 1)).$$
we need to show that

\begin{equation}
(2.3) \quad |a_n| \leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n-1} \sum_{k=1}^{n-1} |a_k| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k - 1)).
\end{equation}

We use the mathematical induction for the proof. When \( n = 2 \), this assertion is true. We assume that the proposition is true for \( n = 2, 3, 4, \ldots, m - 1 \). For \( n = m \), we obtain that

\begin{align*}
|a_m| & \leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \sum_{k=1}^{m-1} |a_k| \\
& = \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \left( \sum_{k=1}^{m-2} |a_k| + |a_{m-1}| \right) \\
& = \frac{m-2}{m-1} \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-2} \sum_{k=1}^{m-2} |a_k| + \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} |a_{m-1}| \\
& \leq \frac{m-2}{(m-1)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) \\
& \quad + \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1} \frac{1}{(m-2)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) \\
& = \frac{1}{(m-1)!} \prod_{k=1}^{m-2} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) (m-2 + 2(\cos(\arg(\alpha)) - |\alpha|)) \\
& = \frac{1}{(m-1)!} \prod_{k=1}^{m-1} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1).
\end{align*}

Thus the inequality (2.3) is true for \( n = m \). By the mathematical induction, we prove that

\begin{equation}
|a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k - 1)) \quad (n = 2, 3, 4 \ldots).
\end{equation}

For the equality, we consider the extremal function \( f(z) \) given by Theorem 2.2. Since

\[ f(z) = \frac{z}{(1 - z)^{2\alpha(\Re(\frac{1}{\alpha}) - 1)}}, \]

if we let

\[ 2\alpha(\Re(\frac{1}{\alpha}) - 1) = j, \]

then \( f(z) \) becomes that

\[ f(z) = z(1 - z)^{-j} = z \left( \sum_{n=0}^{\infty} \binom{-j}{n} (-z)^n \right) = z + \sum_{n=2}^{\infty} \frac{j(j+1) \cdots (j+n-2)}{(n-1)!} z^n. \]
From the above, we obtained

\[ a_n = \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2\alpha (\text{Re}\left(\frac{1}{\alpha}\right) - 1) + k - 1). \]

For \( n = 2 \),

\[ |a_2| = 2|\alpha||\text{Re}\left(\frac{1}{\alpha}\right) - 1| = 2(\cos(\arg(\alpha)) - |\alpha|). \]

Furthermore, for \( n \geq 3 \), we have that

\[ |a_n| = \left| \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2\alpha (\text{Re}\left(\frac{1}{\alpha}\right) - 1) + k - 1) \right| \]

\[ = \frac{1}{(n-1)!} \prod_{k=1}^{n-1} |2\alpha (\text{Re}\left(\frac{1}{\alpha}\right) - 1) + k - 1| \]

\[ \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1). \]

Equality holds true for some real \( \alpha (0 < \alpha < 1) \).

This completes the proof of Theorem 2.3. \( \square \)

**Example 2.4** Let \( \alpha = \frac{1}{2} + \frac{1}{4}i \) in (2.1). Then we have that

\[ f(z) = \frac{z}{(1 - z)^{\frac{6+3i}{10}}}. \]

This function \( f(z) \) maps the unit disk \( U \) onto the following domain.
Example 2.5 If we take \( \alpha = \frac{2}{3} + \frac{1}{4}i \) in (2.1), then we have that

\[
f(z) = \frac{z}{(1 - z)^{\frac{184 + 69i}{438}}}.
\]

This function \( f(z) \) maps the unit disk \( U \) onto the following domain.

References


