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On generic automorphisms of a tree structure

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Abstract

We give a theory $T$ with the strict order property such that for some automorphism $\sigma_0$ of a prime model $M_0$ of $T$, the theory

$$T + \"\sigma \text{ is an automorphism}\" + \"\sigma|M_0 = \sigma_0\"$$

is model complete. Note that $T + \"\sigma \text{ is an automorphism}\"$ has no model companion if $T$ has the strict order property [3]. This seems to have some resemblance with the theory of the rings of Witt Vectors carrying the Frobenius automorphism [1].

We consider each natural number $n$ as the set $\{0, 1, \ldots, n-1\}$. Consider a structure $(M_0, <)$ with

$$M_0 = \{f : n \to n+1 \mid n < \omega, \ f(i) < i+1 \text{ for } i < n\},$$

and $f < g$ if $g$ is a proper extension of $f$ as a map for $f, g \in M_0$.

For each $f \in M_0$ with $\text{dom} \ f = n$, let $f^*$ be a map such that

$$f^*(i) = (f(i) + 1) \mod (i + 1)$$

for $i < n$. Then the map $s : M_0 \to M_0$ defined by $s(f) = f^*$ is an automorphism of $(M_0, <)$. $\epsilon$ denotes the least element of $M_0$ (i.e., $\epsilon$ is the empty sequence). Let $<_1$ be a definable relation on $M_0$ defined by the formula

$$x < y \land \forall z \neg (x < z < y).$$

Let $T_0$ be the theory of $(M_0, <, <_1)$. Note that for any model $M$ of $T_0$, $\text{acl}_M(\emptyset) = M_0$. The root (the least element) of $M_0$ will be denoted by $\epsilon$. 
Proposition 1. Let $M$ be a model of $T_0$. Then the following sentences are valid in $M$:

1. $\forall x \exists y \ x <_1 y$.
2. $\forall x, y \ x < y \rightarrow \exists z \ x <_1 z \leq y$.
3. $\forall x, y \ x < y \rightarrow \exists z \ x \leq z <_1 y$.
4. $\forall x, y, z \ x, y \leq z \rightarrow x < y \lor x = y \lor y < x$.
5. $\forall x, y \exists u, v \ x \not\leq y \rightarrow u <_1 v \leq x \land u \leq y \land v \not\leq y$.
6. Let $n$ be any natural number. If $x < y$ and $x$ has (at least) $n$ childs then $y$ has (at least) $n + 1$ childs.

Theorem 2. The theory

$$T_0 \cup \{\sigma \text{ is a } <\text{-automorphism extending } s\}$$

in the language $\{<, <_1, \sigma\} \cup M_0$ has a model companion. In fact, it is model complete.

We fix models $M \subset M'$ of $T$ and assume that $\sigma$ is a $<\text{-automorphism of } M'$ extending $s$ and $M$ is $\sigma$-invariant.

Lemma 3. If $a, b \in M$ then $\inf_M\{a, b\} = \inf_{M'}\{a, b\}$.

Proof. Let $c = \inf_M\{a, b\}$. If $c = a$ or $c = b$ then there is nothing to prove.

Suppose $c < a, b$. Then we can choose $c_a, c_b \in M$ such that $c <_1 c_a \leq a$, $c <_1 c_b \leq b$, and $c_a$ is incomparable with $c_b$. Now, we show that $c = \inf_{M'}\{a, b\}$. Let $d \in M' - M$ be such that $d < a, b$. Then $d$ is comparable with both $c_a$ and $c_b$. Only the case $d < c_a, c_b$ is possible. Therefore, $d < c$. □

Definition 4. Suppose $a, b \in M' - M$. We say that $a$ and $b$ are dependent over $M$ if there is $c \in M' - M$ such that $c \leq a$ and $c \leq b$. We call such $c$ a witness of the dependence. $a$ and $b$ are dependent over $M$ if and only if $\inf\{a, b\} \in M' - M$.

We say that $a$ and $b$ are independent over $M$ if $a$ and $b$ are not dependent over $M$.

Lemma 5. The dependence over $M$ is an equivalence relation on $M' - M$.

Proof. The reflexivity and the symmetry are trivial. We show the transitivity. Suppose $b$ and $c$ are dependent over $M$ with a witness $u$, and $c$ and $d$ are dependent over $M$ with a witness $v$. Since $u \leq c$ and $v \leq c$, $u$ and $v$ are comparable. Without loss of generality, we can assume that $u \leq v$. Then $u \leq v \leq d$. Therefore, $b$ and $d$ are dependent over $M$ with a witness $u$. □

Lemma 6. If $b \in M' - M_0$ then $b$ and $\sigma^m b$ are independent over $M_0$ for any integer $m \neq 0$. 
Proof. Let $m \neq 0$ be an integer and $b \in M' - M$. Choose $f < b$ such that $f \in M_0$ and $\text{dom } f > m$. Then $f$ and $s^m f$ are incomparable and also $s^m f < \sigma^m b$.

Suppose there is $a \in M' - M_0$ such that $a \leq b$ and $a \leq \sigma^m b$. $f$ and $a$ are comparable by $f < b$ and $a \leq b$. Since $f$ has a finite distance from the root, we have $f < a$. Similarly, $s^m f < a$. Therefore, $f$ and $s^m f$ are comparable. A contradiction. \hfill \Box

Corollary 7. If $a, b \in M' - M$ are dependent over $M$ then $a$ and $\sigma^m b$ are independent over $M$ for any integer $m \neq 0$.

Proof. Suppose $a, b \in M' - M$ are dependent over $M$ and $a$ and $\sigma^m b$ are dependent over $M$ for some integer $m \neq 0$. Suppose $c \leq a, c \leq b$ with $c \in M' - M$, and $d \leq a$, $d \leq \sigma^m b$ with $d \in M' - M$.

Since $c, d \leq a$, $c$ and $d$ are comparable. Therefore, $\inf\{c, d\} \leq \inf\{b, \sigma^m b\}$, and hence $\inf\{b, \sigma^m b\} \in M' - M$ contradicting Lemma 6. \hfill \Box

Definition 8. Suppose $a, b \in M' - M$. We say that $a$ and $b$ are quasi-connected over $M$ if there is $c \in M'$ such that

1. $M' \models c \leq a, b$,
2. $M' \models c \leq y \leq a$ implies $y \in M' - M$, and
3. $M' \models c \leq y \leq b$ implies $y \in M' - M$.

We call $c$ a witness of this property. Note that if $a$ and $b$ are quasi-connected over $M$ then it is dependent over $M$.

Lemma 9. The quasi-connectedness over $M$ is an equivalence relation on $M' - M$.

Proof. The reflexivity and the symmetry are trivial. We show the transitivity. Suppose $b$ and $c$ are quasi-connected over $M$ with a witness $u$ and $c$ and $d$ are quasi-connected over $M$ with a witness $v$. Since $u \leq c$ and $v \leq c$, $u$ and $v$ are comparable. Without loss of generality, we can assume that $u \leq v$. We show that $u$ is a witness for quasi-connectedness of $b$ and $d$ over $M$. If $u \leq w \leq b$ then $w \in M' - M$ since $u$ is a witness for quasi-connectedness of $b$ and $c$.

Suppose $u \leq w \leq d$. Then $w$ and $v$ are comparable. If $w \leq v$ then $u \leq w \leq c$ and thus $w \in M' - M$. If $v < w$ then $v \leq w \leq d$ and thus $w \in M' - M$. \hfill \Box

Lemma 10. Suppose that $B$ is a finite subset of $M' - M$ quasi-connected over $M$, $a_1, \ldots, a_m \in M$ and for each $a_i$ there is $b_i \in B$ such that $b_i < a_i$. Then there is $b \in B$ such that $b < \inf\{a_1, \ldots, a_m\}$.

Proof. Let $a = \inf\{a_1, \ldots, a_m\}$ in $M$. Then $a = \inf\{a_1, \ldots, a_m\}$ in $M'$ by Lemma 3.

Let $b = \inf B$ in $M'$. We have $b \in M' - M$ because $B$ is quasi-connected over $M$. Since $b$ is a lower bound for $\{a_1, \ldots, a_m\}$, we have $b \leq a$. Choose $b_1 \in B$ such that $b_1 < a_1$. Then $b_1$ and $a$ are comparable. If $a \leq b_1$ then $b \leq a \leq b_1$, but this cannot happen since there is no element $y \in M$ such that $b \leq y \leq b_1$. Therefore, $b_1 < a$. \hfill \Box
Lemma 11. (1) Suppose $M' \models a <_1 b$ with $a \in M$ and $b \in M' - M$. Then there is no $a' \in M$ such that $M' \models b < a'$.

(2) If $b \in M' - M$ then there is no $a \in M$ such that $M' \models b < _1 a$.

Proof. (1) Suppose $M' \models a <_1 b < a'$ with $a, a' \in M$ and $b \in M' - M$. Then there must be $a'' \in M$ such that $M \models a <_1 a'' < a'$, and thus $M' \models a <_1 a'' < a'$. But this cannot happen because $b \neq a''$.

(2) Suppose there is $b \in M' - M$ and $a \in M$ such that $M' \models b <_1 a$. Since $M' \models \epsilon < a$, we have $M \models \epsilon < a$. Therefore, $M \models a' <_1 a$ for some $a' \in M$ and thus $M' \models a <_1 a$. But this cannot happen because $b \neq a'$.

Definition 12. Suppose $C$ and $D$ are subsets of $M'$. We write $C < D$ if there is $c \in C$ such that $c \leq d$ for any $d \in D$.

Definition 13. A finite subset $X$ of $M' - M$ is called canonical if the following conditions are satisfied:

1. For any $x, y \in X$, whenever $x$ and $\sigma^m(y)$ with $m \in \mathbb{Z}$ are dependent over $M$ then $m = 0$;

2. if $x, y \in X$ are dependent over $M$ then there is $z \in X$ witnessing the dependence; and

3. if $x, y \in X$ are quasi-connected over $M$ then there is $z \in X$ witnessing the quasi-connectedness.

Definition 14. Let $B$ be a subset of $M'$. $\langle B \rangle_\sigma$ denotes the set $\{\sigma^m(b) \mid b \in B, m \in \mathbb{Z}\}$.

Lemma 15. For any finite subset $X \subset M' - M$ there is a canonical subset $Z \subset M' - M$ such that $X \subset \langle Z \rangle_\sigma$.

Proof. We prove the statement by induction on the number of elements in $X$. It is trivial if $|X| = 0$. Suppose $X = \{a\} \cup X'$ with $|X'| < |X|$. By the induction hypothesis, there is a canonical subset $Y'$ of $M' - M$ such that $X' \subset \langle Y' \rangle_\sigma$.

We split the proof into the following cases.

Case 1. $\sigma^m a$ and $b$ are quasi-connected over $M$ for some $b \in Y'$ and an integer $m$.

Let $b_0$ be the least element in $Y'$ which is quasi-connected to $\sigma^m a$ over $M$. Let $c = \inf\{\sigma^m a, b_0\}$. We claim that $Y = Y' \cup \{\sigma^m a, c\}$ is canonical and has the desired property.

Let $C_{b_0}$ be the quasi-connected component of $Y$ containing $b_0$ and $D_{b_0}$ be the dependent component of $Y$ containing $b_0$. It is easy to see that $\{c\} \cup C_{b_0}$ is a tree. $\{c\} \cup D_{b_0}$ is also a tree. Let $d$ be the least element of $D_{b_0}$. Since $c \leq b_0$ and $d \leq b_0$, $c$ and $d$ are comparable. Therefore, $\{c\} \cup D_{b_0}$ is a tree.
Now, suppose that $\sigma^{m+l}a$ and $b \in Y'$ are dependent over $M$. Then $\sigma^l b_0$ and $\sigma^{m+l}a$ are dependent over $M$ and thus $\sigma^l b_0$ and $b \in Y'$ are dependent over $M$. Since $Y'$ is canonical, we have $l = 0$.

Case 2. Case 1 does not hold but $\sigma^m a$ and $b$ are dependent over $M$ for some $b \in Y'$ and an integer $m$.

Let $b_0$ be the least element in $Y'$ which is dependent to $\sigma^m a$ over $M$. Choose a witness $c \in M' - M$ of dependence of $b_0$ and $\sigma^m a$. $Y = Y' \cup \{\sigma^m a, c\}$ is canonical and has the desired property. The argument is the same as that for Case 1.

Case 3. There is no integer $m$ and $b \in Y'$ such that $\sigma^m a$ and $b$ are dependent over $M$. In this case, $Y = Y' \cup \{a\}$ is canonical and has the desired property. 

\[ \textbf{Lemma 16. Suppose } \{t_1, \ldots, t_n\} \subset M' - M \text{ is canonical. Then any formula in } \text{qftp}_{\langle,\sigma\rangle}(t_1, \ldots, t_n/M) \text{ is realised in } M. \]

\[ \text{Proof. Suppose } \{t_1, \ldots, t_n\} \subset M' - M \text{ is canonical. Let } t \text{ be the tuple } (t_1, \cdots, t_n) \text{ and } \varphi(x) \text{ a formula with } x = (x_1, \ldots, x_n) \text{ belonging to } \text{qftp}_{\langle,\sigma\rangle}(t/M). \text{ Let } N \text{ be a natural number such that if } \sigma^n(x_i) \text{ occurs in } \varphi(x) \text{ then } n \leq N. \text{ Let } A \text{ be a finite subset of } M \text{ such that } \varphi(x) \text{ is over } A. \]

By adding finitely many points of $M$ to $A$ if necessary, we can assume the following:

- If $C$ is a quasi-connected component of $t$ then $\{a\} < C$ for some $a \in A$;

- if $C$ and $C'$ are two quasi-connected components of $t$ with $C < C'$ then there is $a \in A$ such that $C < \{a\} < C'$;

- if $C$ is a quasi-connected component of $t$ and there is $a \in M$ and $c \in M' - M$ quasi-connected to $C$ over $M$ such that $a <_1 c$ then $a \in A$ and $c \in C$;

- if $C$ is a quasi-connected component of $t$ such that $\{a \in A | C < \{a\}\}$ is non-empty then $\inf\{a \in A | C < \{a\}\} \in A$;

- if $a \in A$ is comparable with $t_i$ for some $i$ then $\sigma^m(a) \in A$ for $m \leq N$; and

- if $a \in A$ is comparable with $\sigma^m(t_i)$ for some $i$ and a natural number $m \leq N$ then $\sigma^{-m}(a) \in A$.

We can assume that $t = C_1^\wedge \cdots C_l$ where each $C_i$ is an enumeration of a quasi-connected component of $t$.

Let $a_i$ be the maximum element in $A$ such that $\{a_i\} < C_i$ and $b_i$ be the minimum element in $A$ such that $C_i < \{b_i\}$. Such $a_i$ exists by the assumption on $A$ and such $b_i$ exists if there is $b \in A$ such that $C_i < \{b\}$ by Lemma 10 and the assumption on $A$.

Suppose that there are infinitely many elements $d$ of $M$ connected to $a_i$ such that $a_i < d < C_i$. Choose $a'_i \in M$ connected to $a_i$ with the following properties:

- If $x \in A$ and $M \models \sigma^m b_i \not\leq x$ with $0 \leq m \leq N$ then $M \models \sigma^m a_i \not\leq x$; and

- if $C'$ is a quasi-connected component of $t$ such that $C_i \not\subseteq C'$ then $\{a'_i\} \not\subseteq C'$. 

In the case that $b_i$ exists, choose a tuple $C'_i$ from $M$ such that \( \text{qftp}_{\{<, <\}}(C_i/a'_i, b_i) = \text{qftp}_{\{<, <\}}(C'_i/a'_i, b_i) \). Then we have \( \text{qftp}_{\{<, <\}}(\sigma^m C_i/A) = \text{qftp}_{\{<, <\}}(\sigma^m C'_i/A) \) for \( m = 0, 1, \ldots, N \).

In the case that there is no such $b_i$ for $C_i$, choose a tuple $C'_i$ from $M$ such that \( \text{qftp}_{\{<, <\}}(C_i/a'_i) = \text{qftp}_{\{<, <\}}(C'_i/a'_i) \). Then we have \( \text{qftp}_{\{<, <\}}(\sigma^m C_i/A) = \text{qftp}_{\{<, <\}}(\sigma^m C'_i/A) \) for \( m = 0, 1, \ldots, N \).

Suppose $C_{i_1}, \ldots, C_{i_k}$ are quasi-connected and \( a <_1 \inf C_{i_j} \) for \( j = 1, \ldots, k \). In this case, there is no \( x \in A \) such that $C_{i_j} < \{x\}$ by Lemma 11. We can choose $c'_j \in M - A$ for \( j = 1, \ldots, k \) which are pairwise distinct such that $M \models a_i <_1 c'_j$ and $M \models \sigma^m c'_j \not\leq x$ for \( x \in A \) and \( m \) with \( 0 \leq m \leq N \). Choose a tuple $C'_{i_j}$ for \( j = 1, \ldots, k \) from $M$ such that $\text{qftp}_{\{<, <\}}(C_{i_j}, \inf C_{i_j}) = \text{qftp}_{\{<, <\}}(C'_{i_j}, c'_j)$.

Then we have $\text{qftp}_{\{<, <\}}(C_{i_j}/A) = \text{qftp}_{\{<, <\}}(C'_{i_j}/A)$.

Let $t' = C'_{1} \cdots C'_{k}$.

**Claim 1.** $\text{qftp}_{\{<, <\}}(t'^{\sigma}t'^{\sigma^2}t'^{\sigma^N}t/A) = \text{qftp}_{\{<, <\}}(t'^{\sigma}t'^{\sigma^2}t'^{\sigma^N}t'/A)$

Proof of Theorem 2. We show that $(M, <, \sigma|M)$ is existentially closed in $(M', <, \sigma)$. Choose a finite tuple $(t_1, \ldots, t_n)$ from $M' - M$ and let $\varphi(x_1, \ldots, x_n)$ be a quantifier-free formula of $\{<, \sigma\} \cup M$ realised by $(t_1, \ldots, t_n)$. By Lemma 15, we can choose $t'_1, \ldots, t'_n \in M' - M$ such that $t_i = \sigma^{k_i}(t'_i)$ for each $i$ with some $k_i \geq 0$ and the set \( \{t'_1, \ldots, t'_n\} \) is canonical. We have

\[
M' \models \varphi(\sigma^{k_1}(t'_1), \ldots, \sigma^{k_n}(t'_n)).
\]

By Lemma 16, we can choose $t''_1, \ldots, t''_n \in M$ such that

\[
M \models \varphi(\sigma^{k_1}(t''_1), \ldots, \sigma^{k_n}(t''_n)).
\]

Therefore, $\varphi(x_1, \ldots, x_n)$ is realised in $M$.

**References**

