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Kyoto University
What are o-minimal sheaves

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Abstract
In this small note we present an introduction to o-minimal sheaves and their connection to semi-algebraic and sub-analytic sheaves.

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1 Introduction

O-minimal structures are a class of ordered structures which are a model theoretic (logic) generalization of interesting classical structures such as:

- the field of real numbers;
- the field of real numbers expanded by restricted globally analytic functions ([7]).

More precisely, an ordered structure

\[ \mathcal{M} = (M, (c)_{c \in C}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}}, <) \]

is o-minimal if every definable subset of \( M \) in the structure is already definable in the ordered set \( (M, <) \).

The development of o-minimality has been strongly influenced by real analytic geometry and it is based on: (i) adaptation of methods of real analytic geometry to the o-minimal setting; (ii) construction of new and mathematically interesting examples of o-minimal structures; (iii) new insights originated from model-theoretic methods into the real analytic setting. O-minimal structures provide: a generalization, a uniform treatment and new tools.

Good references on o-minimality are, for example, the book [8] by van den Dries and the notes [3] by Coste. For semialgebraic geometry relevant to this paper the reader should consult the work by Delfs [5], Delfs and Knebusch [6] and the book [2] by Bochnak, Coste and Roy. For subanalytic geometry we refer to the work [1] by Bierstone and Milman.

Given an o-minimal structure

\[ \mathcal{M} = (M, (c)_{c \in C}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}}, <) \]

we have:

- the category Def of definable spaces with continuous definable maps.
- the geometry of Def is called o-minimal geometry.

Examples 1.1 (Special cases of o-minimal geometry)

- \( \mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, <) \) - semi-algebraic geometry (includes real algebraic geometry);
• $\mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, (f)_{f \in \text{an}}, <)$ - restricted globally sub-analytic geometry;

The model theoretic language allows a uniform development of o-minimal geometry in non-standard o-minimal structures. Concrete non-standard o-minimal structures are:

• $\mathbb{R}((t^Q)) = (\mathbb{R}((t^Q)), 0, 1, +, \cdot, <)$ (or any ordered real closed field),

• $\mathbb{R}((t^Q))_{an} = (\mathbb{R}((t^Q)), 0, 1, +, \cdot, (f)_{f \in \text{an}}, <)$

where $\mathbb{R}((t^Q))$ is the field of power series with well ordered supports on which every restricted globally analytic function $f \in \text{an}$ can be interpreted in a canonical way ([9]). There are many important o-minimal expansions

$$\mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, (f)_{f \in \mathcal{F}}, <)$$

of the ordered field of real numbers. For example $\mathbb{R}_{\text{an}}, \mathbb{R}_{\text{exp}}, \mathbb{R}_{\text{an,exp}}, \mathbb{R}_{\text{an}\cdot\exp}, \mathbb{R}_{\text{an}\cdot\exp}$ see resp., [7, 29, 10, 12, 13]. For each such we have $2^\kappa$ many non-isomorphic non-standard o-minimal models for each $\kappa >$ cardinality of the language! There is however a non-standard o-minimal structure

$$\mathcal{M} = (\bigcup_{n \in \mathbb{N}} \mathbb{R}((t^{\frac{1}{n}})), 0, 1, +, \cdot, (f_p)_{p \in \mathbb{R}[[\zeta_1, \ldots, \zeta_n]]}, <)$$

which does not came from a standard one ([23, 17]). O-minimal geometry includes the geometry of all those (standard) tame analytic structures but it goes beyond and includes also a generalization of PL-geometry: any ordered vector space over an ordered division ring

$$\mathcal{M} = (M, 0, +, (\lambda_d)_{d \in D}, <)$$

is an o-minimal structure ([8]).

Following or inspired by the work of:

• Verdier (locally compact topological spaces) - [16, 18, 19].

• Delfs (real algebraic geometry) - [5].

• Kashiwara-Schapira, L. Prelli et al. (sub-analytic geometry) - [22, 20, 21, 25, 26].

• Grothendieck (étale framework) - [28].

we would like to develop sheaf theory in the category Def in a fixed but arbitrary o-minimal structures $\mathcal{M}$. 
2 What are o-minimal sheaves

Recall that our goal is to develop sheaf theory in the category Def in a fixed but arbitrary o-minimal structures \( \mathcal{M} \). Every object of Def is a topological space with topology defined from the ordering of \( \mathcal{M} \). So why not topological sheaf theory? Topological sheaf theory is not suitable, since it gives:

- no information in the non standard setting;
- no new information in the standard setting.

In fact we have to use sites (Grothendieck topologies). Usually the problem is having too many or too few open subsets.

So what are o-minimal sheaves? Let \( X \) be an object of Def and \( k \) a field. An o-minimal sheaf of \( k \)-vector spaces on \( X \), called also an o-minimal \( k \)-sheaf on \( X \), is a contravariant functor:

\[
F : \text{Op}(X_{\text{def}}) \to \text{Mod}(k)
\]

\[
U \mapsto F(U)
\]

\[
(V \subset U) \mapsto (F(U) \to F(V))
\]

\[
s \mapsto s|_V
\]

where \( X_{\text{def}} \) is the o-minimal site on \( X \). Satisfying the following gluing conditions: for \( U \in \text{Op}(X_{\text{def}}) \) and \( \{U_j\}_{j \in J} \in \text{Cov}(U) \) we have the exact sequence

\[
0 \to F(U) \to \Pi_{j \in J} F(U_j) \to \Pi_{j,k \in J} F(U_j \cap U_k).
\]

What is the o-minimal site on \( X \)? The o-minimal site \( X_{\text{def}} \) on \( X \) is the data consisting of:

- The category \( \text{Op}(X_{\text{def}}) \) of open definable subsets of \( X \) with inclusions;
- The collection of admissible coverings \( \text{Cov}(U), \ U \in \text{Op}(X_{\text{def}}) \) such that \( \{U_j\}_{j \in J} \in \text{Cov}(U) \) if \( \{U_j\}_{j \in J} \) covers \( U \), its elements are in \( \text{Op}(X_{\text{def}}) \) and has a finite sub-cover.
This includes semi-algebraic and restricted globally sub-analytic sites and sheaves. What about sub-analytic site and sheaves? If we work in the slightly more general category of locally definable spaces with continuous locally definable maps, then the o-minimal site includes also the sub-analytic site on real analytic manifolds.

The gluing condition

\[ 0 \rightarrow F(U) \rightarrow \Pi_{j \in J} F(U_j) \rightarrow \Pi_{j,k \in J} F(U_j \cap U_k) \]

means:

- if \( s \in F(U) \) and \( s|_{U_j} = 0 \) for each \( j \), then \( s = 0 \);
- if \( s_j \in F(U_j) \) are such that \( s_j = s_k \) on \( U_j \cap U_k \) then they glue to \( s \in F(U) \) (i.e. \( s|_{U_j} = s_j \)).

For \( X \) an object of \( \text{Def} \) and \( k \) a field, we use the following notation:

\( \text{Mod}(k_{X_{\text{def}}}) := k\text{-sheaves in the o-minimal site } X_{\text{def}} \) and \( \text{Mod}(k_X) := \text{topological } k\text{-sheaves on } X \).

**Examples 2.1 (Simple examples)** Let \( X \) be an object of \( \text{Def} \). The following pre-sheaves are in \( \text{Mod}(\mathbb{R}_{X_{\text{def}}}) \):

- \( U \mapsto \mathbb{R}_X(U) := \{ f : U \rightarrow \mathbb{R} | f \text{ locally constant} \} \);
- \( U \mapsto \{ f : U \rightarrow \mathbb{R} | f \text{ bounded} \} \);
- \( U \mapsto C_X(U) := \{ f : U \rightarrow \mathbb{R} | f \text{ continuous} \} \);
- \( U \mapsto \{ f : U \rightarrow \mathbb{R} | f \text{ definable} \} \).

The second and the fourth examples above are not in \( \text{Mod}(\mathbb{R}_X) \).

In our context the gluing condition gives rise to the following gluing criteria. Let \( X \) be an object of \( \text{Def} \) (resp. a real analytic manifold) and \( F \) a presheaf on \( X_{\text{def}} \) (resp. on \( X_{\text{sa}} \) - the sub-analytic site of \( X \)). Assume that

- \( F(\emptyset) = 0 \);
- for all \( U, V \in \text{Op}(X_{\text{def}}) \) (resp. in \( \text{Op}(X_{\text{sa}}) \)) the sequence

\[ 0 \rightarrow F(U \cup V) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V) \]

is exact.
Then $F$ is a sheaf on $X_{\text{def}}$ (resp. $X_{sa}$).

**Examples 2.2 ([20] - Deep examples)** M. Kashiwara and P. Schapira combined classical analytical results of S. Lojasiewicz and the gluing criteria to show that the following pre-sheaves

- tempered distributions $\mathcal{D}b_X$;
- tempered $C^\infty$ functions;
- Whitney $C^\infty$ functions;
- tempered holomorphic $\mathcal{O}^t_X$ functions;

are sheaves on $X_{sa}$. This is very deep and has applications to the theory of $D$-modules.

### 3 Some results

Of course all the classical homological results for sheaves on sites hold in the category Mod($k_{X\text{def}}$). So if we want to obtain specific results on the geometry of objects of Def we have to introduce something more. For this it will be convenient to replace the o-minimal site $X_{\text{def}}$ by the o-minimal spectrum $\tilde{X}$ of $X$. See [14]. This method was also used in the semi-algebraic context but never in the sub-analytic case where everything is standard - [2, 4, 5].

The o-minimal spectrum $\tilde{X}$ of $X$ is the set of ultrafilters of definable subsets of $X$ equipped with the topology generated by the open subsets of the form $\tilde{U}$ where $U \in \text{Op}(X_{\text{def}})$. This is a spectral topological space - [3, 14, 24].

**Example 3.1 (The connection to real algebraic geometry)** If $R$ is a real closed field and $X$ an affine real algebraic variety over $R$ with coordinate ring $R[X]$, then $\tilde{X} \simeq \text{Spec} R[X]$ (the real spectrum of the commutative ring $R[X]$).

The tilde operation determines the tilde functor $\text{Def} \rightarrow \tilde{\text{Def}}$ which determines morphisms of sites

$$\nu_X : \tilde{X} \rightarrow X_{\text{def}}$$

given by the functor $\nu'_X : \text{Op}(X_{\text{def}}) \rightarrow \text{Op}(\tilde{X}) : U \mapsto \tilde{U}$. 

**Theorem 3.2** ([14]) *The functor Def $\rightarrow \tilde{\text{Def}}$ induces an isomorphism of categories*

$$\text{Mod}(k_{X_{\text{def}}}) \rightarrow \text{Mod}(k_{\tilde{X}}): F \mapsto \tilde{F},$$

where $\text{Mod}(k_{\tilde{X}})$ is the category of sheaves of $k$-modules on the topological space $\tilde{X}$.

The isomorphism is the inverse image $\nu_{X}^{-1}$ and its inverse is the direct image $\nu_{X*}$. The canonical isomorphism extends to the derived categories

$$D^{*}(k_{X_{\text{def}}}) \rightarrow D^{*}(k_{\tilde{X}}): I \mapsto \tilde{I}$$

where $D^{*}(k_{\tilde{X}}) = D^{*}(\text{Mod}(k_{\tilde{X}}))$ and $(*) = b, +, -$.

**Corollary 3.3** *The functors*

\[
\begin{align*}
\text{RHom}_{k_{X_{\text{def}}}}(\bullet, \bullet): & \ D^{-}(k_{X_{\text{def}}})^{\text{op}} \times D^{+}(k_{X_{\text{def}}}) \rightarrow D^{+}(k), \\
\text{RHom}_{k_{X_{\text{def}}}}(\bullet, \bullet): & \ D^{-}(k_{X_{\text{def}}})^{\text{op}} \times D^{+}(k_{X_{\text{def}}}) \rightarrow D^{+}(k_{X_{\text{def}}}), \\
f^{-1}: & \ D^{*}(k_{Y_{\text{def}}}) \rightarrow D^{*}(k_{X_{\text{def}}}) \quad (* = b, +, -), \\
f_{*}: & \ D^{+}(k_{X_{\text{def}}}) \rightarrow D^{+}(k_{Y_{\text{def}}}), \\
\otimes_{k_{X_{\text{def}}}}^{L} \bullet: & \ D^{*}(k_{X_{\text{def}}}) \times D^{*}(k_{X_{\text{def}}}) \rightarrow D^{*}(k_{X_{\text{def}}}) \quad (* = b, +, -)
\end{align*}
\]

*commute with the tilde functor.*

In the paper [14] can develop o-minimal sheaf cohomology by setting

$$H^{*}(X; F) := H^{*}(\tilde{X}; \tilde{F})$$

where $X$ is a definable space and $F$ is a sheaf in $\text{Mod}(k_{X_{\text{def}}})$ and prove the following results:

**Theorems 3.4** ([14])

- Vanishing Theorem.
- Vietoris-Begle Theorem.
- Eilenberg-Steenrod Axioms.

The vanishing theorem above has the following application to sub-analytic sheaves:

**Theorem 3.5** ([27]) *Let $X$ be a real analytic manifold. The homological dimension of $\text{Mod}(k_{X_{\text{s.a.}}})$ is finite.*
After developing the theory of definably compact supports one obtains the following result conjectured by Delfs in the semi-algebraic case:

**Theorem 3.6 ([15] - Global Verdier duality)** Let $X$ be definably normal, definably locally compact, definable space. There exists $D^*$ in $D^+(k_{X_{def}})$ and a natural isomorphism

$$R\text{Hom}_{k_{X_{def}}}(\mathcal{F}^*, D^*) \simeq R\text{Hom}_k(R\Gamma_c(X, \mathcal{F}^*), k)$$

as $\mathcal{F}^*$ varies through $D^+(k_{X_{def}})$.

This is a general form of Poincaré duality:

**Corollary 3.7 ([15] - Poincaré and Alexander duality)** Let $X$ be definably normal, definably locally compact, definable manifold of dimension $n$.

- If $X$ has an orientation $k$-sheaf $\mathcal{O}r_X$, then
  $$H^p(X; \mathcal{O}r_X) \simeq H^{n-p}_c(X; k)^\vee.$$ 

- If $X$ is $k$-orientable and $Z$ is a closed definable subset, then
  $$H^p_Z(X; k_X) \simeq H^{n-p}_c(Z; k)^\vee.$$ 

With L. Prelli we are working on developing the formalism of the six operations on o-minimal sheaves in Def:

$$Rf_*, f^{-1}, \otimes^L, R\text{Hom}, Rf_!!, f^{!!}$$

Such formalism was developed for sub-analytic sheaves by Kashiwara-Schapira using the complicated theory of ind-sheaves and later a direct construction was given by L. Prelli. However, both methods do not generalize to o-minimal sheaves since they rely on the formalism of the six operations on topological sheaves in locally compact topological spaces (Verdier).

**References**


