A note on lowness for Robinson theories

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Abstract
We show following two theorems. Theorem A: for thick simple existentially universal domain, the equality of Lascar strong types is definable by an existential type. Theorem B: for thick low existentially universal domain, Lascar strong types equal strong types. Theorem A is already proved by Ben-Yaacov [2].

1 Preliminaries

Definition 1.1 We say that an \( L \)-structure \( M \) is \( \kappa \)-existentially universal domain (e.u.domain) if

- if \( \Sigma(x) \) is a partial existential type over \( A \) (\( |A| < \kappa \)) which is finitely satisfiable in \( M \), then \( \Sigma \) is satisfiable in \( M \), and

- for \( |A|, |B| < \kappa \), and \( f : A \to B \) : a bijection such that \( \text{etp}(a) \subset \text{etp}(f(a)) \) for all tuples \( a \) from \( A \), \( f \) extends to an automorphism of \( M \).

Remark 1.1 An e.u.domain \( M \) is an existentially closed model for the universal theory of \( M \), \( \text{Th}(M)_\forall \).

Let \( M \) be a \( \kappa \)-e.u.domain for a enoughly big cardinal \( \kappa \). Put \( T = \text{Th}(M)_\forall \). \( M, N, \ldots \) denote existentially closed models of \( T \), \( a, b, \ldots \) denote finite tuples in \( M \), and \( A, B, \ldots \) denote small subsets of \( M \).

Definition 1.2 Let \( \Sigma(x, B) \) be an existential type over \( B \).

1. We sat that \( \Sigma(x, B) \) divides over \( A \) if there exists an existentially indiscernible sequence \( (B_i : i < \omega) \) over \( A \) with \( B_0 = B \) such that \( \bigcup_{i<\omega} \Sigma(x, B_i) \) is not realized in \( M \).
2. We say that $\Sigma(x)$ forks over $A$ if there exists a small set of dividing $(/A)$ existential formulas $\Psi$ (with parameters) such that $\mathcal{M} \models \Sigma \rightarrow \bigvee \Psi$.

**Remark 1.2**

- If $\Sigma(x)$ divides over $A$, then there is an existential formula $\varphi(x)$ such that $\Sigma \vdash \varphi(x)$ and $\varphi(x)$ divides over $A$.

- It is not known whether if $\Sigma$ forks over $A$, then there are an existential formula $\theta$ where $\Sigma \vdash \theta$ and dividing $(/A)$ existential formulas $\psi_1, \ldots, \psi_n$ such that $\mathcal{M} \models \theta \rightarrow \bigvee_{i=1}^{n} \psi_i$.

**Definition 1.3** We say that $\mathcal{M}$ is simple if for all $a \in \mathcal{M}$, $A \subset \mathcal{M}$, there exists $B \subset A$ with $|B| \leq |T| + \aleph_0$ such that $\text{etp}(a/A)$ does not fork over $B$.

**Fact 1.1** [3] Suppose that $\mathcal{M}$ is simple. Then, $\Sigma$ forks over $A$ if and only if $\Sigma$ divides over $A$.

**Definition 1.4**

1. We say that $\text{lstp}(a) = \text{lstp}(b)$ if for any bounded $\emptyset$-invariant equivalence relation $E(x, y), E(a, b)$ holds.

2. We say that $d(a, b) \leq 1$ if there is an existentially indiscernible sequence $I$ such that $a, b \in I$.

3. We say that $d(a, b) \leq n$ if there exist $a_0, \ldots, a_n$ with $a_0 = a, a_n = b$ such that $d(a_i, a_{i+1}) \leq 1$ for any $i < n$.

4. We say that $d(a, b) < \omega$ if $d(a, b) \leq n$ for some $n < \omega$.

**Fact 1.2** [3] $\text{lstp}(a) = \text{lstp}(b)$ if and only if $d(a, b) < \omega$.

**Fact 1.3** [3] If $(a_i : i < \lambda)$ is an enoughly long sequence and $A \subset \mathcal{M}$, then there is an existentially indiscernible sequence $(b_i : i < \omega)$ such that for any $n < \omega$, there are $i_0 < \cdots < i_{n-1} < \lambda$ such that $\text{etp}(b_0, \ldots, b_{n-1}/A) = \text{etp}(a_{i_0}, \ldots, a_{i_{n-1}}/A)$.

**Fact 1.4** [3] Suppose that $\mathcal{M}$ is simple. Then, for all $a, A \subset B$, there exists $a'$ such that

- $\text{lstp}(a'/A) = \text{lstp}(a/A)$ and

- $\text{etp}(a'/B)$ does not fork over $A$.

We write $a \downarrow b$ to mean that $\text{etp}(a/b)$ does not fork over $\emptyset$. 
Fact 1.5 (Independence theorem for simple e.u.domain, [3]) Suppose that $\mathcal{M}$ is simple and $a_1, a_2, b_1, b_2$ satisfy the following:

- lstp($a_1$) = lstp($a_2$),
- $a_1 \downarrow b_1$, $a_2 \downarrow b_2$, $b_1 \downarrow b_2$.

Then, there exists a such that

- $a \models$ etp($a_1/b_1$) $\cup$ etp($a_2/b_2$)
- $a \downarrow b_1b_2$.

2 Proof of Theorem A

In this section, we prove Theorem A. For simplicity, we show over $\emptyset$.

Definition 2.1 We say that $\mathcal{M}$ is thick if "$d(x, y) \leq 1$" is definable by an existential type. If $\mathcal{M}$ is thick, then we assume that $q_1(x, y)$ defines "$d(x, y) \leq 1$".

Lemma 2.1 Suppose that $\mathcal{M}$ is thick. Then, "$d(x, y) \leq 2$" is definable by an existential type.

Proof: It is defined by $\{\exists z \varphi(x, z) \land \varphi(z, y) | \varphi(x, y) \in q_1(x, y)\}$.

Lemma 2.2 Suppose that $\mathcal{M}$ is thick and simple. Then, the following are equivalent:

1. lstp($a$) $=$ lstp($b$)
2. $d(a, b) \leq 2$
3. $q_1(x,a) \cup q_1(x,b)$ does not fork over $\emptyset$

Proof: $(3 \rightarrow 2 \rightarrow 1)$ is trivial. $(1 \rightarrow 2)$ Let $c$ be a tuple such that lstp($c$) = lstp($a$) = lstp($b$) and $c \downarrow ab$. Take $a'$ such that etp($a'a$) = etp($ac$). Then lstp($a'$) = lstp($a$) and $a' \downarrow a$. So, by independence theorem, we can get $a_2$ such that $a_2 \models$ etp($a/c$) $\cup$ etp($a'/a$) and $a_2 \downarrow ac$.

Iterating this, we can get a sequence $(a_i : i < \omega)$ such that etp($a_i a_j$) = etp($ac$) for each $j < i < \omega$. By compactness and Fact 1.3, we can assume this sequence is existentially indiscernible. So, we get existentially indiscernible sequences $I, J$ such that $a, c \in I$ and $b, c \in J$.  

Theorem A [2] Suppose that $\mathcal{M}$ is thick and simple. Then, "$\text{lstp}(x) = \text{lstp}(y)$" is definable by an existential type.

Proof: By above lemmas.

3 Proof of Theorem B

In this section, we prove Theorem B. Again for simplicity, we show over $\emptyset$.

Definition 3.1 We say that $\text{stp}(a) = \text{stp}(b)$ if for any definable (by an existential formula over $\emptyset$) finite equivalence relation $E(x, y)$, $E(a, b)$ holds.

Definition 3.2 1. Let $\varphi(x, y)$ be an existential formula. An existential formula $\psi(y_0, \ldots, y_{k-1})$ where $\text{lh}(y_i) = \text{lh}(y)$ for each $i < k$ is said to be a $k$-inconsistency witness for $\varphi$ if $\mathcal{M} \models \forall y_0 \ldots y_{k-1} (\psi(y_0, \ldots, y_{k-1}) \rightarrow \neg \exists x \bigwedge_{i<k} \varphi(x, y_i))$.

2. Let $\Sigma(x)$ be an existential type and $\varphi(x, y)$ be an existential formula.

- We say that $D(\Sigma, \varphi) \geq 0$ if $\Sigma$ is satisfiable.
- We say that $D(\Sigma, \varphi) \geq n + 1$ if there is a natural number $k$, a $k$-inconsistency witness $\psi$, and an existentially indiscernible sequence $(b_i : i < \omega)$ such that $D(\Sigma(x) \cup \{\varphi(x, b_i)\}, \varphi) \geq n$ for each $i < \omega$ and $\mathcal{M} \models \psi(b_{i_0}, \ldots b_{i_{k-1}})$ for all $i_0, \ldots, i_{k-1} < \omega$.

3. We say that $\mathcal{M}$ is low if

- $\mathcal{M}$ is simple and
- $D(x = x, \varphi) < \omega$ for any existential formula $\varphi$.

Lemma 3.1 Suppose that $\mathcal{M}$ is thick and low. Then,

1. $\{a : \varphi(x, a) \text{ divides over } \emptyset\}$ is definable by an existential type.

2. $\{(a, b) : \varphi(x, a) \land \varphi(x, b) \text{ does not divide over } \emptyset\}$ is definable by an existential type if it is restricted to $(p \otimes p)^{\mathcal{M}} = \{(a, b) : a, b \models p, a \downarrow b\}$. So, it is definable by an existential universal formula if it is restricted to $(p \otimes p)^{\mathcal{M}}$

Proof: (1) Note that by lowness, for any $\varphi(x, y)$ there is an existential formula $\psi$ such that for all $a$, if $\varphi(x, a)$ divides over $\emptyset$, then $\varphi$ divides by an existentially indiscernible sequence in which any $k$-elements satisfies $\psi$.

(2) For $a, b \models p$ where $a \downarrow b$, the following are equivalent:
1. $\varphi(x, a) \land \varphi(x, b)$ does not divide over $\emptyset$

2. there exist $a^*$ and $b^*$ such that
   
   - $\mathcal{M} \models \varphi(a^*, a)$ and $a^* \not\vdash a$;
   - $\mathcal{M} \models \varphi(b^*, b)$ and $b^* \not\vdash b$;
   - lstp($a^*$) = lstp($b^*$)

By Theorem A, "lstp($a^*$) = lstp($b^*$)" is expressible by an existential type. "$a^* \not\vdash a$" is expressible by "$D(\text{etp}(a/a^*), \varphi, \psi) \geq D(p, \varphi, \psi)$" for any $\varphi, \psi$.

We sat that $E_{p(x), \varphi(x,y)}(b, c)$ if for all $a \models p$ with $a \not\vdash bc$, $\varphi(x, a) \land \varphi(x, b)$ does not divide over $\emptyset$ if and only if $\varphi(x, a) \land \varphi(x, c)$ does not divide over $\emptyset$.

**Lemma 3.2** Suppose that $\mathcal{M}$ is thick and low. For any $a \models p$ where $\varphi(x, a)$ does not divide over $\emptyset$, $E_{p(x), \varphi(x,y)}$ is a definable (by an existential formula) finite equivalence relation on $(p^2)\mathcal{M}$.

**Proof:** We can check that $E_{p, \varphi}$ is a bounded equivalence relation boundedness is by "lstp($x$) = lstp($y$) $\Rightarrow E_{p, \varphi}(x, y)$". On the other hand, by the above lemma $\neg E_{p, \varphi}$ is definable by an existential type. So, $E_{p, \varphi}$ is a finite equivalence relation. Let $a_1, \ldots, a_n$ be representations of classes. Then $\bigcup\{\neg E(x, a_i) : i \leq n\}$ is not satisfiable. For simplicity, we assume $n = 3$. There exists an existential formula $\varphi(x, y)$ such that

1. $\neg E(x, a_i) \vdash \varphi(x, a_i)$ for each $i \leq 3$

2. $\mathcal{M} \models \neg \exists x \varphi(x, a_1) \land \varphi(x, a_2) \land \varphi(x, a_3)$.

Put $\psi(x, y) = \neg \varphi(x, y)$. Note that $\mathcal{M} \models \forall x (\psi(x, a_1) \leftrightarrow \varphi(x, a_2) \land \varphi(x, a_3))$. So, $\psi(x, a_1)$ is also existential. By a symmetric argument, $\psi(x, a_2), \psi(x, a_3)$ are all existential. Then we have

$$E(x, y) \leftrightarrow \bigwedge_{i \leq 3} (\psi(x, a_i) \leftrightarrow \psi(y, a_i)).$$

We can omit parameters $a_i$'s because this does not depend on a choice of representations and $\psi(x, a_i)$ is existential universal.

**Theorem B** Suppose that $\mathcal{M}$ is thick and low. Then, $\text{stp} = \text{lstp}$

**Proof:** If $\text{stp}(a) = \text{stp}(b)$, then by the above lemma $a, b \models E_{p, \varphi}$ for any $\varphi$. Take $c$ such that $\text{lstp}(c) = \text{lstp}(a)$ and $c \not\vdash ab$. Then, $q_1(x, a) \cup q_1(x, c)$ does not divide by Lemma 3. Then, $q_1(x, b) \cup q_1(x, c)$ does not divide by $E_{p, \varphi}(a, b)$. Again by Lemma 3, we have $\text{lstp}(b) = \text{lstp}(c)$. 

References


