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On geometry of quasi-minimal structures

前園 久智 (Hisatomo MAESONO)
早稲田大学メディアネットワークセンター
(Media Network Center, Waseda University)

Abstract

Itai, Tsuboi and Wakai investigated the geometric properties of qusai-minimal structures by using the countable closure [1]. I considered another closure operator in such structures.

1. Quasi-minimal structure and the countable closure

We recall some definitions.

Definition 1 An uncountable structure M is called *quasi-minimal* if every definable subset of M with parameters is at most countable or co-countable.

I introduce the examples in [1] and [2].

Example 2

1. $M = (\mathcal{Q}^{\omega}, +, \sigma, 0)$ *where* σ *is the shift function* ;
for $x = (x_0, x_1, x_2, \cdots)$, $\sigma(x) = (x_1, x_2, x_3, \cdots)$

2. $M_0 = (2^{\omega}, E_i\ (i < \omega))$ *such that* $E_i(x, y) \Longleftrightarrow x(i) = y(i)$ *for* $x, y \in 2^{\omega}$. *Let* $M' \prec M_0$ *be a countable elementary substructure and fix* $a \in M'$. *And let* $M_1 = (M' \cup B, E_i(i < \omega))$ *where* $|B| > \omega$ *and* $stp(b) = stp(a)$ *for all* $b \in B$. *Then* M_1 *is quasi-minimal.*

Definition 3 Let M be quasi-minimal. Then a type $p(x)$ defined by $p(x) = \{\psi(x) \in L(M) : |\psi^M| \geqq \omega_1\}$ is a complete type.
We call the type $p(x)$ the *main type* of M.

Definition 4 Let M be an uncountable structure and $A \subset M$.
The *n-th countable closure* $\mathrm{ccl}_n(A)$ of A is inductively defined as follows :
$\mathrm{ccl}_0(A) = A$ and
$\mathrm{ccl}_{n+1}(A) = \bigcup\{\phi^M : \phi(x) \in L(\mathrm{ccl}_n(A)),\ \phi^M \text{ is countable}\}$
We put $\mathrm{ccl}(A) = \bigcup_{n\in\omega}\mathrm{ccl}_n(A)$ (*the countable closure of* A).

Definition 5 Let X be an infinite set and cl a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X. If the function cl satisfies the following properties, we say (X, cl) is a *pregeometry*.
(I) $A \subset B \Longrightarrow A \subset \mathrm{cl}(A) \subset \mathrm{cl}(B)$,
(II) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,
(III) (Finite character) $b \in \mathrm{cl}(A) \Longrightarrow b \in \mathrm{cl}(A_0)$ for some finite $A_0 \subset A$,
(IV) (Exchange axiom)
$b \in \mathrm{cl}(A \cup \{c\}) - \mathrm{cl}(A) \Longrightarrow c \in \mathrm{cl}(A \cup \{b\})$.

It is shown that the countable closure is a closure operator in [1].

Fact 6 *Let M be a quasi-minimal structure. Then (M, ccl) satisfies the first three properties* (I) *through* (III) *of pregeometry.*

The exchange axiom (IV) does not hold in (M, ccl) generally. In [1], Itai, Tsuboi and Wakai showed some conditions for M such that (M, ccl) satisfies the exchange axiom.

Theorem 7 *Let M be a quasi-minimal structure. Then (M, ccl) satisfies the axioms of pregeometry under some conditions.*

And we recall the next theorem from [1].

Theorem 8 *Let M be a quasi-minimal structure. And $Th(M)$ is ω-stable. Then M can be elementarily embedded to an ω-saturated quasi-minimal stucture M'.*

The notion of quasi-minimal structures is a generalization of minimal structures. Thus the countable closure is the canonical closure operator for quasi-minimal structures. However, I tried to divide the countable closure by some P-closure.

2. P-closure in quasi-minimal structures

First we recall some definitions from [6].

Definition 9 A family P of partial types is *A-invariant* if it is invariant under A-automorphisms (where A is a subset of a sufficiently large saturated model as usual).
Let P be an A-invarint family of partial types.
A partial type q over A is *P-internal* if for every realization a of q, there is $B \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\lower0.2ex\hbox{$|$}}_A a$, types $\bar{p}$ from P based on B, and realizations $\bar{c}$ of $\bar{p}$, such that $a \in \mathrm{dcl}(B\bar{c})$.

A patial type q is *P-analysable* if for any $a \models q$, there are $(a_i : i < \alpha) \in \mathrm{dcl}(A, a)$ such that $\mathrm{tp}(a_i/A, \{a_j : j < i\})$ is P-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A, \{a_i : i < \alpha\})$.

A complete type $q \in S(A)$ is *foreign* to P if for all $a \models q$, $B \downarrow_A a$, and realizations $\bar{c}$ of extensions of types in P over B, we always have $a \downarrow_{AB} \bar{c}$.

Definition 10 Let P be an $\emptyset$-invariant family of types.

A partial type q is *co-foreign* to P if every type in P is foreign to q.

The *P-closure* $\mathrm{cl}_P(A)$ of a set A is the collection of all element a such that $\mathrm{tp}(a/A)$ is P-analysable and co-foreign to P. (The P-analysable assumption could be modified or even omitted, resulting in a larger P-closure.)

Fact 11 *P-closure satisfies the axioms* (I) *and* (II) *of pregeometry.*

The axiom (III) and the exchange axiom (IV) do not hold in general.

We define P-closures in stable quasi-minimal structures. We argue under the assumptions in the following.

Assumptions

M is an ω-saturated quasi-minimal structure such that $Th(M)$ is ω-stable.

We may assume that the main type $p(x) \in S(M)$ strongly based on $\emptyset$.

The set P of types is defined by

$P = \{q \in S(A) : q$ is a conjugate of $p\lceil A$ for some finite $A \subset M\}$.

We can prove the next fact.

Fact 12 *Under the assumptions as above, the P-closure* cl_P *is a closure operator in* M.

(M, cl_P) *satisfies the axioms* (I) *through* (III) *of pregeometry.*

And $\mathrm{acl}(A) \subset \mathrm{cl}_P(A) \subset \mathrm{ccl}(A)$ *for* $A \subset M$.

If we omit the P-analysability assumption from cl_P*, then* $\mathrm{cl}_P(A) = \mathrm{ccl}(A)$.

Remark 13 *In Example 2.1,* $\mathrm{cl}_P(A) = \mathrm{ccl}(A)$ *for* $A \subset M$ *under the P-analysability assumption. By the argument in [3], we can show the same fact for ($\omega-$stable) quasi-minimal groups in general.*

3. p-closure for regular types p

We recall some definitions from [4].

Definition 14 Let $p(x)$, $q(x)$ be complete types over A. We say that *p is almost orthogonal to q* if whenever a realizes p, and b realizes q, then tp(a/Ab) does not fork over A.

Let $p(x) \in S(A)$, $q \in S(B)$ are stationary types.

We say that p *is orthogonal to* q if whenever $C \supset A \cup B$, then $p \mid C$ is almost orthgonal to $q \mid C$.

And we say that p *is hereditarily orthogonal to* q if every extension of p is orthogonal to q.

Definition 15 Let $p(x) \in S(A)$ be a non-algebraic stationary type.

We say that p is *regular* if for any forking extension q of p, p is orthogonal to q.

In the following, let p be a regular type over some domain.

Definition 16 Let $q(x) \in S(X)$ be a strong type, where p is non-orthogonal to X.

We say that q *is p-simple* if there is a set $B \supset A \cup X$, some realization a of $q \mid B$ and a set Y of realizations of p such that $stp(a/BY)$ is hereditarily orthogonal to p.

And we say that q *is p-semi-regular* if q is p-simple and domination equivalent to some non-zero power $p^{(n)}$ of p.

Definition 17 Let $q = stp(a/X)$ be p-simple. Then the *p-weight of* q, $w_p(q)$ is defined to be

$\min\{ \kappa :$ there is $B \supset A \cup X$, there is a' realizing $q \mid B$, and there is J, an independent set of realizations of $p \mid B$, such that $stp(a'/BJ)$ is hereditarily orthogonal to p and $|J| = \kappa \}$

We define the *p-closure of* X, denoted $cl_p(X)$, the set $\{ b : stp(b/X)$ *is p-simple and* $w_p(b/X) = 0 \}$

We try to argue p-closure in quasi-minimal structures.

We can check the next fact easily.

Fact 18 *Let* M *be a quasi-minimal structure. And* $Th(M)$ *is* ω*-stable. Then we may assume that the main type* $p \in S(M)$ *is a regular type.*

It is well known that for regular types p of stable theory, $(p^{\mathcal{C}}, cl_p)$ is pregeometry (where $\mathcal{C}$ is the big model).

For quasi-minimal structure M of stable theory and the main type p of M, we consider cl_p.

We can prove the next fact like Fact. 12.

Fact 19 *Let* M *be a quasi-minimal structure of* ω*-stable theory. Then* cl_p *is a closure operator, i.e.* (M, cl_p) *satisfies the axioms* (I) *through* (III) *of pregeometry.*

And $acl(A) \subset cl_p(A) = ccl(A)$ *for* $A \subset M$.

4. Further problem

We recall some definitions and theorems from [4] again.

Definition 20 Let (S, cl) be pregeometry.
(S, cl) is *modular* if for any closed sets X, $Y \subset S$, X is independent from Y over $X \cap Y$.
Equivalently, for any finite-dimensional closed sets X, Y,
$dim(X) + dim(Y) - dim(X \cap Y) = dim(X \cup Y)$.
(S, cl) is *locally modular* if for some $a \in S$, $(S, cl_{\{a\}})$ (the localization of S at $\{a\}$) is modular.

The next theorems are well-known.

Theorem 21 *Let $p \in S(\emptyset)$ be a stationary, minimal locally modular type. Then p is trivial, or p is non-trivial modular (in which case the geometry on p is projective over a division ring), or p is non-modular in which case the geometry associated to p (over $\emptyset$) is affine geometry over a division ring.*

Theorem 22 *Let $p \in S(\emptyset)$ be a stationary, regular, locally modular type over $\emptyset$. Then the geometry of (p^C, cl_p) is either trivial, or affine or projective geometry over some division ring.*

There are examples of quasi-minimal structures whose main type is locally modular. (See Example 2.1)

Question
Let p be the main type of a (ω-)stable quasi-minimal structure.
And let p be a locally modular regular type.
Does its geometry (p^C, cl_p) have characteristics?

Apology and acknowlegement

I did not know the paper [3] by A.Pillay and P.Tanović until Kirishima meeting. Some participants told me about their work. The content of my talk is not shown in their paper on the surface.

References

[1] M.Itai, A.Tsuboi and K.Wakai, *Construction of saturated quasi − minimal structure*, J. Symbolic Logic, vol. 69 (2004) pp. 9-22
[2] M.Itai and K.Wakai, *ω-saturated quasi-minimal models of* $Th(Q^\omega, +, \sigma, 0)$, Math. Log. Quart, vol. 51 (2005) pp. 258-262

[3] A.Pillay and P.Tanović, *Generic stability, regularity, quasi–minimality*, preprint
[4] A. Pillay, *Geometric stability theory*, Oxford Science Publications, 1996
[5] F.O.Wagner, *Stable groups*, Cambridge University Press, 1997
[6] F.O.Wagner, *Simple theories*, Kluwer Academic Publishers, 2000