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ON 3-AMPLENESS IN ROSY THEORIES

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ABSTRACT. In rosy theories we introduce a geometric notion of independence, strong non-3-ampleness. And we show that strong non-3-ampleness implies non-3-ampleness, and non-2-ampleness (=CM-triviality) implies strong non-3-ampleness.

1. INTRODUCTION

There is a simple characterization of CM-triviality. By using the characterization, we could show that any rosy CM-trivial theory has weak canonical bases, and CM-triviality in the real sort implies geometric elimination of imaginaries [Y]. We want to know whether these results can be extended in case of non-3-ampleness or not, so our first motivation is to find a simple characterization of non-3-ampleness like in case of CM-triviality. This paper is organized as follows. In the second section, we review the definition of CM-triviality (=non-2-ampleness) and non-3-ampleness for each \( n < \omega \) in rosy theories. In the third section, trying to find a simple characterization, we offer another geometric notion (we call it strong non-3-ampleness). We show that strong non-3-ampleness implies non-3-ampleness, and non-2-ampleness (=CM-triviality) implies strong non-3-ampleness. But, for now there are no examples of non-3-ampleness. We also raise up some problems on non-3-ampleness.

Our notation is standard. Let \( T \) be a rosy theory. (i.e. having a good independence relation \( \perp \)) We work in \( \mathcal{M}^{eq} \), the eq-structure, consisting of imaginary elements, where \( \mathcal{M} \) is a sufficiently saturated model of \( T \). \( \bar{a}, \bar{b}, \ldots \subset \omega \mathcal{M} \) denote finite sequences in \( \mathcal{M}^{eq} \). \( A, B, \ldots \) denote small subsets of \( \mathcal{M}^{eq} \) and \( \overline{\mathcal{M}} \) denotes \( \mathcal{M} \cup B \). For \( a \in \mathcal{M}^{eq} \) and \( A \subset \mathcal{M}^{eq} \), we write \( a \in acl^{eq}(A) \) if the orbit of \( a \) by automorphisms fixing \( A \) pointwise is finite. In rosy theories \( [A] \), we have that \( a \perp_{b} c \) implies \( acl^{eq}(ab) \cap acl^{eq}(bc) = acl^{eq}(b) \).

2. REVIEW OF ROSY CM-TRIVIALITY AND NON-\( n \)-AMPLENESS

CM-triviality is a geometric notion of the nonforking independence relation. In 1988, it is introduced by Hrushovski where he disproves Zilber's conjecture on strongly minimal sets [H]. CM-triviality forbids a point-line-plane incident system. Hrushovski also offered three characterizations of CM-triviality in stable theories. The following is the simplest characterization for rosy CM-triviality.

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Definition 2.1. A rosy theory $T$ is CM-trivial, if $\bar{a} \downarrow_{A \cap \text{acl}^{eq}(\bar{a}, B)} B$ holds for any $\bar{a}, A, B \subseteq \mathcal{M}^{eq}$ such that $\bar{a} \downarrow_{A} B$ and $A, B$ are algebraically closed.

The Weak Canonical Base $\text{wcb}(\bar{a}/B)$ of $\text{tp}(\bar{a}/B)$ has the following properties, where $B$ is an algebraically closed subset of $\mathcal{M}^{eq}$:

- $\bar{a} \downarrow_{\text{wcb}(\bar{a}/B)} B$
- $\text{wcb}(\bar{a}/B)$ is algebraically closed.
- $\bar{a} \downarrow_{A} B \Rightarrow \text{wcb}(\bar{a}/B) \subseteq \text{acl}^{eq}(A) \subseteq \mathcal{M}^{eq}$

The weak canonical base is the smallest algebraically closed subset $C$ of $B$ such that $\bar{a} \downarrow_{C} B$. As in [P2], rosy theories do not necessarily have weak canonical bases. But any rosy CM-trivial theory has weak canonical bases, so we have the following characterization [Y].

Fact 2.2. Let $T$ be rosy. The following are equivalent.

1. $T$ is CM-trivial.
2. $T$ has weak canonical bases and $\text{wcb}(\bar{a}/A) \subseteq \text{wcb}(\bar{a}/B)$ holds for any $\bar{a}, A, B \subseteq \mathcal{M}^{eq}$ such that $\text{acl}^{eq}(\bar{a}, A) \cap B = A$ with $A = \text{acl}^{eq}(A)$ and $B = \text{acl}^{eq}(B)$.

We use the following notations to briefly write the definition of $n$-ampleness.

$$a \wedge b := \text{acl}^{eq}(a) \cap \text{acl}^{eq}(b)$$

$$a_{<i} := a_{0}, a_{1}, \ldots, a_{i-1}$$

$$a_{<0} := \emptyset$$

Definition 2.3. $T$ is not $n$-ample, if $a_{n} \downarrow_{c} a_{0}$ holds for any $c, a_{0}, a_{1}, \ldots, a_{n}$ such that $ca_{<i}a_{i} \wedge ca_{<i}a_{i+1} = \text{acl}^{eq}(ca_{<i})$, $a_{i} \downarrow_{a_{i-1}c} a_{<i}$ for $i = 1, 2, \ldots, n$, where $c$ looks like constants.

The following is proved in [P1].

Remark 2.4. (1) one-basedness ($a \downarrow_{a \wedge b} b$ for any $a, b$) is equivalent to non-1-ampleness: $ca_{0} \wedge ca_{1} = \text{acl}^{eq}(c)$ ($a_{1} \downarrow_{a_{0}} a_{0}$) implies $a_{1} \downarrow_{c} a_{0}$.

(2) CM-triviality ($a \downarrow_{b} c \Rightarrow a \downarrow_{b \wedge ac} c$) is equivalent to non-2-ampleness: $ca_{0} \wedge ca_{1} = \text{acl}^{eq}(c)$, $ca_{0}a_{1} \wedge ca_{0}a_{2} = \text{acl}^{eq}(ca_{0})$

$a_{2} \downarrow_{a_{1}} a_{0}$, $a_{1} \downarrow_{a_{0}} a_{0}$ imply $a_{2} \downarrow_{c} a_{0}$.

(3) Non-$n$-ampleness implies non-$(n + 1)$-ampleness for each $n < \omega$.

3. STRONG NON-3-AMPLENESS

Definition 3.1. A rosy theory $T$ is not 3-ample, if $a_{3} \downarrow_{c} a_{0}$ holds for any $c, a_{0}, a_{1}, a_{2}, a_{3} \subseteq \mathcal{M}^{eq}$ such that $a_{0}c \wedge a_{1}c = \text{acl}^{eq}(c)$, $a_{0}a_{1}c \wedge a_{0}a_{2}c = \text{acl}^{eq}(a_{0}c)$ $a_{0}a_{1}a_{2}c \wedge a_{0}a_{1}a_{3}c = \text{acl}^{eq}(a_{0}a_{1}c)$, $a_{3} \downarrow_{a_{2}c} a_{0}a_{1}$, $a_{2} \downarrow_{a_{1}c} a_{0}$.

The following remark is a non-3-ample’s version of Fact 2.2 under assuming the existence of weak canonical bases.

Remark 3.2. If $T$ has weak canonical bases, then the following are equivalent.

1. $T$ is not 3-ample.
(2) \( \text{wcb}(a_3/\text{wcb}(a_2/\text{wcb}(a_1)) \subseteq \text{acl}^\text{eq}(\text{wcb}(a_3/\text{wcb}(a_2/\text{wcb}(a_1)))) \) holds for any \( a_0, a_1, a_2, a_3, c \subseteq \mathcal{M}^\text{eq} \) such that \( a_0c \land a_1c = \text{acl}^\text{eq}(c) \), \( a_0a_1c \land a_0a_2c = \text{acl}^\text{eq}(a_0c) \), \( a_0a_1a_2c \land a_0a_3c = \text{acl}^\text{eq}(a_0a_1c) \), \( a_3 \leftarrow \downarrow a_2 \downarrow a_1 \downarrow a_0 \).

Proof. (1) \( \Rightarrow \) (2): Clear. 

(1) \( \Leftarrow \) (2): We may assume \( c = \emptyset \). As \( a_3 \downarrow \downarrow a_0a_1 \) and \( \text{wcb}(a_3/\text{wcb}(a_2/\text{wcb}(a_1)) \subseteq \text{wcb}(a_3/\text{wcb}(a_2/\text{wcb}(a_1))) \), we have \( \text{wcb}(a_3/\text{wcb}(a_2/\text{wcb}(a_1)) \subseteq a_0 \land a_2 \). On the other hand, as \( a_2 \downarrow a_1 \), we have \( a_0 \land a_2 \subseteq a_0a_1 \land a_1a_2 \subseteq \text{acl}^\text{eq}(a_1) \). As \( a_0 \land a_1 = \text{acl}^\text{eq}(\emptyset) \), we have \( \text{wcb}(a_3/\text{wcb}(a_2/\text{wcb}(a_1)) \subseteq a_0 \land a_2 \subseteq a_0 \land a_1 = \text{acl}^\text{eq}(\emptyset) \). \( \square \)

Now we consider the following notion.

**Definition 3.3.** We say that \( T \) is strongly non-3-ample, if \( a_3 \downarrow \downarrow a_0a_1 \), \( a_0a_1 \land a_0a_2 = \text{acl}^\text{eq}(a_0) \), \( a_0a_1a_2 \land a_0a_1a_3 = \text{acl}^\text{eq}(a_0a_1) \), \( a_3 \downarrow \downarrow a_1a_2 \downarrow a_1a_3 \). and let \( b := a_0a_2 \land a_1a_2 \land a_0a_1a_3 \).

We need to show \( b = \text{acl}^\text{eq}(\emptyset) \).

**Claim 1** \( b \subseteq \text{acl}^\text{eq}(a_1) \): As \( a_2 \downarrow a_1 \), we have \( a_0a_1 \land a_1a_2 = \text{acl}^\text{eq}(a_1) \). Then we have

\[
\begin{align*}
b &= a_0a_2 \land a_1a_2 \land a_0a_1a_3 \\
&\subseteq a_0a_1a_2 \land a_1a_2 \land a_0a_1a_3 \\
&= (a_0a_1a_2 \land a_0a_1a_3) \land a_1a_2 \\
&= a_0a_1 \land a_1a_2 = \text{acl}^\text{eq}(a_1)
\end{align*}
\]

**Claim 2** \( b \subseteq \text{acl}^\text{eq}(a_0) \): As \( b \subseteq \text{acl}^\text{eq}(a_1) \),

\[
b \subseteq a_1 \land a_0a_2 \land a_1a_2 \land a_0a_1a_3 \\
&\subseteq a_1 \land a_0a_2 \land a_0a_1a_3 = \text{acl}^\text{eq}(a_0)
\]

By two claims, we have \( b \subseteq a_0 \land a_1 = \text{acl}^\text{eq}(\emptyset) \). \( \square \)

**Remark 3.5.** Assume that \( \text{acl}^\text{eq}(A(B \land C)) = AB \land AC \) for any \( A, B, C \subseteq \mathcal{M}^\text{eq} \).

We usually have that \( \text{acl}^\text{eq}(A(B \land C)) \subseteq AB \land AC \). Then non-3-ampleness coincides with strong non-3-ampleness.

**Proof.** Let \( b = a_0a_2 \land a_1a_2 \land a_0a_1a_3 \). As \( a_3 \downarrow \downarrow a_0a_1 \), \( a_2 \downarrow a_1 \), we have

\[
a_3 \downarrow \downarrow a_0a_1a_2 \downarrow a_0.
\]

So we need to show

1. \( a_0b \land a_1b = \text{acl}^\text{eq}(b) \)
2. \( a_0a_1b \land a_0a_2b = \text{acl}^\text{eq}(a_0b) \)
3. \( a_0a_1a_2b \land a_0a_1a_3b = \text{acl}^\text{eq}(a_0a_1b) \)


The proof of $a_0b \land a_1b = \acl^\eq(b)$:
\[
a_0b \land a_1b \subseteq (a_0a_2 \land a_0a_1a_2 \land a_0a_1a_3) \land (a_0a_1a_2 \land a_1a_2 \land a_0a_1a_3) \\
\subseteq a_0a_2 \land a_1a_2 \land a_0a_1a_3 = \acl^\eq(b)
\]

The proof of $a_0a_1b \land a_0a_2b = \acl^\eq(a_0b)$: We use our assumption in the last equation.
\[
a_0a_1b \land a_0a_2b \subseteq (a_0a_1a_2 \land a_0a_1a_3) \land (a_0a_2 \land a_0a_1a_2 \land a_0a_1a_2a_3) \\
\subseteq a_0a_2 \land a_0a_1a_3 = \acl^\eq(a_0b)
\]

The proof of $a_0a_1a_2b \land a_0a_1a_3b = \acl^\eq(a_0a_1b)$: We also use our assumption in the last equation.
\[
a_0a_1a_2b \land a_0a_1a_3b \subseteq (a_0a_1a_2 \land a_0a_1a_3) \land (a_0a_1a_2a_3 \land a_0a_1a_3) \\
\subseteq a_0a_1a_2 \land a_0a_1a_3 = \acl^\eq(a_0a_1b)
\]

\[\square\]

Remark 3.6. Non-2-ampleness (=CM-triviality) implies strong non-3-ampleness.

Proof. We will show that $a_3 \triangleleft a_2 \land a_0a_1$ implies $a_3 \triangleleft a_0a_2 \triangleleft a_1a_2 \land a_0a_1a_3 = a_0a_1$. Put $A = \acl^\eq(a_0a_2)$ and $B = \acl^\eq(a_1a_2)$. Clearly we have $a_3 \triangleleft a_2 \triangleleft AB$. As $\acl^\eq(a_2) \subseteq A \cap B \subseteq AB$, we have $a_3 \triangleleft A \cap B \triangleleft AB$. In particular, we have $a_3 \triangleleft a_0a_1a_2 \land a_0a_1a_3$. By CM-triviality, we get $a_3 \triangleleft a_0a_2 \triangleleft a_1a_2 \land a_0a_1a_3 \triangleleft a_0a_1$, as desired.

\[\square\]

We have the following implications.
non-1-ampleness(\(\Leftrightarrow\) one-basedness) \(\Rightarrow\) non-2-ampleness(\(\Leftrightarrow\) CM-triviality) \(\Rightarrow\) strong non-3-ampleness \(\Rightarrow\) non-3-ampleness \(\Rightarrow\) non-4-ampleness \(\Rightarrow\) ...

In [E] an $n$-ample (relational) structure $M_n$ is constructed for each $n < \omega$, but it is unknown whether $M_n$ is not $(n + 1)$-ample. For now, $n$-ample and non-$(n + 1)$-ample structure is not discovered for each $n \geq 2$. (Generic relational structures are 1-ample and non-2-ample.)

Problem 3.7. It is shown that free group is 2-ample [P3]. Is free group non-3-ample? (We need the characterization of non-forking in the free group to check non-3-ampleness.)

Problem 3.8. Does non-3-ample theory have weak canonical bases? (I think No.) We need to check Adler's criterion [A]: $a \triangleleft B \triangleleft C$, $a \triangleleft C \triangleleft B$ \(\Rightarrow\) $a \triangleleft B \cap C \triangleleft BC$ for any $a, b, c$ such that $B$ and $C$ are algebraically closed subsets of $M^\eq$.

Problem 3.9. Is strong non-3-ampleness with weak canonical bases equivalent to CM-triviality?

Let $T = \Th(\mathbb{R}, +, \cdot, \pi|_{(-1,1)}(\ast))$, where $\pi|_{(-1,1)}(x) := \pi \cdot x$ for $x \in (-1,1)$. $T$ is an $\omega$-minimal theory with elimination of imaginaries. As $T$ does not have weak canonical bases, $T$ is 2-ample. And $T$ does not interpret fields by [LP][PS], so it is possible that $T$ is not $n$-ample for some $n < \omega$.

Problem 3.10. Is $T$ non-3-ample? (We need the characterization of non-forking in $T$.)

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