Uniformly definable subrings of some infinite algebraic extensions of the rationals

鹿児島国際大学国際文化学部 福崎賢治 (Kenji Fukuzaki)
Faculty of Intercultural Studies,
The international University of Kagoshima

Abstract
We consider the formulas used by Julia Robinson in her proof that number fields are first order undecidable. We extend the result of [1]. We prove that it defines subrings in some infinite algebraic extensions of the rationals. As an application we discuss undecidabilities of those infinite algebraic extensions.

1 Introduction

In 1959 Julia Robinson [8] proved that any number field, as well as the corresponding ring of algebraic integers, is undecidable, by showing that \( \mathbb{N} \) is \( \emptyset \)-definable (in the ring language) in the ring, and the ring is \( \emptyset \)-definable in its number field.

She first considered the formula

\[
\varphi_m(s, u, t) : \exists x, y, z(1 - sut^{2m} = x^2 - sy^2 - uz^2),
\]

where \( m \) is a positive integer such that \( p^m \not\mid 2 \) for all prime ideals \( p \) of a given number field \( F \), that is, \( m \) is an integer greater than all the ramification indices of prime ideals of \( F \) which divide 2. Then she proved that for a given prime \( p_1 \) of \( F \) there are \( a, b \in F \) such that \( \varphi_m(a, b, t) \) defines a finite intersection of valuation rings \( \bigcap_{p \in \Delta} \mathcal{O}_p \) where \( \Delta \) is a finite set of primes of \( F \) containing \( p_1 \). (We actually can define the valuation ring of \( p_0 \) using two \( \varphi_m(s, t, u) \) with some choice of those parameters.) We denote by \( \varphi_m(a, b, F) \) the solution set of \( \varphi_m(a, b, t) \) in \( F \), that is, \( \varphi_m(a, b, F) = \{ \alpha \in F : F \models \varphi_m(a, b, \alpha) \} \). It is easy to see that \( \bigcap_{a, b \in F} \varphi_m(a, b, F) = 0 \).

Therefore in order to define the ring of algebraic integers \( \mathfrak{o}_F \) in a given number field \( F \), J. Robinson considered the intersection of all \( \varphi_m(a, b, F) \) containing \( \mathbb{Z} \), which is defined by \( \psi_m(t) : \)

\[
\forall s, u(\forall c(\varphi_m(s, u, c) \rightarrow \varphi_m(s, u, c + 1)) \rightarrow \varphi_m(s, u, t)).
\]

Note that \( \varphi_m(s, u, t) \leftrightarrow \varphi_m(s, u, -t) \). We denote by \( \psi_m(F) \) the solution set of \( \psi_m(t) \) in \( F \) as before. It is possible to define \( \mathfrak{o}_F \) since \( \mathbb{Z} \subseteq \psi_m(F) \subseteq \mathfrak{o}_F \) and \( F \) has an integral basis over the rationals \( \mathbb{Q} \). (The defining formula of \( \mathfrak{o}_F \) depends on \( F \).

In this paper we calculate the solution set of \( \psi_2(t) \) in some infinite algebraic extensions of \( \mathbb{Q} \).
2 Construction of $\psi(t)$

Let $F$ be a number field (a finite algebraic extension of the rationals $\mathbb{Q}$) and let $\mathcal{O}_F$ be the ring of algebraic integers of $F$. $F^*$ will denote the set of non-zero elements of $F$. By $p$ we denote a place of $F$ and by $F_p$ the completion of $F$ with respect to $p$. Since non-archimedean places of $F$ are $p$-adic valuations for some prime ideal $p$ of $F$, we use the same letter $p$ for both the place and the prime ideal. The ring of integers of $F_p$ is denoted by $(\mathcal{O}_F)_p$, its maximal ideal is also denoted by $p$. Let $p$ be a prime ideal of $F$ and $a \in F$. By $\nu_p(a)$ we denote the order of $a$ at $p$. Given $a, b \in F^*$, we use Hilbert symbol $(a, b)_p$, which is defined to be $+1$ if $ax^2 + by^2 = 1$ is solvable in $F_p$, otherwise defined to be $-1$. For $a, b \in F^*$ we denote by $S_F(a, b)$ the set of places $p$ of $F$ such that $(a, b)_p = -1$. We know that it contains even number of places of $F$.

The following lemma is well-known.

**Lemma 1** A nonzero element $h$ of $F$ can be represented by the ternary quadratic form $x^2 - ay^2 - bz^2$ in $F$ if and only if $h/(-ab) \notin F_p^{*2}$ for any place $p$ such that $(a, b)_p = -1$.

This follows from the properties of quaternary quadratic forms and the Hasse-Minkowski theorem on quadratic forms. See [7, p. 187].

**Lemma 2** Given even number of distinct prime ideals $p_1, \ldots, p_{2k}$ of $F$ there are $a$ and $b$ in $F^*$ such that $S_F(a, b) = \{p_1, \ldots, p_{2k}\}$ and $\nu_{p_i}(a) = 1$, $\nu_{p_i}(b) = 0$ for $i = 1, \ldots, 2k$.

**Proof.** By weak approximation, we get an element $a$ of $F^*$ with $\nu_{p_i}(a) = 1$ for all $i$. We know by [7, 71:19. Theorem p. 203] that there is $b \in F^*$ such that $S_F(a, b) = \{p_1, \ldots, p_{2k}\}$. In the proof of [7, 71:19. Theorem p. 203], we can take $b$ with $\nu_{p_i}(b) = 0$ for $i = 1, \ldots, 2k$. \hfill $\square$

J. Robinson actually proved in [8, Lemma 9] that given a prime ideal $p_1$ of $F$ there are relatively prime elements $a$ and $b$ in $\mathcal{O}_F$ such that $(a) = p_1 \cdots p_{2k}$, where $p_1, \ldots, p_{2k}$ are distinct prime ideals that include every prime ideal dividing 2, and $b$ is a totally positive prime element such that $(a, b)_p = -1$ iff $p|a$.

**Lemma 3** Let $a, b, c \in F$. If $a$ and $b$ satisfy Lemma 2 and $m$ be a positive integer such that $p^m \not| 2$ for every prime ideal $p$. Then

$$1 - abc^2 = x^2 - ay^2 - bz^2$$

is solvable for $x, y$ and $z$ in $F$ iff $\nu_{p_i}(c) \geq 0$ for each $i$.

**Proof.** By Lemma 1, $h = 1 - abc^2m$ can be represented by $x^2 - ay^2 - bz^2$ iff $h/(-ab) \notin F_{p_i}^{*2}$ for $1 \leq i \leq 2k$.

If $\nu_{p_i}(c) \geq 0$ for each $i$, then we have $\nu_{p_i}(h/(-ab)) = -1$, hence $h/(-ab)$ is not a square of $F_{p_i}^*$ for each $i$. 
Suppose \( \nu_{p_i}(c) < 0 \) for some \( i \). We know in \( F_p \) that \( (1 + p^r)^2 = 1 + 2p^r \) if \( p^r \subseteq 2p \) by [7, p. 163]. Noting \( h/(ab) = c^{2m}(1 - 1/(abc^{2m})) \), we see that \( h/(ab) \) is a square of \( F_p \) since \( \nu_{p_i}(1/(ab)^{2m}) \geq 2m - 1 \) and \( p^{2m-1} \subseteq 2p \).

Thus we have that if \( a \) and \( b \) satisfy Lemma 2, \( \varphi_m(a, b, F) = \bigcap_{1 \leq i \leq 2k} \mathcal{O}_{p_i} \), and \( \forall c(\varphi_m(a, b, c) \to \varphi_m(a, b, c + 1) \) holds in \( F \) since \( \varphi_m(a, b, F) \) is a ring containing \( \mathbb{Z} \).

For a given \( c \in F^* \) there are \( a, b \in F^* \) such that \( c \not\in \varphi_m(a, b, F) \) since we can construct \( a, b \in F^* \) such that \( 1 - 1/(abc^{2m}) \) is a square of \( F_p \) for some \( p \) with \( (a, b)_p = -1 \). Noting \( 0 \in \varphi_m(a, b, F) \) for all \( a, b \) we have \( \bigcap_{a, b \in F} \varphi_m(a, b, F) = 0 \).

Nevertheless we have that \( \psi_m(F) \) is a subset of \( \mathfrak{o}_F \) containing \( \mathbb{Z} \) since \( \psi_m(F) \) is the intersection of all the solution set of

\[
\forall c(\varphi_m(a, b, c) \to \varphi_m(a, b, c + 1) \to \varphi_m(a, b, t)).
\]

If the premise of the above formula fails, the solution set is \( F \).

We don’t know what \( \psi_m(F) \) is. But we can show what \( \psi_2(K) \) is, if \( K \) is a certain infinite algebraic extension of \( \mathbb{Q} \).

**Remark 4** For a given prime ideal \( p_1 \) we can define the valuation ring of \( p_1 \). Take three prime ideal \( p_1, p_2, p_3 \) of \( F \) and \( a, b, c, d \in \mathfrak{o}_F \) such that \( S_F(a, b) = \{p_1, p_2\} \) and \( S_F(c, d) = \{p_1, p_3\} \), then we easily see that \( \varphi_m(a, b, F) + \varphi_m(c, d, F) \) defines \( \mathcal{O}_{p_1} \).

### 3 The solution set of \( \psi(t) \) in some infinite algebraic extensions

Let \( F \) be a number field and let \( \mathcal{F} \) be an infinite set of finite Galois extensions \( M \) of \( F \) such that \( [M : F] \) is odd and every prime ideal of \( M \) dividing 2 is unramified in \( M/\mathbb{Q} \). (We say that 2 is unramified in \( M/\mathbb{Q} \). Note \( p^2 \not| 2 \) for all prime ideals \( p \) of \( M \).) Let \( K \) be the composite field of all fields in \( \mathcal{F} \). Then \( K \) is an infinite Galois extension of \( F \) and every finite Galois subextension \( M \) has odd extension degree over \( \mathbb{Q} \). We denote by \( \mathcal{O}_K \) the ring of algebraic integers of \( K \).

In this section we will prove that the solution set \( \psi_2(K) \) of \( \psi_2(t) \) in \( K \) is a subset of \( \mathcal{O}_K \) containing \( \mathbb{Z} \).

We need the following lemma, which is proved in [2, pp. 272,337].

**Lemma 5** Let \( M, L \) be number fields with \( L \supset M \) and let \( \mathfrak{p} \supset p \) be primes of \( L \) and \( M \) respectively. For \( \alpha \in L_{\mathfrak{p}}^* \), let \( a = N_{L_{\mathfrak{p}}/M_{\mathfrak{p}}}(\alpha) \) and \( b \in M_{\mathfrak{p}} \). Then we have \( (\alpha, b)_{\mathfrak{p}} = (a, b)_p \).

The next lemma follows from Lemma 5.
Lemma 6 Let $L$ be a finite Galois extension of a number field $M$ with $[L : M]$ odd. Let $\mathfrak{p}$ be a prime ideal of $M$ and let $\mathfrak{P}$ be a prime of $L$ lying over $\mathfrak{p}$. Then for $a, b \in M^*$, we have $(a, b)_\mathfrak{p} = 1$ iff $(a, b)_\mathfrak{P} = 1$.

Proof. Since $L/M$ is a Galois extension, the local degree at $\mathfrak{P}$ divides the degree of $L/M$, that is, $[(L)_\mathfrak{P} : (M)_\mathfrak{p}]|[L : M]$ (see [7, p. 32]). Let $u$ be the local degree at $\mathfrak{P}$. Then $N_{(L)_\mathfrak{P}/(M)_\mathfrak{p}}(a) = a^u$ and $(a, b)_\mathfrak{P} = (a^u, b)_\mathfrak{p} = (a, b)_\mathfrak{p}^u$. Since $u$ is odd, it follows that $(a, b)_\mathfrak{p} = 1$ iff $(a, b)_\mathfrak{P} = 1$.

We recall that $\varphi_2(s, u, t)$ is

$$\exists x, y, z(1 - su^4 = x^2 - sy^2 - uz^2)$$

and $\psi_2(t)$ is

$$\forall s, u(\forall c(\varphi(s, u, c) \rightarrow \varphi(s, u, c + 1)) \rightarrow \varphi_2(s, u, t)).$$

Lemma 7 Let $M$ be a subfield of $K$ with $M/F$ finite and Galois. Let $a, b, \alpha \in M$ with $ab \neq 0$. Then

$$M |\models \varphi(a, b, \alpha) \iff K |\models \varphi(a, b, \alpha).$$

Proof. If $M |\models \varphi(a, b, \alpha)$, then we have trivially $K |\models \varphi(a, b, \alpha)$.

If $M |\models \neg \varphi(a, b, \alpha)$, then $(1 - ab\alpha^4)/(-ab) \in M^*_\mathfrak{p}$ for some $\mathfrak{p}$ a place of $M$ such that $(a, b)_\mathfrak{p} = -1$. Let $L$ be any subfield of $K$ with $L/M$ finite and Galois and let $\mathfrak{P}$ be a place of $M$ lying above $\mathfrak{p}$. Since $[L : M]$ is odd we have $(a, b)_\mathfrak{P} = -1$ and $(1 - ab\alpha^4)/(-ab) \in L^*_\mathfrak{P}$. Hence $L |\models \neg \varphi(a, b, \alpha)$ and $K |\models \neg \varphi(a, b, \alpha)$. Note that for archimedean places $\mathfrak{p} \subset \mathfrak{P}$, it is also true that $(a, b)_\mathfrak{p} = 1$ iff $(a, b)_\mathfrak{P} = 1$.

Theorem 8 The solution set $\psi_2(K)$ of $\psi_2(t)$ in $K$ is a subset of $\mathcal{O}_K$ containing $\mathbb{Z}$ ($\mathbb{Z} \subseteq \psi_2(K) \subseteq \mathcal{O}_K$).

Proof. We have trivially $\mathbb{Z} \subseteq \psi_2(K)$. Let $t \in K \backslash \mathcal{O}_K$. We show that there are $a, b \in K$ such that

$$K |\models \neg \varphi_2(a, b, t) \land \forall c(\varphi_2(a, b, c) \rightarrow \varphi_2(a, b, c + 1)).$$

We fix a subfield $M$ of $K$ such that $[M : F]$ is finite and $t \in M$. Then we have $\nu_{\mathfrak{p}_i}(t) < 0$ for some prime $\mathfrak{p}_1$ of $M$. Take a prime $\mathfrak{p}_2 \neq \mathfrak{p}_1$ of $M$. By Lemma 2, there are $a$ and $b$ in $M^*$ such that $\nu_{\mathfrak{p}_1}(a) = 1$, $\nu_{\mathfrak{p}_2}(b) = 0$ and $(a, b)_{\mathfrak{p}_i} = -1$ for $i = 1, 2$, and $t \not\in \varphi_2(a, b, M)$. By Lemma 7, $1 - abt^4 = x^2 - ay^2 - bz^2$ is not solvable for $x, y, z$ in $K$.

Let $c$ in $K$ and suppose $K |\models \varphi_2(a, b, c)$. Take a subfield $L$ of $K$ such that $L$ contains $c$ and $L/M$ is a finite Galois extension, then we have $L |\models \varphi_2(a, b, c)$ by
Lemma 7. Let \( h = 1 - abc^4 \) and \( h' = 1 - ab(c+1)^4 \). Let \( \mathfrak{P}_1, \ldots, \mathfrak{P}_k \) be all the primes of \( L \) lying above \( p_1 \) and \( \mathfrak{P}_{k+1}, \ldots, \mathfrak{P}_{k+s} \) be all the primes of \( L \) lying above \( p_2 \). By Lemma 5, we have \( S_L(a, b) = \{ \mathfrak{P}_1, \ldots, \mathfrak{P}_{k+s} \} \), that is, \( \mathfrak{P}_i \) are all the primes \( \mathfrak{P} \) of \( L \) such that \( (a, b)_{\mathfrak{P}} = -1 \). \( k \) and \( s \) are odd since \( L/M \) is Galois with odd extension degree. We will show that for all \( \mathfrak{P}_i \), \( h'/(-ab) \) is not a square of \( L_{\mathfrak{P}_i} \), assuming \( h/(-ab) \) is not. Take one \( \mathfrak{P} = \mathfrak{P}_i \). We will break into cases according to whether or not \( \mathfrak{P} \) divides 2.

Case 1: \( \mathfrak{P} \parallel 2 \).

As mentioned before we have \( (1 + p^r)^2 = 1 + 2p^r \) if \( p^r \subseteq 2p \) by [7, p. 163]. Hence we have \( (1 + \mathfrak{P})^2 = 1 + \mathfrak{P} \). If \( \nu_{\mathfrak{P}}(c) \geq 0 \), then \( h' = 1 - ab(c+1)^4 \) is a square of \( L_{\mathfrak{P}} \) since \( \nu_{\mathfrak{P}}(-ab(c+1)^4) > 0 \). Since \( (a, b)_{\mathfrak{P}} = (a, -ab)_{\mathfrak{P}} = -1 \) we have \(-ab\) is not a square of \( L_{\mathfrak{P}} \), hence \( h'/(-ab) \) is also not.

We consider the case \( \nu_{\mathfrak{P}}(c) < 0 \). Since \( h/(-ab) = c^4(1 - 1/(abc^4)) \) it follows that \( \nu_{\mathfrak{P}}(-abc^4) \geq 0 \). Let \( \mathfrak{P} \) lie above \( p_i \) and let \( e = e(\mathfrak{P}/p_i) \) be the ramification index of \( \mathfrak{P} \). \( e \) must be odd since \( L/M \) is Galois with odd extension degree. Hence we have \( \nu_{\mathfrak{P}}(-abc^4) > 0 \). Then we have \( \nu_{\mathfrak{P}}(-ab(c+1)^4) = \nu_{\mathfrak{P}}(-ab) + 4\nu_{\mathfrak{P}}(c) = \nu_{\mathfrak{P}}(-abc^4) > 0 \), hence \( h' = 1 - ab(c+1)^4 \) is a square of \( L_{\mathfrak{P}} \) and \( h'/(-ab) \) is not.

Case 2: \( \mathfrak{P} \parallel 2 \).

Since 2 is unramified in \( L/\mathbb{Q} \) we have \( \nu_{\mathfrak{P}}(2) = 1 \) and \( \nu_{\mathfrak{P}}(-ab) = 1 \). Furthermore we know \( (1 + \mathfrak{P})^2 = 1 + \mathfrak{P}^3 \) by [7, p. 163]. If \( \nu_{\mathfrak{P}}(c) < 0 \) then \( h/(-ab) = c^4(1 - 1/(abc^4)) \) would be a square of \( L_{\mathfrak{P}} \); hence we have \( \nu_{\mathfrak{P}}(c) \geq 0 \). It follows that \( \nu_{\mathfrak{P}}(h'/(-ab)) = -1 \) and \( h'/(-ab) \) is not a square of \( L_{\mathfrak{P}} \).

Example 9 1. Let \( F = \mathbb{Q}(\zeta_l) \) and \( \mathscr{P} \) be a set of all \( M_n = \mathbb{Q}(\zeta_{l^n}) \) \((n > 1)\), where \( l \) is an odd integer > 1 and \( \zeta_{l^n} \) is a primitive \( l^n \)-th root of unity. \( K = \bigcup_n M_n \).

2. Let \( F = \mathbb{Q} \) and \( \mathscr{P} \) be a set of all \( \mathbb{Q}(\cos(2\pi/l^n)) \), where \( n \in \mathbb{N} \) and \( l \) is an odd prime with \( l \equiv -1 \) \((\mod 4)\). \( K = \mathbb{Q}(\{\cos(2\pi/l^n) : n \in \mathbb{N}, l \) a prime, \( l \equiv -1 \) \((\mod 4)\)}).

Remark 10 In the proof of Theorem 8, we have \( \varphi_2(a, b, M) = \mathcal{O}_{M_{\mathfrak{P}_1}}^{M} \cap \mathcal{O}_{M_{\mathfrak{P}_2}}^{M} \). Here \( \mathcal{O}_{M_{\mathfrak{P}_i}}^{M} \) denotes the valuation ring of \( p_i \) in \( M \). But it is not necessarily true that \( \varphi_2(a, b, L) = \bigcap_{i} \mathcal{O}_{\mathfrak{P}_i}^{L} \). Actually we have \( \varphi_2(a, b, M) \subseteq \bigcap_{i} \mathcal{O}_{\mathfrak{P}_i}^{L} \subseteq \varphi_2(a, b, L) \).

Nevertheless we can prove \( \varphi_2(a, b, L) = \bigcap_{i} \mathcal{O}_{\mathfrak{P}_i}^{L} \) for \( K = \bigcup_n \mathbb{Q}(\zeta_{l^n}) \), where \( l \) is an odd prime and \( \zeta_{l^n} \) is a primitive \( l^n \)-th root of unity.

4 The structure of \( \psi(K) \)

In this section we let \( F = \mathbb{Q} \), that is, let \( K \) be the composite of all fields in \( \mathscr{P}_0 \) where \( \mathscr{P}_0 \) is a set of infinitely many finite Galois extensions \( M \) of \( \mathbb{Q} \) such that \( [M : \mathbb{Q}] \) is odd.
and 2 is unramified in $M/\mathbb{Q}$. We let $\mathcal{F}$ be the family of all finite Galois subextensions of $K$. Then every $M$ also has odd extension degree over $\mathbb{Q}$ and 2 is unramified in $M/\mathbb{Q}$. We write $\varphi$ and $\psi$ instead of $\varphi_2$ and $\psi_2$ respectively.

We shall investigate what $\psi(K)$ is. For $a, b \in K$ we let $T_{a,b}$ be the set of elements $\alpha$ of $K$ such that

$$K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \rightarrow \varphi(a, b, \alpha).$$

Then we have $\psi(\Omega_K) = \bigcap_{a,b \in K} T_{a,b}$. We easily see $T_{a,b} = K$ for $a, b$ with $ab = 0$. So we shall investigate what $T_{a,b}$ is, for $a, b \in K^*$. We recall that for $a, b \in M^*$, $M \models \neg\varphi(a, b, \alpha)$ iff $\alpha^4 - 1/ab \in M_p^{*2}$ for some $p \in S_M(a, b)$. Hence we easily see the following: for $a, b \in K^*$, if $S_M(a, b) = \emptyset$ for some $M \in \mathcal{F}$ with $a, b \in M$, then $\varphi(a, b, K) = T_{a,b} = K$ by Lemma 6. So we shall investigate what $T_{a,b}$ is, for $a, b \in K^*$ such that for some $M \in \mathcal{F}$ with $a, b \in M$, $S_M(a, b) \neq \emptyset$.

From now on we use the following notation. For a number field $M$, the ring of integers of $M_p$ is denoted by $(\mathcal{O}_M)_p$, its maximal ideal is also denoted by $\mathfrak{p}$, its residue field $(\mathcal{O}_M)_p/\mathfrak{p}$ by $(\bar{M})_p$, and the group of units of $(\mathcal{O}_M)_p$ by $(U_M)_p$. For $\alpha \in \mathcal{M}$, we denote by $\bar{\alpha}$ its residue class in $(\bar{M})_p$. Furthermore we usually let $\mathfrak{p}$ lie above a rational prime $p$. Note that $(\bar{M})_p \simeq (\mathcal{O}_M)_p/\mathfrak{p} \simeq \mathbb{F}_p^f$, where $f$ is the residue degree of $M$ at $\mathfrak{p}$.

**Lemma 11** Let $a, b \in K^*$ such that

$$K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1))$$

holds. Then for every $M \in \mathcal{F}$ with $a, b \in M$, every $\mathfrak{p} \in S_M(a, b)$ is not archimedean.

This is proved similarly as Lemma 14 in [1].

**Lemma 12** Let $M \in \mathcal{F}$. Let $a, b \in M^*$, $\alpha \in \mathcal{O}_M$ and $\mathfrak{p}_0 \in S_M(a, b)$ with $\mathfrak{p}_0 \parallel 2$ such that

1. $K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1))$ and
2. $\alpha^4 - 1/ab \in M_{p_0}^{*2}$ hold.

Then $\nu_{\mathfrak{p}_0}(-ab) = 0$ and $\nu_{\mathfrak{p}_0}(\alpha) = 0$.

This is also proved similarly as Lemma 15 in [1].

Now we will prove the following lemma on finite fields.

**Lemma 13** Let $p$ be an odd prime and $q = p^f$. Let $\mathbb{F}_q$ be a finite field with $q$ elements other than $\mathbb{F}_3, \mathbb{F}_5$. We let $\eta$ be the quadratic character of $\mathbb{F}_q$, that is, $\eta(0) = 0, \eta(c) = 1$ if $c \in \mathbb{F}_q^{*2}$ and $\eta(c) = -1$ otherwise.

Then for all $a \in \mathbb{F}_q^*$ with $\eta(a) = -1$,

1. there are $b \in \mathbb{F}_q$ and $j \in \mathbb{F}_p$ such that $\eta(b^4 + a)\eta((b + j)^4 + a) = -1$.

   Exceptional cases are, $\mathbb{F}_3$ and $a = 2$, and, $\mathbb{F}_5$ and $a = 2$. 


Proof. We will first prove the following; for all \( a \in \mathbb{F}_q^* \) with sufficiently large \( q \), we can take \( j = 1 \) in the statement (†). We use Weil’s Theorem [5, p. 225, Theorem 5.41], from which we have that for \( a \in \mathbb{F}_q^* \),

\[
\left| \sum_{c \in \mathbb{F}_q} \eta\{(c^4 + a)((c + 1)^4 + a)\} \right| \leq 7q^{1/2}.
\]

Thus if \( q \) satisfies inequality \( 7q^{1/2} < q - 8 \) then for all \( a \in \mathbb{F}_q^* \) there is \( b \in \mathbb{F}_q \) such that \( \eta(b^4 + a)\eta((b + 1)^4 + a) = -1 \). Hence for all \( \mathbb{F}_q \) with \( q > 64 \) the assertion holds. For the small values of \( q \leq 64 \) we can check the assertion directly. \( \Box \)

Note that in the statement (†) we cannot always take \( j = 1 \) if \( q \leq 64 \); for example in \( \mathbb{F}_7 \) there is no \( b \) such that \( \eta(b^4 + 5)\eta((b + 1)^4 + 5) = -1 \) but in \( \mathbb{F}_7 \) \( \eta(1^4 + 5)\eta(1 + 2)^4 + 5 = -1 \) holds. Note also that we need the assumption \( \eta(a) = -1 \) for \( \mathbb{F}_9 \) since for \( a = 1, 2 \), for which \( \eta(a) = 1 \), the statement (†) does not hold.

Lemma 14 Let \( M \in \mathcal{F} \). Let \( a, b \in M^* \). Suppose that \( S_M(a, b) \) contains a non-archimedean place \( p_0 \) such that \( p_0 \not\models \varphi(a, b, c) \rightarrow \varphi(a, b, c + 1) \).

Then \( K \models \neg \forall c \varphi(a, b, c) \rightarrow \varphi(a, b, c + 1) \).

The proof is similar to that of Lemma 16 in [1].

Proposition 15 Let \( M \in \mathcal{F} \). For \( a, b \in M^* \), if \( S_M(a, b) \) contains no primes dividing 2, then we have \( \mathcal{O}_K \subseteq T_{a,b} \), that is,

\( K \models \forall c \varphi(a, b, c) \rightarrow \varphi(a, b, c + 1) \rightarrow \varphi(a, b, \alpha) \) for all \( \alpha \in \mathcal{O}_K \).

Proof. We first note the following; if we take \( N \in \mathcal{F} \) such that \( a, b \in N^* \) then \( S_N(a, b) \) also contains no primes dividing 2 by Lemma 6. Suppose not. Then there is \( \alpha \in \mathcal{O}_K \) such that

\( K \models \forall c \varphi(a, b, c) \rightarrow \varphi(a, b, c + 1) \) but \( K \models \neg \varphi(a, b, \alpha) \).

Take \( N \in \mathcal{F} \) such that \( a, b, \alpha \in N \). We have by Lemma 7,

\( N \models \forall c \varphi(a, b, c) \rightarrow \varphi(a, b, c + 1) \) but \( N \models \neg \varphi(a, b, \alpha) \).

Then there is a \( p_0 \in S_N(a, b) \) such that \( \alpha^4 - 1/ab \in N_{p_0}^{\times 2} \).

We see that \( p_0 \) is not archimedean by Lemma 11 and that \( \nu_{p_0}(-ab) = 0 \) and \( \nu_{p_0}(\alpha) = 0 \) by Lemma 12. If \( (\bar{N})_{p_0} \neq \mathcal{F}_3, \mathcal{F}_5 \), we get a contradiction by Lemma 14.

Suppose that \( (\bar{N})_{p_0} = \mathcal{F}_5 \). Since \( (a, b)_{p_0} = -1 \) and \( N \models \psi(1) \), we have \( -1/ab \in N_{p_0}^{\times 2} \) and \( 1 - 1/ab \in N_{p_0}^{\times 2} \), hence \(-1/ab \equiv 2 \pmod{p_0} \). Since \( \nu_{p_0}(\alpha) = 0 \), we have
$\alpha^4 \equiv 1 \pmod{p_0}$. Then we have $\alpha^4 - 1/ab \equiv 3 \pmod{p_0}$, hence $\alpha^4 - 1/ab \not\in N_{p_0}^{*2}$, a contradiction.

Suppose that $(\tilde{N})_{p_0} = \mathcal{F}_3$. We first deal with the case where $p_0$ is not ramified in $N/Q$. Then 3 is a prime element of $N_{p_0}$ and we can write $-1/ab = 2 + s_13 + s_23^2 + \cdots$, where $s_1 \in \{0, 1, 2\}$. We note that $N \not\models \varphi(a, b, n)$ for all $n \in \mathbb{N}$. If $s_1 = 0$, then $2^4 - 1/ab = (s_2 + 2)3^2 + \cdots$, $7^4 - 1/ab = s_23^2 + \cdots$ and $11^4 - 1/ab = (s_2 + 1)3^2 + \cdots$. Thus we have one of these three must be contained in $N_{p_0}^{*2}$, a contradiction. Likewise if $s_1 = 1$, then $4^4 - 1/ab = (s_2 + 2)3^2 + \cdots$, $13^4 - 1/ab = s_23^2 + \cdots$ and $5^4 - 1/ab = (s_2 + 1)3^2 + \cdots$. And if $s_1 = 2$, then $1^4 - 1/ab = (s_2 + 1)3^2 + \cdots$, $8^4 - 1/ab = s_23^2 + \cdots$ and $10^4 - 1/ab = (s_2 + 1)3^2 + \cdots$. Thus in the case where $p_0$ is not ramified in $N/Q$, we get contradictions.

Secondly, we deal with the case where $p_0$ is ramified in $N/Q$. Let $\nu_{p_0}(3) = e$ and let $\pi$ be a prime element of $N_{p_0}$. We can write $-1/ab = 2 + s_1\pi + s_2\pi^2 + \cdots$, where $s_i \in \{0, 1, 2\}$. We may write $\alpha = 1 + c_1\pi + c_2\pi^2 + \cdots$ where $c_i \in \{0, 1, 2\}$, since if $\alpha \equiv 2 \pmod{p_0}$ then $-\alpha \equiv 1 \pmod{p_0}$. Since $N \models \neg \varphi(a, b, \alpha - n)$, we have $N \models \neg \varphi(a, b, \alpha - n)$ for all $n \in \mathbb{N}$. But $(\alpha - 1)^4 - 1/ab \equiv 2 \pmod{p_0}$, hence there must be another prime $p_1 \in S_N(a, b)$ with $(\alpha - 1)^4 - 1/ab \in N_{p_1}^{*2}$. $p_1$ must be a prime lying above 3 and $\alpha \equiv 2 \pmod{p_1}$. And we have $(\alpha - (3k + 1))^4 - 1/ab \equiv 2 \pmod{p_0}$ and $(\alpha - (3k + 2))^4 - 1/ab \equiv 0 \pmod{p_0}).$ Likewise $(\alpha - (3k + 1))^4 - 1/ab \equiv 0 \pmod{p_1}$ and $(\alpha - (3k + 2))^4 - 1/ab \equiv 2 \pmod{p_1}$. Since there are finitely many primes in $S_N(a, b)$, we must have for some $k$ $(\alpha - (3k + 2))^4 - 1/ab \equiv 0 \pmod{p_0}$ and $(\alpha - (3k + 2))^4 - 1/ab \in N_{p_0}^{*2}$.

We have $s_1 + c_1 \equiv 0 \pmod{p_0}$ since $\alpha^4 - 1/ab = (s_1 - c_1)\pi + \cdots$. And we have $s_1 - c_1 \equiv 0 \pmod{p_0}$ since $(\alpha - (3k + 2))^4 - 1/ab = (s_1 - c_1)\pi + \cdots$. Thus we have $s_1 \equiv 0 \pmod{p_0}$ and $c_1 \equiv 0 \pmod{p_0}$. Likewise we have $s_2 \equiv 0 \pmod{p_0}$ and $c_2 \equiv 0 \pmod{p_0}$. We can proceed to $\pi^{e-1}$. It follows that $-1/ab = 2 + s_e\pi^e + s_{e+1}\pi^{e+1} + \cdots$. Then we have $2^4 - 1/ab = (s_e + 2)3^2 + \cdots$, $7^4 - 1/ab = s_e3^2 + \cdots$ and $11^4 - 1/ab = (s_e + 1)3^2 + \cdots$, a contradiction.

We will deal with primes dividing 2.

**Lemma 16** Let $M \in \mathcal{F}$. Let $a, b \in M^*$, $\alpha \in \mathcal{O}_M$ and $p_0 \in S_M(a, b)$ with $p_0|2$ such that

1. $K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1))$ and

2. $\alpha^4 - 1/ab \in M_{p_0}^{*2}$ hold.

Then $\nu_{p_0}(-ab) = \pm 2$.

The proof is similar to that of Lemma 18 in [1].

We shall prove a similar result to Lemma 14.
Let $M \in \mathscr{P}$ and $a, b \in M^*$. Suppose that $S_n(a, b)$ contains a $p_0$ such that $p_0|2$ and $\nu_{p_0}(-ab) = -2$.

Then $K_l \models \neg \forall c (\varphi(a, b, c) \rightarrow \varphi(a, b, c+1))$.

The proof is similar to that of Lemma 19 in [1]

Thus we get the following proposition. The proof is similar to that of Proposition 15.

Proposition 18 Let $l$ be an odd prime such that $l \equiv -1 \pmod{4}$. For $a, b \in F_n^*$, if $S_n(a, b)$ contains no primes $p$ such that $p|2$ and $\nu_p(-ab) = 2$, then we have $\mathfrak{O}_{K_l} \subseteq T_{a,b}$, that is,

$$K_l \models \forall c (\varphi(a, b, c) \rightarrow \varphi(a, b, c+1)) \rightarrow \varphi(a, b, \alpha) \text{ for all } \alpha \in \mathfrak{O}_{K_l}.$$ 

Since $\psi(K) = \bigcap_{a,b \in K^*} T_{a,b} \subseteq \mathfrak{O}_K$, Proposition 18 implies $\psi(K) = \bigcap_{(a,b) \in \Delta} T_{a,b}$, where $\Delta$ is the set of $(a, b) \in K^* \times K^*$ such that for some $M$ with $a, b \in M$, $S_M(a, b)$ contains a prime $p$ with $p|2$ and $\nu_p(-ab) = 2$. Such $a$ and $b$ exist, for example, let $a = 2$ and $b = 10$.

Let $M \in \mathscr{P}$ and $(2) = p_1 \cdots p_k$ in $M$. Put $P_M = \bigcap_i ((1 + p_i) \cup p_i)$. Then $P_M$ is a subring of $\mathfrak{o}_M$ containing 1. Let $P_K = \bigcup \{P_M : M \in \mathscr{P}\}$. $P_K$ is a subring of $\mathfrak{O}_K$ containing 1.

Theorem 19 $\psi(K) = P_K$.

The proof is similar to that of Proposition 20 in [1].

Example 20 1. $K_l = \bigcup_n \mathbb{Q}(\cos(2\pi/l^n))$ with $l$ a prime and with $l \equiv -1 \pmod{4}$.

2. $K_W = \prod_{l \in W} K_l$. ($W = \{l \text{ a prime : } l \equiv -1 \pmod{4}\}$)

3. $K_0 = \mathbb{Q}(\{\cos(2\pi/l) : l \text{ a prime, } l \equiv -1 \pmod{4}\})$.

5 Undecidability results

Let $K_l = \bigcup_n \mathbb{Q}(\cos(2\pi/l^n))$. In [1] we proved that if $l$ is a prime such that $l \equiv -1 \pmod{4}$ and 2 is a prime of $\mathfrak{O}_{K_l}$, then $K_l$ is undecidable. But in 2000 C.R. Videla [12] proved that $K_l$ is undecidable for every prime $l$. He considered $K/F$ a pro-$p$ Galois extension over a number field $F$ and using Rumely’s formula in [6] he proved that $\mathfrak{O}_{K_l}$ is definable with parameters. Then he also used the results of Kronecker and J. Robinson.

Kronecker [3] determined all sets of conjugate algebraic integers in the interval $c - 2 \leq x \leq c + 2$, provided that $c$ is a rational integer; they have the form

$$x = c + 2 \cos(2k\pi/m) \text{ with } 0 \leq k \leq m/2 \text{ and } (k, m) = 1.$$
Note that if $m = 1, 2, 3, 4$, then $x = c + 2, c - 2, c \pm 1, c$ respectively. Furthermore it is known that an interval of length less than 4 can contain only finitely many complete sets of conjugate algebraic integers. (See [11].)

Therefore we see that the interval $(0,4)$ contains infinitely many complete conjugate sets of totally real algebraic integers and that no sub-interval does.

These facts are used by J. Robinson in [9]. Her results concerns the integral closure of $\mathbb{Z}$ inside totally real fields, not necessarily finite over $\mathbb{Q}$. She calls such a ring a totally real algebraic integer ring. In 1962 she proved the following: The natural numbers can be defined arithmetically in any totally real algebraic integer ring $A$ such that there is a smallest interval $(0,s)$ with $s$ real or $\infty$, which contains infinitely many complete conjugate sets of numbers of $A$. But we can say more. We recall that $\mathbb{Z}^{tr}$ denotes the ring of all totally real algebraic integers.

**Theorem 21** Let $R$ be a subring of $\mathbb{Z}^{tr}$ containing $\mathbb{Z}$ such that there is a smallest interval $(0,s)$ with $s$ real or $\infty$, which contains infinitely many complete conjugate sets of numbers of $R$. Here $s$ need not be in $R$. Then $\mathbb{N}$ is definable in $R$.

In particular such a ring is undecidable.

The proof of J. Robinson just works. See [9, pp. 300–301].

Thus it follows that for every positive integer $l > 1$, $\mathfrak{O}_{K}$ is undecidable, from which Videla proved that $K_{l}$ is undecidable. Note that even if the defining formula contains parameters it is possible to define $\mathbb{N}$. See [12].

We give alternative proof of this fact in the case where $l$ is a prime with $l \equiv -1 \pmod{4}$. We know that $\psi(K_{l})$ is a subring of $\mathbb{Z}^{tr}$ containing $\mathbb{Z}$ if $l$ is a prime such that $l \equiv -1 \pmod{4}$. Furthermore we know by [11, p. 312], that $2 + 2 \cos(2\pi/l^{n})$ are units in $\mathfrak{O}_{K}$ and that $1 + 2 \cos(2\pi/l^{n})$ are units in $\mathfrak{O}_{K}$ if $l \neq 3$, and $|N_{F_{n}/\mathbb{Q}}(1 + 2 \cos(2\pi/3^{n}))| = 3$ for $n \geq 2$. Hence we see that $2 + 2 \cos(2\pi/l^{n})$ are not in $\psi(K_{l})$ if $l^{n} \neq 3$. On the other hand $4 + 4 \cos(2\pi/l^{n})$ are in $\bigcap_{i} \mathfrak{P}_{i}^{(2)}$, hence in $\psi(K_{l})$. Thus we see that the interval $(0,8)$ contains infinitely many complete conjugate sets of numbers of $\psi(K_{l})$ and the interval $(0,4)$ does not. We show that $(0,8)$ is actually such a smallest interval for $\psi(K_{l})$.

**Lemma 22** Let $l$ be an odd prime such that $l \equiv -1 \pmod{4}$. Then $(0,8)$ is a smallest interval of the form $(0,c)$ which contains infinitely many complete conjugate sets of numbers of $\psi(K_{l})$.

**Proof.** We know that $K_{l}$ has only finitely many primes lying above 2. (See Lemma 13 in [1].) Thus $\psi(K_{l}) = P_{K_{l}} = \bigcap_{i}((1 + \mathfrak{P}_{i}) \cup \mathfrak{P}_{i})$, where $\mathfrak{P}_{1}, \ldots, \mathfrak{P}_{k}$ are primes of $K_{l}$ lying above 2. We easily see that $\psi(K_{l})$ is a union of $2^{k}$ cosets of $\mathfrak{O}_{K_{l}}/2\mathfrak{O}_{K_{l}}$. Suppose that $(0,8)$ is not such a smallest interval. Then some interval $(0,\delta)$ with $\delta < 8$ contains infinitely many complete conjugate sets of numbers of $\psi(K_{l})$. Then we have that some coset, say $\alpha + 2\mathfrak{O}_{K_{l}}$, contains infinitely many complete conjugate
sets of numbers. It follows that an interval of length less than 4 contains infinitely many complete conjugate sets of algebraic integers, a contradiction.

Let \( K_\Delta = \prod_{l \in \Delta} K_l \) where \( \Delta \) is a finite set of primes. From the result of Videla we deduce that \( K_\Delta \) is undecidable. If \( \Delta \) is a finite set of primes with \( l \equiv -1 \pmod{4} \), then we can give another proof similarly.

Nevertheless we can give a new undecidable infinite algebraic extension of \( \mathbb{Q} \) by our method. Let \( V \) be a set of Sophie Germain primes, that is, a prime \( p \) such that \( 2p + 1 \) is again a prime. It is considered that there are infinitely many Sophie Germain primes but it is not proved. Let \( K_V = \mathbb{Q}(\{\cos(2\pi/l) : l \in V\}) \). Then we have \( \psi(K_V) = (1 + 2\mathfrak{O}_{K_V}) \cup \mathfrak{O}_{K_V} \), hence \( K_V \) is undecidable.

References


