A note on lowness for Robinson theories (New developments of independence notions in model theory)

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A note on lowness for Robinson theories

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Abstract

We show following two theorems. Theorem A: for thick simple existentially universal domain, the equality of Lascar strong types is definable by an existential type. Theorem B: for thick low existentially universal domain, Lascar strong types equal strong types. Theorem A is already proved by Ben-Yaacov [2].

1 Preliminaries

Definition 1.1 We say that an $L$-structure $M$ is $\kappa$-existentially universal domain (e.u.domain) if

- if $\Sigma(x)$ is a partial existential type over $A (|A| < \kappa)$ which is finitely satisfiable in $M$, then $\Sigma$ is satisfiable in $M$, and

- for $|A|, |B| < \kappa$, and $f : A \rightarrow B$: a bijection such that $\text{etp}(a) \subset \text{etp}(f(a))$ for all tuples $a$ from $A$, $f$ extends to an automorphism of $M$.

Remark 1.1 An e.u.domain $M$ is an existentially closed model for the universal theory of $M$, $\text{Th}(M)_\forall$.

Let $\mathcal{M}$ be a $\kappa$-e.u.domain for a enoughly big cardinal $\kappa$. Put $T = \text{Th}_\forall(\mathcal{M})$. $M, N, \ldots$ denote existentially closed models of $T$, $a, b, \ldots$ denote finite tuples in $\mathcal{M}$, and $A, B, \ldots$ denote small subsets of $\mathcal{M}$.

Definition 1.2 Let $\Sigma(x, B)$ be an existential type over $B$.

1. We say that $\Sigma(x, B)$ divides over $A$ if there exists an existentially indiscernible sequence $(B_i : i < \omega)$ over $A$ with $B_0 = B$ such that $\bigcup_{i<\omega} \Sigma(x, B_i)$ is not realized in $\mathcal{M}$.
2. We say that $\Sigma(x)$ forks over $A$ if there exists a small set of dividing $(/A)$ existential formulas $\Psi$ (with parameters) such that $\mathcal{M} \models \Sigma \rightarrow \bigvee_i \psi_i$.

**Remark 1.2**

- If $\Sigma(x)$ divides over $A$, then there is an existential formula $\varphi(x)$ such that $\Sigma \vdash \varphi(x)$ and $\varphi(x)$ divides over $A$.

- It is not known whether if $\Sigma(x)$ divides over $A$, then there is an existential formula $\psi(x)$ such that $\Sigma \vdash \psi(x)$ and $\psi(x)$ divides over $A$.

**Definition 1.3**

We say that $\mathcal{M}$ is simple if for all $a \in \mathcal{M}$, $A \subset \mathcal{M}$, there exists $B \subset A$ with $|B| \leq |T| + \aleph_0$ such that etp$(a/A)$ does not fork over $B$.

**Fact 1.1** [3] Suppose that $\mathcal{M}$ is simple. Then, $\Sigma$ forks over $A$ if and only if $\Sigma$ divides over $A$.

**Definition 1.4**

1. We say that lstp$(a) = \text{lstp}(b)$ if for any bounded $\emptyset$-invariant equivalence relation $E(x, y), E(a, b)$ holds.

2. We say that $d(a, b) \leq 1$ if there is an existentially indiscernible sequence $I$ such that $a, b \in I$.

3. We say that $d(a, b) \leq n$ if there exist $a_0, \ldots, a_n$ with $a_0 = a, a_n = b$ such that $d(a_i, a_{i+1}) \leq 1$ for any $i < n$.

4. We say that $d(a, b) < \omega$ if $d(a, b) \leq n$ for some $n < \omega$.

**Fact 1.2** [3] lstp$(a) = \text{lstp}(b)$ if and only if $d(a, b) < \omega$.

**Fact 1.3** [3] If $(a_i : i < \lambda)$ is an enoughly long sequence and $A \subset \mathcal{M}$, then there is an existentially indiscernible sequence $(b_i : i < \omega)$ such that for any $n < \omega$, there are $i_0 < \cdots < i_{n-1} < \lambda$ such that etp$(b_0, \ldots, b_{n-1}/A) = \text{etp}(a_{i_0}, \ldots, a_{i_{n-1}}/A)$.

**Fact 1.4** [3] Suppose that $\mathcal{M}$ is simple. Then, for all $a, A \subset B$, there exists $a'$ such that

- lstp$(a'/A) = \text{lstp}(a/A)$ and

- etp$(a'/B)$ does not fork over $A$.

We write $a \downarrow b$ to mean that etp$(a/b)$ does not fork over $\emptyset$. 
Fact 1.5 (Independence theorem for simple e.u.domain, [3]) Suppose that $\mathcal{M}$ is simple and $a_1, a_2, b_1, b_2$ satisfy the following:

- $\text{lstp}(a_1) = \text{lstp}(a_2)$,
- $a_1 \downarrow b_1$, $a_2 \downarrow b_2$, $b_1 \downarrow b_2$.

Then, there exists $a$ such that

- $a \models \text{etp}(a_1/b_1) \cup \text{etp}(a_2/b_2)$
- $a \downarrow b_1 b_2$.

2 Proof of Thorem A

In this section, we prove Theorem A. For simplicity, we show over $\emptyset$.

Definition 2.1 We say that $\mathcal{M}$ is thick if "$d(x, y) \leq 1$" is definable by an existential type. If $\mathcal{M}$ is thick, then we assume that $q_1(x, y)$ defines "$d(x, y) \leq 1$".

Lemma 2.1 Suppose that $\mathcal{M}$ is thick. Then, "$d(x, y) \leq 2$" is definable by an existential type.

Proof: It is defined by $\{ \exists z \varphi(x, z) \land \varphi(z, y) | \varphi(x, y) \in q_1(x, y) \}$.

Lemma 2.2 Suppose that $\mathcal{M}$ is thick and simple. Then, the following are equivalent:

1. $\text{lstp}(a) = \text{lstp}(b)$
2. $d(a, b) \leq 2$
3. $q_1(x, a) \cup q_1(x, b)$ does not fork over $\emptyset$

Proof: $(3 \rightarrow 2 \rightarrow 1)$ is trivial. $(1 \rightarrow 2)$ Let $c$ be a tuple such that $\text{lstp}(c) = \text{lstp}(a) = \text{lstp}(b)$ and $c \downarrow ab$. Take $a'$ such that $\text{etp}(a'a) = \text{etp}(ac)$. Then $\text{lstp}(a') = \text{lstp}(a)$ and $a' \downarrow a$. So, by independence theorem, we can get $a_2$ such that $a_2 \models \text{etp}(a/c) \cup \text{etp}(a'/a)$ and $a_2 \downarrow ac$.

Iterating this, we can get a sequence $(a_i : i < \omega)$ such that $\text{etp}(a_ia_j) = \text{etp}(ac)$ for each $j < i < \omega$. By compactness and Fact 1.3, we can assume this sequence is existentially indiscernible. So, we get existentially indiscernible sequences $I, J$ such that $a, c \in I$ and $b, c \in J$. 
Theorem A [2] Suppose that $\mathcal{M}$ is thick and simple. Then, "$\text{lstp}(x) = \text{lstp}(y)$" is definable by an existential type.

Proof: By above lemmas.

3 Proof of Theorem B

In this section, we prove Theorem B. Again for simplicity, we show over $\emptyset$.

Definition 3.1 We say that $\text{stp}(a) = \text{stp}(b)$ if for any definable (by an existential formula over $\emptyset$) finite equivalence relation $E(x, y), E(a, b)$ holds.

Definition 3.2 1. Let $\varphi(x, y)$ be an existential formula. An existential formula $\psi(y_0, \ldots, y_{k-1})$ where $lh(y_i) = lh(y)$ for each $i < k$ is said to be a $k$-inconsistency witness for $\varphi$ if $\mathcal{M} \models \forall y_0 \cdots y_{k-1} (\psi(y_0, \ldots, y_{k-1}) \rightarrow -\exists x \bigwedge_{i<k} \varphi(x, y_i))$.

2. Let $\Sigma(x)$ be an existential type and $\varphi(x, y)$ be an existential formula.
   - We say that $D(\Sigma, \varphi) \geq 0$ if $\Sigma$ is satisfiable.
   - We say that $D(\Sigma, \varphi) \geq n + 1$ if there is a natural number $k$, a $k$-inconsistency witness $\psi$, and an existentially indiscernible sequence $(b_i : i < \omega)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi) \geq n$ for each $i < \omega$ and $\mathcal{M} \models \psi(b_{i_0}, \ldots, b_{i_{k-1}})$ for all $i_0, \ldots, i_{k-1} < \omega$.

3. We say that $\mathcal{M}$ is low if
   - $\mathcal{M}$ is simple and
   - $D(x = x, \varphi) < \omega$ for any existential formula $\varphi$.

Lemma 3.1 Suppose that $\mathcal{M}$ is thick and low. Then,

1. $\{a : \varphi(x, a) \text{ divides over } \emptyset\}$ is definable by an existential type.

2. $\{(a, b) : \varphi(x, a) \land \varphi(x, b) \text{ does not divide over } \emptyset\}$ is definable by an existential type if it is restricted to $(p \otimes p)^{\mathcal{M}} = \{(a, b) : a, b \models p, a \downarrow b\}$. So, it is definable by an existential universal formula if it is restricted to $(p \otimes p)^{\mathcal{M}}$.

Proof: (1) Note that by lowness, for any $\varphi(x, y)$ there is an existential formula $\psi$ such that for all $a$, if $\varphi(x, a)$ divides over $\emptyset$, then $\varphi$ divides by an existentially indiscernible sequence in which any $k$-elements satisfies $\psi$.

(2) For $a, b \models p$ where $a \downarrow b$, the following are equivalent:
1. \( \varphi(x, a) \land \varphi(x, b) \) does not divide over \( \emptyset \)

2. there exist \( a^* \) and \( b^* \) such that

- \( \mathcal{M} \models \varphi(a^*, a) \) and \( a^* \parallel a \);
- \( \mathcal{M} \models \varphi(b^*, b) \) and \( b^* \parallel b \);
- \( \text{lstp}(a^*) = \text{lstp}(b^*) \)

By Theorem A, "\( \text{lstp}(a^*) = \text{lstp}(b^*) \)" is expressible by an existential type. "\( a^* \parallel a \)" is expressible by "\( D(\text{etp}(a/a^*), \varphi, \psi) \geq D(p, \varphi, \psi) \)" for any \( \varphi, \psi \).

We sat that \( E_{p(x), \varphi(x, y)}(b, c) \) if for all \( a \models p \) with \( a \vdash bc \), \( \varphi(x, a) \land \varphi(x, b) \) does not divide over \( \emptyset \) if and only if \( \varphi(x, a) \land \varphi(x, c) \) does not divide over \( \emptyset \).

**Lemma 3.2** Suppose that \( \mathcal{M} \) is thick and low. For any \( a \models p \) where \( \varphi(x, a) \) does not divide over \( \emptyset \), \( E_{p(x), \varphi(x, y)} \) is a definable (by an existential formula) finite equivalence relation on \( (p^2)^{\mathcal{M}} \).

**Proof:** We can check that \( E_{p, \varphi} \) is a bounded equivalence relation boundedness is by "\( \text{lstp}(x) = \text{lstp}(y) \Rightarrow E_{p, \varphi}(x, y) \)". On the other hand, by the above lemma \( \neg E_{p, \varphi} \) is definable by an existential type. So, \( E_{p, \varphi} \) is a finite equivalence relation. Let \( a_1, \ldots, a_n \) be representations of classes. Then \( \bigcup \{ \neg E(x, a_i) : i \leq n \} \) is not satisfiable. For simplicity, we assume \( n = 3 \). There exists an existential formula \( \varphi(x, y) \) such that

1. \( \neg E(x, a_i) \vdash \varphi(x, a_i) \) for each \( i \leq 3 \)

2. \( \mathcal{M} \models \neg \exists x \varphi(x, a_1) \land \varphi(x, a_2) \land \varphi(x, a_3) \).

Put \( \psi(x, y) = \neg \varphi(x, y) \). Note that \( \mathcal{M} \models \forall x (\psi(x, a_1) \leftrightarrow \varphi(x, a_2) \land \varphi(x, a_3)) \).

So, \( \psi(x, a_1) \) is also existential. By a symmetric argument, \( \psi(x, a_2), \psi(x, a_3) \) are all existential. Then we have

\[
E(x, y) \leftrightarrow \bigwedge_{i \leq 3} (\psi(x, a_i) \leftrightarrow \psi(y, a_i)).
\]

We can omit parameters \( a_i \)'s because this does not depend on a choice of representations and \( \psi(x, a_i) \) is existential universal.

**Theorem B** Suppose that \( \mathcal{M} \) is thick and low. Then, \( \text{stp} = \text{lstp} \)

**Proof:** If \( \text{stp}(a) = \text{stp}(b) \), then by the above lemma \( a, b \models E_{p, \varphi} \) for any \( \varphi \). Take \( c \) such that \( \text{lstp}(c) = \text{lstp}(a) \) and \( c \parallel ab \). Then, \( q_1(x, a) \cup q_1(x, c) \) does not divide by Lemma 3. Then, \( q_1(x, b) \cup q_1(x, c) \) does not divide by \( E_{p, \varphi}(a, b) \). Again by Lemma 3, we have \( \text{lstp}(b) = \text{lstp}(c) \).
References


