<table>
<thead>
<tr>
<th>Title</th>
<th>Morley's theorem on Omitting types (New developments of independence notions in model theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>YANAGAWA, MAKOTO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2010), 1718: 70-74</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/170344">http://hdl.handle.net/2433/170344</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Morley's theorem on Omitting types

MAKOTO YANAGAWA
Tokyo University of Science

Abstract

We think about Morley's omitting types theorem for countable first-order theory. Then I introduce the result of having been related to Morley's theorem shown by [4].

1 Introduction

Definition. ($\mathfrak{U}$-number) $\mathfrak{U}_0 = \omega$, $\mathfrak{U}_{\alpha+1} = 2^{\mathfrak{U}_\alpha}$, $\mathfrak{U}_\delta = \sup_{\alpha < \delta} \mathfrak{U}_\alpha$.

Fact. (Erdős-Rado) Let $\alpha$ be infinite cardinal and $n < \omega$. Then $\mathfrak{U}_n^+ \rightarrow (\omega^+)^n_\omega + 1$, $\mathfrak{U}_{\alpha+n}^+ \rightarrow (\mathfrak{U}_\alpha^+)_{\omega+n}^+$.

Note. $\alpha \rightarrow (\beta)^n_\alpha$ means whenever $|X| = \alpha$ and given any function $f$ from $[X]^n$ into $\gamma$, there exists a subset $Y$ of $X$ with $|Y| = \beta$ and an $i < \gamma$ such that for all $\overline{y} \in [Y]^n$, $f(\overline{y}) = i$.

Theorem. (Stretching) Let $\mathcal{L}$ be countable language, $M$ be a model of theory $T$ of $\mathcal{L}$, $(A,<)$ be an infinite set of indiscernibles in $M$, and $(B,<)$ be an arbitrary infinite linearly ordered set. Then there exist a model $N$ of $T$ such that $(B,<)$ is a set of indiscernibles in $N$, and for any $a_1 < \cdots < a_n \in A$ and $b_1 < \cdots < b_n \in B$, $tp(a_1,\ldots,a_n) = tp(b_1,\ldots,b_n)$.

Proof. Put $\Sigma := \{t(x_1,\ldots,x_n) : t$ is term in $\mathcal{L}\}$. We define an equivalence relation $\sim$ on $\Sigma$ as follows. If $t(x_1,\ldots,x_n), t'(x_1,\ldots,x_n) \in \Sigma$, define $t \sim t'$ iff for any $a_1 < \cdots < a_n \in A, M \models t(a_1,\ldots,a_n) = t'(a_1,\ldots,a_n)$. Put $\overline{N} := \{t(b_1,\ldots,b_n) : t(x_1,\ldots,x_n) \in \Sigma, b_1 < \cdots < b_n \in B\}$. We define an equivalence relation $\approx$ on $\overline{N}$ as follows. If $t(b_1,\ldots,b_n), t'(b_1',\ldots,b_n') \in \overline{N}$, define $t \approx t'$ iff $t_0(z_1,\ldots,z_s) \sim t'_0(z_1,\ldots,z_s)$, where $\{x_1,\ldots,x_n\} \cup \{x_1',\ldots,x_m'\}$ and $t_0(z_1,\ldots,z_s) := t(x_1,\ldots,x_n), t'_0(z_1,\ldots,z_s) := t'(x_1',\ldots,x_m')$.

Put $N := \{t(b)^\approx : t(b) \in \overline{N}\}$. Note that for any $t_1(b_1)^\approx, \ldots, t_n(b_n)^\approx \in N$ there exists, for some $\overline{b} \in B$ and $t_i' \in \Sigma$ such that $t_i(\overline{b})^\approx = t_i'(\overline{b})^\approx$. We treat $B$ as a subset of $N$ by identifying each $b \in B$ with $b^\approx$.

$N$ can be made into a $\mathcal{L}$-structure by defining constants, functions and relations as follows:

(Contrasts) $N \models c_N = c^\approx$.

(Functions) $N \models F(t_1(\overline{b})^\approx, \ldots, t_n(\overline{b})^\approx) = (F(t_1(\overline{b}), \ldots, t_n(\overline{b})))^\approx$.

(Relations) $N \models R(t_1(\overline{b})^\approx, \ldots, t_n(\overline{b})^\approx)$

: $\overset{\text{def}}{=} \text{for all } a_1 < \cdots < a_m \in A, M \models R(t_1(\overline{a}), \ldots, t_n(\overline{a}))$.

This definition does not depend on the choice of representatives of the equivalence classes under $\approx$. 
By induction on the complexity of formulas and use Skolem function it can be shown that for any \( b_1 < \cdots < b_n \in B \) and \( \phi(x_1, \ldots, x_n) \in \mathcal{L} \),

\[
N \models \phi(b_1, \ldots, b_n) \iff \text{ for all } a_1 < \cdots < a_n \in A,
M \models \phi(a_1, \ldots, a_n).
\]

By indiscernibility of \( \langle A, < \rangle, \langle B, < \rangle \) is a set of indiscernibles in \( N \) and for any \( a_1 < \cdots < a_n \in A \) and \( b_1 < \cdots < b_n \in B \), \( tp(a_1, \ldots, a_n) = tp(b_1, \ldots, b_n) \). In particular, \( N \equiv M \), hence \( N \) is a model of \( T \).

## 2 Morley's Theorem

**Theorem.** (Morley's omitting types theorem) Let \( T \) be a theory of countable language \( \mathcal{L} \), \( \Gamma \) a set of partial types in finitely many variables over \( \emptyset, \mu = (2^\omega)^+ \). Suppose \( \{M_\alpha : \alpha < \mu\} \) is a sequence of models of \( T \) such that

1. \( |M_\alpha| > 2_\alpha \),
2. \( M_\alpha \) omits each member of \( \Gamma \).

Then for every \( \lambda \geq \omega \), there is a model \( N \) with \( |N| = \lambda \) of \( T \) such that \( N \) omits each member of \( \Gamma \).

**Proof.** Assume to simplify an argument \( T \) has built-in Skolem functions and the set of formulas \( \Gamma \) in the unary. Let \( C = \langle c_i : i < \omega \rangle \) be a sequence of new constant symbols, \( \mathcal{L}' = \mathcal{L} \cup C \).

Now we construct the consistent \( \mathcal{L}' \)-theory \( \Phi \) as following properties:

1. \( T \cup \{c_i \neq c_j : i < j < \omega\} \subset \Phi \);
2. for each term \( t(x_1, \ldots, x_n) \) and \( p \in \Gamma \), there is a \( \phi_p \in p \) such that for all \( i_1 < \cdots < i_n < \omega \),
   \[ -\phi_p(t(c_{i_1}, \ldots, c_{i_n})) \in \Phi \];
3. for any \( \psi(x_1, \ldots, x_n) \in \mathcal{L} \) if \( i_1 < \cdots < i_n < \omega \) and \( j_1 < \cdots < j_n < \omega \),
   \[ \psi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \ldots, c_{j_n}) \in \Phi \].

**Notation.** \( F := \{(M_\alpha, A_\alpha) : \alpha < \mu\} \) is a sequence such that \( M_\alpha \) is satisfied the hypotheses of the theorem and \( A_\alpha \) is subset of \( M_\alpha \) with \( |A_\alpha| > 2_\alpha \).

We say that \( F' = \{(M'_\alpha, B'_\alpha) : \alpha < \mu\} \) is subsequence of \( F \) if for each \( M'_\alpha \) there is \( \beta \geq \alpha \) such that \( M'_\alpha = M_\beta \) and \( B_\alpha \subset A_\beta \) with \( |B_\alpha| > 2_\alpha \).

Fix a linear ordering of each \( M_\alpha \) in an arbitrary fashion denoting them all by \(<\).

**Claim 1.** Fix a term \( t(x_1, \ldots, x_n) \). There is subsequence \( F' \) of \( F \) as following property: for each \( p \in \Gamma \) there is a \( \phi_p \in p \) such that for any \( (M'_\alpha, B'_\alpha) \in F' \), if \( i_1 < \cdots < i_n < \omega \) and \( b_{i_j} \in B'_\alpha \) then \( M'_\alpha \models -\phi_p(t(b_{i_1}, \ldots, b_{i_n})) \).
Proof of claim 1. Note $|\Gamma| \leq 2^\omega$. Let $N_\alpha = M_{\alpha+n}$. Define, for all $\alpha < \mu$, $f_\alpha : [A_{\alpha+n}]^n \rightarrow \mathcal{L}^\Gamma (\overline{a} \mapsto f_\alpha(\overline{a}))$ where $f_\alpha(\overline{a}) : \Gamma \rightarrow \mathcal{L} (p \mapsto (f_\alpha(\overline{a}))(p) := \phi_{\overline{a},p} \in p)$ such that $N_\alpha \models \neg \phi_{\overline{a},p}(t(\overline{a}))$ such a $\phi_{\overline{a},p}$ exists since $N_\alpha$ omits $p$.

Now $|A_{\alpha+n}| > \beth_\alpha$ and for $\alpha \geq 3$, $\beth_\alpha \geq |\mathcal{L}|$.

By Erdős-Rado Theorem, $(\beth_\alpha)^+ \rightarrow (\beth_\alpha^+)^n_{\mathcal{L}}$. Thus we obtain $B_\alpha \subset A_{\alpha+n}$ and $\phi_{\overline{a},p} \in \mathcal{L}$ such that

1. $|B_\alpha| > \beth_\alpha$, 
2. for all $\overline{b} \in [B_\alpha]^n$, $N_\alpha \models \neg \phi_{\overline{a},p}(t(\overline{b}))$.

Namely, for all $\overline{b} \in [B_\alpha]^n$, $f_\alpha(\overline{b})$ is constant.

As $\mu = (2^\omega)^+$, by Erdős-Rado, there is subsequence $\{M'_\alpha : \alpha < \mu\}$ of $\{N_\alpha : \alpha < \mu\}$ such that for all $\overline{b} \in [B'_\alpha]^n$ and $p \in \Gamma$, $(f_\alpha(\overline{b}))(p) = \text{constant}$. Thus $\{(M'_\alpha, B'_\alpha) : \alpha < \mu\}$ and $\phi_p := (f_\alpha(\overline{b}))(p)$ are required.

Claim 2. Fix a $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$. There is subsequence $F'$ of $F$ as following property: for any $(M'_\alpha, B'_\alpha) \in F'$ if $i_1 < \cdots < i_n < \mu$, $j_1 < \cdots < j_n < \mu$ and $b_{i_1}, b_{j_1} \in B_\alpha$

$$M'_\alpha \models \psi(b_{i_1}, \ldots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \ldots, b_{j_n}).$$

Proof of Claim 2. Define, for all $\alpha < \mu$, $h_\alpha : [A_\alpha]^n \rightarrow 2$ as follows:

$$h_\alpha(\overline{a}) = \begin{cases} 0 & \text{if } M_\alpha \models \psi(\overline{a}), \\ 1 & \text{otherwise.} \end{cases}$$

By Erdős-Rado theorem, there is $B_\alpha \subset A_\alpha$ such that $|B_\alpha| > \beth_\alpha$ and for all $\overline{b} \in [B_\alpha]^n$, $h_\alpha(\overline{b}) = \text{constant}$. Thus, $\{(M_\alpha, B_\alpha) : \alpha < \mu\}$ is required.

Let $\{t_i : i < \omega, t_i \text{ is a term of } \mathcal{L}\}$ and $\{\psi_i : i < \omega, \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$ be enumerations of all the terms of $\mathcal{L}$ and all the $\mathcal{L}$-formula, respectively. Now we construct $\Phi$ by induction on $i < \omega$. Suppose $F_0 := \{(M_\alpha, M_\alpha) : \alpha < \mu\}$ and $\Phi_0 := T \cup \{c_i \neq c_j : i < j < \omega\}$. Clearly, for any $(M_\alpha, M_\alpha) \in F_0$, $M_\alpha \models \Phi_0$ and $|M_\alpha| > \beth_\alpha$.

Case 1 ($i < \omega$ is even). Assume we have found $F_i$ and $\Phi_i$. We take new term $\overline{t}(x_1, \ldots, x_n) \in \{t_i : i < \omega, t_i \text{ is a term of } \mathcal{L}\}$, by claim 1, there is subsequence $F_{i+1}$ of $F_i$ as following property: for each $p \in \Gamma$, there is a $\phi_p \in p$ such that for any $(M'_\alpha, B'_\alpha) \in F_{i+1}$, if $i_1 < \cdots < i_n < \omega$ and $b_{i_1}, b_{i_2} \in B_\alpha$,

$$M'_\alpha \models \neg \phi_p(t(b_{i_1}, \ldots, b_{i_n})).$$

We put $\Phi_{i+1} = \Phi_i \cup \{\neg \phi_p(t(c_{j_1}, \ldots, c_{j_n})) : p \in \Gamma, i_1 < \cdots < i_n < \omega\}$.

Case 2 ($i < \kappa$ is odd). Assume we have found $F_i$ and $\Phi_i$. We take new formula $\psi(x_1, \ldots, x_n) \in \{\psi_i : i < \omega, \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$, by claim 2, there is subsequence $F_{i+1}$ of $F_i$ as following property: for any $(M'_\alpha, B'_\alpha) \in F_{i+1}$ if $i_1 < \cdots < i_n < \mu$, $j_1 < \cdots < j_n < \mu$ and $b_{j_1}, b_{j_2} \in B_\alpha$,

$$M'_\alpha \models \psi(b_{i_1}, \ldots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \ldots, b_{j_n}).$$
We put $\Phi_{i+1} := \Phi_i \cup \{\psi(c_{i_1}, \ldots, c_{i_n}) \iff \psi(c_{j_1}, \ldots, c_{j_n}) : i_1 < \cdots < i_n < \omega, \ j_1 < \cdots < j_n < \omega\}$.

If put $\Phi := \bigcup_{i<\omega} \Phi_i$ then it is required $\mathcal{L}^\ast$-theory. We take any $(M_\alpha, A_\alpha) \in F := \cap_{i<\omega} F_i.$ By construction $M_\alpha \models \Phi.$

Let $A$ be the set of all interpretation $C = \{c_i : i < \omega\}$ in $M_\alpha$, $N$ be Skolem closure of $A$ in $M_\alpha$. Thus $N$ is model of $T$, omitting $\Gamma$, indiscernibles in $M_\alpha$, and $|N| = \omega$.

Take any $\lambda \geq \omega$. By stretching theorem, there is a model of $T$ which the cardinality of $\lambda$ such that omitting $\Gamma$. Note that if $|\Gamma| \leq \omega$ then it is sufficient $\mu = \omega_1$, see [2].

It is known that Morley's theorem is proved in infinitary logic, and it is effective means to show existence of models in infinitary logic that the compactness theorem is false generally, see [1], [3].

## 3 Related Result

The following result is related to Morley's omitting type theorem. This theorem says the thing that is stronger than Morley's theorem under a certain condition.

**Theorem.** (Tsuboi) Let $T$ be a countable complete $\mathcal{L}$-theory and $\Gamma$ a set of complete types with $|\Gamma| < 2^\omega$. Suppose that for each $\alpha < \omega_1$, there is a model $M_\alpha \models T$ with the following properties:

1. $|M_\alpha| > \beth_\alpha$,

2. $M_\alpha$ omits each member of $\Gamma$.

Then for each $\lambda \geq \omega$ there is a model $N$ omitting $\Gamma$ and with $|N| = \lambda$.

**Proof.** Let $X = \omega_1$ and $\{I_i : i \in X\}$ be a set of infinite indiscernible sequences and $\{t_n; n < \omega\}$ be an enumeration of all the $\mathcal{L}$-terms. We may assume that $t_n$ has $n$-variables. We will say that the set $\{I_i : i \in X\}$ is $t_n$-uniform if the following condition holds: If $i, j \in X$, then $tp(t_n(I_i)) = tp(t_n(I_j))$ where $tp(t_n(I_i)) := tp(t_n(a_0, \ldots, a_{n-1}))$ ($a_0 < \cdots < a_{n-1} \in I_i$). We will say that $\{I_i : i \in X\}$ is essentially $t_n$-uniform if there is an uncountable subset $Y$ of $X$ such that $\{I_i : i \in Y\}$ is $t_n$-uniform. For a formula $\phi(x)$, define $X^{\phi,t_n} := \{i \in X : \phi(x) \in tp(t_n(I_i))\}$. Put $X_\emptyset = \omega_1$, and for each $i \in X_\emptyset$ we fix a sequence $I_\emptyset(i)$ enumerating the universe $M_i$.

Using the argument in the paper([4]), for $\eta \in 2^{<\omega}$ and $k < \omega$, we can inductively choose $X_\eta \subset \omega_1$, $\{I_\eta(i) : i \in X_\eta\}$ and formulas $\phi_{\eta,k}$ with the following properties:

1. If $\eta < \nu$, then
   
   (a) $X_\nu$ is an uncountable subset of $X_\eta$;
   
   (b) $I_\nu(i)$ is a subsequence of $I_\eta(i)$ for each $i \in X_\nu$.

2. $i < j \Rightarrow |I_\eta(i)| < |I_\eta(j)|$, and $\sup\{|I_\eta(i)| : i \in X_\eta\} \geq \beth_\omega$. 


3. If $\eta \in 2^n$ then
   
   (a) each $I_\eta(i)$ is an infinite indiscernible sequence;
   (b) $\{I_\eta(i) : i \in X_\eta\}$: essentially $t_n$-uniform $\Rightarrow$ it is $t_n$-uniform.

4. If $\eta \in 2^n$ and $k \leq n$ then
   
   $\{I_\eta(i) : i \in X_\eta\}$:
   essentially $t_n^{-}$ uniform $\Rightarrow$ it is $t_n$-uniform.

For all $\nu \in 2^\omega$, we define the following:

1. $K_\nu$ is the set of all $n < \omega$ such that $\{I_{\nu|n}(i) : i \in X_{\nu|n}\}$ is not $t_n$-uniform;
2. for $n \in K_\nu$, $\Delta_\nu^n(x) := \bigcup_{n \leq m < \omega} \{\phi_{\nu|m,n}(x) : \nu(m) = 0\} \cup \bigcup_{n \leq m < \omega} \{\neg \phi_{\nu|m,n}(x) : \nu(n) = 1\};$
3. $\Phi_\nu := \{\{x_i\}_{i < \omega} : \text{is indiscernible}\} \cup \bigcup_{n \in K_\nu} \Delta_\nu^n(t_n(\overline{x}_n)) \cup \bigcup_{n \not\in K_\nu} p_{\nu|n}(t_n(\overline{x}_n));$
4. $F_\nu := \{(M_i^\nu, I_\nu(i)) : i \in X_\nu\}$ ($I_\nu(i) \subset M_i^\nu$).

We can take $\nu \in 2^\omega$ well, see [4], such that if $\{c_i : i < \omega\}$ realizing $\Phi_\nu$ in $M_0^\nu$, and $N$ is Skolem closure of $\{c_i : i < \omega\}$ in $M_0^\nu$ then $N$ omits $\Gamma$. The rest of the statement is clear from Stretching Theorem.

References