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On generic automorphisms of a tree structure

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Abstract

We give a theory $T$ with the strict order property such that for some automorphism $\sigma_0$ of a prime model $M_0$ of $T$, the theory

$$T + \ "\sigma \text{ is an automorphism} + \ "\sigma|M_0 = \sigma_0"$$

is model complete. Note that $T + \ "\sigma \text{ is an automorphism}"$ has no model companion if $T$ has the strict order property [3]. This seems to have some resemblance with the theory of the rings of Witt Vectors carrying the Frobenius automorphism [1].

We consider each natural number $n$ as the set $\{0, 1, \ldots, n - 1\}$. Consider a structure $(M_0, <)$ with

$$M_0 = \{ f : n \to n + 1 \mid n < \omega, \ f(i) < i + 1 \text{ for } i < n \},$$

and $f < g$ if $g$ is a proper extension of $f$ as a map for $f, g \in M_0$.

For each $f \in M_0$ with $\text{dom} \ f = n$, let $f^s$ be a map such that

$$f^s(i) = (f(i) + 1) \mod (i + 1)$$

for $i < n$. Then the map $s : M_0 \to M_0$ defined by $s(f) = f^s$ is an automorphism of $(M_0, <)$. $\epsilon$ denotes the least element of $M_0$ (i.e., $\epsilon$ is the empty sequence). Let $<_1$ be a definable relation on $M_0$ defined by the formula

$$x < y \land \forall z \neg(x < z < y).$$

Let $T_0$ be the theory of $(M_0, <, <_1)$. Note that for any model $M$ of $T_0$, $\text{acl}_M(\emptyset) = M_0$. The root (the least element) of $M_0$ will be denoted by $\epsilon$. 
Proposition 1. Let $M$ be a model of $T_0$. Then the following sentences are valid in $M$:

(1) $\forall x \exists y \ x <_1 y$.
(2) $\forall x, y \ x < y \rightarrow \exists z \ x <_1 z \leq y$.
(3) $\forall x, y \ x < y \rightarrow \exists z \ x \leq z <_1 y$.
(4) $\forall x, y, z \ x, y \leq z \rightarrow x < y \lor x = y \lor y < x$.
(5) $\forall x, y \exists u, v \ x \not\leq y \rightarrow u <_1 v \leq x \land u \leq y \land v \not\leq y$.
(6) Let $n$ be any natural number. If $x < y$ and $x$ has (at least) $n$ childs then $y$ has (at least) $n + 1$ childs.

Theorem 2. The theory

$$T_0 \cup \{ \sigma \text{ is a } <\text{-automorphism extending } s \}$$

in the language $\{<,<_1, \sigma\} \cup M_0$ has a model companion. In fact, it is model complete.

We fix models $M \subset M'$ of $T$ and assume that $\sigma$ is a $<\text{-automorphism of } M'$ extending $s$ and $M$ is $\sigma$-invariant.

Lemma 3. If $a, b \in M$ then $\inf_M\{a, b\} = \inf_{M'}\{a, b\}$.

Proof. Let $c = \inf_M\{a, b\}$. If $c = a$ or $c = b$ then there is nothing to prove.

Suppose $c < a, b$. Then we can choose $c_a, c_b \in M$ such that $c <_1 c_a \leq a$, $c <_1 c_b \leq b$, and $c_a$ is incomparable with $c_b$. Now, we show that $c = \inf_{M'}\{a, b\}$. Let $d \in M' - M$ be such that $d < a, b$. Then $d$ is comparable with both $c_a$ and $c_b$. Only the case $d < c_a, c_b$ is possible. Therefore, $d < c$.

Definition 4. Suppose $a, b \in M' - M$. We say that $a$ and $b$ are dependent over $M$ if there is $c \in M' - M$ such that $c \leq a$ and $c \leq b$. We call such $c$ a witness of the dependence. $a$ and $b$ are dependent over $M$ if and only if $\inf\{a, b\} \in M' - M$.

We say that $a$ and $b$ are independent over $M$ if $a$ and $b$ are not dependent over $M$.

Lemma 5. The dependence over $M$ is an equivalence relation on $M' - M$.

Proof. The reflexivity and the symmetry are trivial. We show the transitivity. Suppose $b$ and $c$ are dependent over $M$ with a witness $u$, and $c$ and $d$ are dependent over $M$ with a witness $v$. Since $u \leq c$ and $v \leq c$, $u$ and $v$ are comparable. Without loss of generality, we can assume that $u \leq v$. Then $u \leq v \leq d$. Therefore, $b$ and $d$ are dependent over $M$ with a witness $u$.

Lemma 6. If $b \in M' - M_0$ then $b$ and $\sigma^mb$ are independent over $M_0$ for any integer $m \neq 0$. 
Proof. Let \( m \neq 0 \) be an integer and \( b \in M' - M \). Choose \( f < b \) such that \( f \in M_0 \) and \( \text{dom } f \supset m \). Then \( f \) and \( s^m f \) are incomparable and also \( s^m f < \sigma^m b \).

Suppose there is \( a \in M' - M_0 \) such that \( a < b \) and \( a \leq \sigma^m b \). \( f \) and \( a \) are comparable by \( f < b \) and \( a \leq b \). Since \( f \) has a finite distance from the root, we have \( f < a \). Similarly, \( s^m f < a \). Therefore, \( f \) and \( s^m f \) are comparable. A contradiction.

Corollary 7. If \( a, b \in M' - M \) are dependent over \( M \) then \( a \) and \( \sigma^m b \) are independent over \( M \) for any integer \( m \neq 0 \).

Proof. Suppose \( a, b \in M' - M \) are dependent over \( M \) and \( a \) and \( \sigma^m b \) are dependent over \( M \) for some integer \( m \neq 0 \). Suppose \( c \leq a, c \leq b \) with \( c \in M' - M \), and \( d \leq a, d \leq \sigma^m b \) with \( d \in M' - M \).

Since \( c, d \leq a \) and \( d \) are comparable. Therefore, \( \min\{c, d\} \leq \inf\{b, \sigma^m b\} \), and hence \( \inf\{b, \sigma^m b\} \in M' - M \) contradicting Lemma 6.

Definition 8. Suppose \( a, b \in M' - M \). We say that \( a \) and \( b \) are quasi-connected over \( M \) if there is \( c \in M' \) such that

1. \( M' \models c \leq a, b \),
2. \( M' \models c \leq y \leq a \) implies \( y \in M' - M \), and
3. \( M' \models c \leq y \leq b \) implies \( y \in M' - M \).

We call \( c \) a witness of this property. Note that if \( a \) and \( b \) are quasi-connected over \( M \) then it is dependent over \( M \).

Lemma 9. The quasi-connectedness over \( M \) is an equivalence relation on \( M' - M \).

Proof. The reflexivity and the symmetry are trivial. We show the transitivity. Suppose \( b \) and \( c \) are quasi-connected over \( M \) with a witness \( u \) and \( c \) and \( d \) are quasi-connected over \( M \) with a witness \( v \). Since \( u \leq c \) and \( v \leq c \), \( u \) and \( v \) are comparable. Without loss of generality, we can assume that \( u \leq v \). We show that \( u \) is a witness for quasi-connectedness of \( b \) and \( d \) over \( M \). If \( u \leq w \leq b \) then \( w \in M' - M \) since \( u \) is a witness for quasi-connectedness of \( b \) and \( c \).

Suppose \( u \leq w \leq d \). Then \( w \) and \( v \) are comparable. If \( w \leq v \) then \( u \leq w \leq c \) and thus \( w \in M' - M \). If \( v < w \) then \( v \leq w \leq d \) and thus \( w \in M' - M \).

Lemma 10. Suppose that \( B \) is a finite subset of \( M' - M \) quasi-connected over \( M \), \( a_1, \ldots, a_m \in M \) and for each \( a_i \) there is \( b_i \in B \) such that \( b_i < a_i \). Then there is \( b \in B \) such that \( b < \inf\{a_1, \ldots, a_m\} \).

Proof. Let \( a = \inf\{a_1, \ldots, a_m\} \) in \( M \). Then \( a = \inf\{a_1, \ldots, a_m\} \) in \( M' \) by Lemma 3.

Let \( b = \inf B \) in \( M' \). We have \( b \in M' - M \) because \( B \) is quasi-connected over \( M \). Since \( b \) is a lower bound for \( \{a_1, \ldots, a_m\} \), we have \( b \leq a \). Choose \( b_1 \in B \) such that \( b_1 < a_1 \). Then \( b_1 \) and \( a \) are comparable. If \( a \leq b_1 \) then \( b \leq a \leq b_1 \), but this cannot happen since there is no element \( y \in M \) such that \( b \leq y \leq b_1 \). Therefore, \( b_1 < a \).
Lemma 11.  (1) Suppose $M' \models a <_{1} b$ with $a \in M$ and $b \in M' - M$. Then there is no $a' \in M$ such that $M' \models b < a'$.

(2) If $b \in M' - M$ then there is no $a \in M$ such that $M' \models b <_{1} a$.

Proof. (1) Suppose $M' \models a <_{1} b < a'$ with $a, a' \in M$ and $b \in M' - M$. Then there must be $a'' \in M$ such that $M \models a < a'' < a'$, and thus $M' \models a < a'' < a'$. But this cannot happen because $b \neq a''$.

(2) Suppose there is $b \in M' - M$ and $a \in M$ such that $M' \models b <_{1} a$. Since $M' \models \epsilon < a$, we have $M \models \epsilon < a$. Therefore, $M \models a' <_{1} a$ for some $a' \in M$ and thus $M' \models a' <_{1} a$. But this cannot happen because $b \neq a'$.

Definition 12. Suppose $C$ and $D$ are subsets of $M'$. We write $C < D$ if there is $c \in C$ such that $c \leq d$ for any $d \in D$.

Definition 13. A finite subset $X$ of $M' - M$ is called canonical if the following conditions are satisfied:

(1) For any $x, y \in X$, whenever $x$ and $\sigma^m(y)$ with $m \in \mathbb{Z}$ are dependent over $M$ then $m = 0$;

(2) if $x, y \in X$ are dependent over $M$ then there is $z \in X$ witnessing the dependence; and

(3) if $x, y \in X$ are quasi-connected over $M$ then there is $z \in X$ witnessing the quasi-connectedness.

Definition 14. Let $B$ be a subset of $M'$. $\langle B \rangle_{\sigma}$ denotes the set $\{\sigma^m(b) \mid b \in B, m \in \mathbb{Z}\}$.

Lemma 15. For any finite subset $X \subset M' - M$ there is a canonical subset $Z \subset M' - M$ such that $X \subset \langle Z \rangle_{\sigma}$.

Proof. We prove the statement by induction on the number of elements in $X$. It is trivial if $|X| = 0$. Suppose $X = \{a\} \cup X'$ with $|X'| < |X|$. By the induction hypothesis, there is a canonical subset $Y'$ of $M' - M$ such that $X' \subset \langle Y' \rangle_{\sigma}$.

We split the proof into the following cases.

Case 1. $\sigma^m a$ and $b$ are quasi-connected over $M$ for some $b \in Y'$ and an integer $m$.

Let $b_0$ be the least element in $Y'$ which is quasi-connected to $\sigma^m a$ over $M$. Let $c = \inf \{\sigma^m a, b_0\}$. We claim that $Y = Y' \cup \{\sigma^m a, c\}$ is canonical and has the desired property.

Let $C_{b_0}$ be the quasi-connected component of $Y$ containing $b_0$ and $D_{b_0}$ be the dependent component of $Y$ containing $b_0$. It is easy to see that $\{c\} \cup C_{b_0}$ is a tree. $\{c\} \cup D_{b_0}$ is also a tree. Let $d$ be the least element of $D_{b_0}$. Since $c \leq b_0$ and $d \leq b_0$, $c$ and $d$ are comparable. Therefore, $\{c\} \cup D_{b_0}$ is a tree.
Now, suppose that $\sigma^{m_1}a$ and $b \in Y'$ are dependent over $M$. Then $\sigma^l b_0$ and $\sigma^{m_1}a$ are dependent over $M$ and thus $\sigma^l b_0$ and $b \in Y'$ are dependent over $M$. Since $Y'$ is canonical, we have $l = 0$.

Case 2. Case 1 does not hold but $\sigma^{m}a$ and $b$ are dependent over $M$ for some $b \in Y'$ and an integer $m$.

Let $b_0$ be the least element in $Y'$ which is dependent to $\sigma^{m}a$ over $M$. Choose a witness $c \in M' - M$ of dependence of $b_0$ and $\sigma^{m}a$. $Y = Y' \cup \{\sigma^{m}a, c\}$ is canonical and has the desired property. The argument is the same as that for Case 1.

Case 3. There is no integer $m$ and $b \in Y'$ such that $\sigma^{m}a$ and $b$ are dependent over $M$. In this case, $Y = Y' \cup \{a\}$ is canonical and has the desired property. ☐

**Lemma 16.** Suppose $\{t_1, \ldots, t_n\} \subset M' - M$ is canonical. Then any formula in $\text{qftp}_{\{<,1}\}}(t_1, \ldots, t_n/M)$ is realised in $M$.

*Proof.* Suppose $\{t_1, \ldots, t_n\} \subset M' - M$ is canonical. Let $t$ be the tuple $(t_1, \cdots, t_n)$ and $\varphi(x)$ a formula with $x = (x_1, \ldots, x_n)$ belonging to $\text{qftp}_{\{<,1\}}(t/M)$. Let $N$ be a natural number such that if $\sigma^{m_1}(x_i)$ occurs in $\varphi(x)$ then $m \leq N$. Let $A$ be a finite subset of $M$ such that $\varphi(x)$ is over $A$.

By adding finitely many points of $M$ to $A$ if necessary, we can assume the following:

- If $C$ is a quasi-connected component of $t$ then $\{a\} < C$ for some $a \in A$;
- if $C$ and $C'$ are two quasi-connected components of $t$ with $C < C'$ then there is $a \in A$ such that $C < \{a\} < C'$;
- if $C$ is a quasi-connected component of $t$ and there is $a \in M$ and $c \in M' - M$ quasi-connected to $C$ over $M$ such that $a <_1 c$ then $a \in A$ and $c \in C$;
- if $C$ is a quasi-connected component of $t$ such that $\{a \in A \mid C < \{a\}\}$ is non-empty then $\inf \{a \in A \mid C < \{a\}\} \in A$;
- if $a \in A$ is comparable with $t_i$ for some $i$ then $\sigma^{m_1}(a) \in A$ for $m \leq N$; and
- if $a \in A$ is comparable with $\sigma^{m_1}(t_i)$ for some $i$ and a natural number $m \leq N$ then $\sigma^{-m_1}(a) \in A$.

We can assume that $t = C_1 \cdots C_l$ where each $C_i$ is an enumeration of a quasi-connected component of $t$.

Let $a_i$ be the maximum element in $A$ such that $\{a_i\} <_1 C_i$ and $b_i$ be the minimum element in $A$ such that $C_i < \{b_i\}$. Such $a_i$ exists by the assumption on $A$ and such $b_i$ exists if there is $b \in A$ such that $C_i < \{b\}$ by Lemma 10 and the assumption on $A$.

Suppose that there are infinitely many elements $d$ of $M$ connected to $a_i$ such that $a_i < d < C_i$. Choose $a'_i \in M$ connected to $a_i$ with the following properties:

- If $x \in A$ and $M \models \sigma^{m_1}b_1 \not\leq x$ with $0 \leq m \leq N$ then $M \models \sigma^{m_1}a_i \not\leq x$; and
- if $C'$ is a quasi-connected component of $t$ such that $C_i \not\leq C'$ then $\{a'_i\} \not\leq C'$.
In the case that $b_i$ exists, choose a tuple $C_i'$ from $M$ such that \( \text{qftp}_{\langle,\langle} (C_i/a'_i, b_i) = \text{qftp}_{\langle,\langle} (C_i'/a'_i, b_i) \). Then we have \( \text{qftp}_{\langle,\langle} (\sigma^m C_i/A) = \text{qftp}_{\langle,\langle} (\sigma^m C_i'/A) \) for \( m = 0, 1, \ldots, N \).

In the case that there is no such $b_i$ for $C_i$, choose a tuple $C_i'$ from $M$ such that \( \text{qftp}_{\langle,\langle} (C_i/a'_i) = \text{qftp}_{\langle,\langle} (C_i'/a'_i) \). Then we have \( \text{qftp}_{\langle,\langle} (\sigma^m C_i/A) = \text{qftp}_{\langle,\langle} (\sigma^m C_i'/A) \) for \( m = 0, 1, \ldots, N \).

Let \( t' = C_1' \cdots C_k' \).

**Claim 1.** \( \text{qftp}_{\langle,\langle} (t^\sigma \sigma^2 t^\sigma \cdots \sigma^N t/A) = \text{qftp}_{\langle,\langle} (t^\sigma t^\sigma^2 t^\sigma \cdots \sigma^N t'/A) \)

\[ \square \]

**Proof of Theorem 2.** We show that \((M, <, \sigma|M)\) is existentially closed in \((M', <, \sigma)\).

Choose a finite tuple \((t_1, \ldots, t_n)\) from \(M' - M\) and let \( \varphi(x_1, \ldots, x_n) \) be a quantifier-
free formula of \(\langle, \sigma \rangle \cup M\) realised by \((t_1, \ldots, t_n)\). By Lemma 15, we can choose \(t_1', \ldots, t_n' \in M' - M\) such that \( t_i = \sigma^{k_i}(t'_i) \) for each \( i \) with some \( k_i \geq 0 \) and the set \( \{t_1', \ldots, t_n'\} \) is canonical. We have

\[
M' \models \varphi(\sigma^{k_1}(t_1'), \ldots, \sigma^{k_n}(t_n')).
\]

By Lemma 16, we can choose \(t_1'', \ldots, t_n'' \in M\) such that

\[
M \models \varphi(\sigma^{k_1}(t_1''), \ldots, \sigma^{k_n}(t_n'')).
\]

Therefore, \( \varphi(x_1, \ldots, x_n) \) is realised in \( M \).

\[ \square \]

**References**

