数理解析研究所講究録
第1718巻 2010年 1-26

Geometry and Categoricity*

John T. Baldwin†
University of Illinois at Chicago

July 15, 2010

1 Introduction

The introduction of strongly minimal sets [BL71, Mar66] began the idea of the analysis of models of categorical first order theories in terms of combinatorial geometries. This analysis was made much more precise in Zilber's early work (collected in [Zil91]). Shelah introduced the idea of studying certain classes of stable theories by a notion of independence, which generalizes a combinatorial geometry, and characterizes models as being prime over certain independent trees of elements. Zilber's work on finite axiomatizability of totally categorical first order theories led to the development of *geometric stability theory*. We discuss some of the many applications of stability theory to algebraic geometry (focusing on the role of infinitary logic). And we conclude by noting the connections with non-commutative geometry.

This paper is a kind of Whig history- tying into a (I hope) coherent and apparently forward moving narrative what were in fact a number of independent and sometimes conflicting themes. This paper developed from talk at the Boris-fest in 2010. But I have tried here to show how the ideas of Shelah and Zilber complement each other in the development of model theory. Their analysis led to frameworks which generalize first order logic in several ways. First they are led to consider more powerful logics and then to more 'mathematical' investigations of classes of structures satisfying appropriate properties.

We refer to [Bal09] for expositions of many of the results; that's why that book was written. It contains full historical references. Only a few minor remarks in this paper are new.

1.1 Zilber's Thesis

Zilber's approach to categoricity begins is based on the intuition that fundamental structures are canonical. That is, truly significant mathematical structures can be character-

*This paper is an expansion of my talk at 'Boris Fest', the Oxford conference on Geometric Model Theory in March 2010. It provides background for my actual talks in Kirishima which discussed in detail the beginnings of Shelah's study of excellence; that material as well as details of many older results referred to in the paper appears in Chapters 6,7, 18, and 19 of [Bal09].

†Baldwin was partially supported by NSF-0500841.

ized in an appropriate logic. This notion is organizational. In the situation here the relevant notion of 'characterize' is taken as categoricity in power[1]

Conversely, characterizable structures are 'fundamental'. We won't explore that direction much here. But, the key idea is that each categorical structure is built from 'fundamental structures' is reasonably standard way. This is the whole point of the structural side of stability theory. This idea motivates the 'canonicity conjectures' below. The thesis is made specific by several concrete problems.

Characterizability Problem:

Find an *axiomatization* for $\mathrm{Th}(\mathbb{C}, +, \cdot, \exp)$.

Canonicity Conjectures:

Zilber Conjecture:

Every strongly minimal first order theory is

1. disintegrated
2. group-like
3. field-like

Cherlin-Zilber Algebraicity Conjecture

1. Every simple ω-stable group is (i.e. is interpretable in) an algebraic group over an algebraically closed field.
   This led to a supporting conjecture:
2. There an ω-stable field of finite Morley rank with a definable proper subgroup of the multiplicative group.[2] ?

This paper will proceed in two stages. In the first, Section 2, we discuss progress on the study of complex exponentiation and in particular the connection with Shelah's analysis of categoricity for $L_{\omega_1,\omega}$. The notion of axiomatization in the characterizability problem depends on the framework in which the investigation is set. We discuss the analysis of these structures in the framework of $L_{\omega_1,\omega}(Q)$ and through the development of more 'logic-free' approaches: 'quasi-minimal excellent classes' and 'abstract elementary classes'. However, we remark recent work by [BP] which places the result in the context of Shelah's first order main-gap theorem.

In the second stage, Section 3, we discuss the challenges to the 'canonicity conjectures' posed by the Hrushovski Construction. Namely, we consider certain structures that arose in working on the conjectures and the attempts to reformulate those examples by using that construction in a positive way. In [Hru93], Hrushovski proves: There

[1] Perhaps the key point is that the canonicity notion must entail some constraints on the definable sets. This consequence of categoricity in power for first order theories is well known; the investigations reported here extend the conclusion to $L_{\omega_1,\omega}(Q)$. The 0-minimality of the real field plays a similar role; there aren't any clear structural consequences simply from categoricity in second order logic.

[2] This problem has been solved [BHMPW07]; but the existence of a bad group, a nonsolvable connected group with finite Morley rank all of whose proper connected subgroups are nilpotent, remains open. An epic amount of work, analogous to the study of finite simple groups has been done on the Cherlin-Zilber conjecture; it too remains unsolved. Blossier, Martin-Pizzaro and Wagner have shown the constructed bad field does not contain a bad group. And much recent work has shown that the existence of bad groups has little to do with the algebraicity conjecture.

is a strongly minimal set which is not locally modular (i.e. neither disintegrated nor group-like) and not field-like. And in [BHMPW07], Baudisch, Hils, Martin-Pizarro, and Wagner use the same construction technique to prove: There is an ω-stable field of finite Morley rank with a definable proper subgroup of the multiplicative group.

Zilber's responses to these challenges came in two forms. The first is to strengthen the hypotheses by introducing the less classically syntactic notion of a *Zariski Structure*, discussed in Subsection 3.1.

The second is to weaken the conclusion: Replace first order interpretable in $(\mathbb{C}, +, \cdot)$ by 'analytically' definable. This approach has focused infinite Morley rank ω-stable fields and superstable fields that arose in the attempt to find a bad field. We will consider this approach and the connections with non-commutative Geometry in Subsection 3.2. We will close the paper by placing Zilber's thesis in the context of Shelah's general approach to classification.

1.2 Frameworks and Homogeneity

Most model theorists in the last thirty years of the twentieth century and especially those focusing on interesting algebraic structures worked in the setting of first order logic. The studies described here employ a more general formulation. There are several alternative frameworks, which provide different notions of 'axiomatizing' the class. We describe them briefly and indicate the role of various notions of homogeneity.

Among the most natural generalizations of first order logic is $L_{\omega_1,\omega}$. Extend the syntax of first order logic by allowing infinite conjunctions and disjunctions and interpret the new connectives in the natural way. A second useful extension is to allow a quantifier Q where $(Qx)\phi(x)$ is true if $\phi(x)$ has uncountably many solutions. (See [BF85] for an account of many extensions of first order logic.)

Of course, $L_{\omega_1,\omega}$ fails the compactness theorem. More subtly it fails the upwards Löwenheim-Skolem theorem, the amalgamation property (for models of a complete sentence), and the downwards Lówenheim-Skolem theorem for theories (although it is true for individual sentences). Each of these problems must be addressed in studying categoricity in these logics. A major result discussed below is that categoricity up to $\aleph_\omega$ (plus weak set theory) implies the existence of arbitrarily large models for a sentence of $L_{\omega_1,\omega}$.

The model theory of a sentence in $L_{\omega_1,\omega}(Q)$ can be transformed into the study of models of a first order theory (in an expanded vocabulary) which omits a specified family of types. This transformation is discussed in detail in Chapter 6 of [Bal09]. A key idea is to replace each formula $\phi(\mathbf{x})$ by a predicate $P_\phi(\mathbf{x})$. Inductively, replace an infinite conjunction ϕ of the form $\bigwedge \phi_i$ by adding the axioms $P_\phi(\mathbf{x}) \rightarrow P_{\phi_i}(\mathbf{x})$ and omitting the type $\{P_{\phi_i}(\mathbf{x}) : i < \omega\} \cup \{\neg P_\phi(\mathbf{x})\}$. This technique allows the translation of an $L_{\omega_1,\omega}(Q)$-sentence into the study of a first order theory omitting types.

It is more difficult to require that the class of types omitted is *all* non-principal types. For this, a result of Keisler says that if $\psi \in L_{\omega_1,\omega}(Q)$ has less than $2^{\aleph_1}$ models in $\aleph_1$, for each countable fragment Δ of $L_{\omega_1,\omega}(Q)$ each model of ψ realizes only countably many Δ-types over the empty set. Then Shelah, using the Lopez-Escobar theorem shows this implies ψ has a model of cardinality $\aleph_1$ that is small for $L_{\omega_1,\omega}(Q)$ (realizes only countably many $L_{\omega_1,\omega}(Q)$-types over $\emptyset$). Now the transformation in the previous paragraph can be extended to translate the models of a $\psi \in L_{\omega_1,\omega}$ that is $\aleph_1$-categorical to the atomic models of a first order theory in an expanded language.

Trying to find a uniform account of constructions that occurred in various logics provides one motivation for Shelah's notion of an abstract elementary class. An abstract elementary class consists of a class of structures $\boldsymbol{K}$ and a relation of *strong substructure*, $\prec_{\boldsymbol{K}}$, between members of the class which satisfies the conditions below.

Definition 1.1 *An abstract elementary class (AEC)* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *is a collection of structures*[3] *for a fixed vocabulary* τ *that satisfy, where* $A \prec_{\boldsymbol{K}} B$ *means in particular* A *is a substructure of* B*,*

1. *If* $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ *and* $A \subseteq B$ *then* $A \prec_{\boldsymbol{K}} B$*;*
2. *Closure under direct limits of* $\prec_{\boldsymbol{K}}$*-embeddings;*
3. *Downward Löwenheim-Skolem. If* $A \subset B$ *and* $B \in \boldsymbol{K}$ *there is an* A' *with* $A \subseteq A' \prec_{\boldsymbol{K}} B$ *and* $|A'| \leq |A| = \mathrm{LS}(\boldsymbol{K})$.

The invariant $\mathrm{LS}(\boldsymbol{K})$, is a crucial property of the class. The class of well-orderings satisfies the other axioms (under end extension) but is not an AEC.

First order logic and $L_{\omega_1,\omega}$ with $\prec_{\boldsymbol{K}}$ as elementary submodel in the respective logic are natural examples of AEC. In order to incorporate the Q-quantifier, some contortion is necessary (Section 6.4 of [Bal09].) We will see below that the important examples considered in this paper are further examples of AEC. Shelah and others have carried out extensive investigations about general AEC. See for example [She09, She10, Bal09, GV06, HK06]. Shelah's presentation theorem shows that any AEC can be given as a class of models of a first order theory which omit a family of types. (This is not reversible.)

The fundamental construction enabled by the notion of an AEC with the amalgamation property is that of a 'monster' or homogeneous-universal model. The notion extends Jónsson's construction of such models by allowing the notion of submodel to vary. There is an important distinction concerning the definition of homogeneity.

Definition 1.2 (Set versus model homogeneity) *1. The class* $\boldsymbol{K}$ *satisfies the* amalgamation property for models *if for any situation*[4] *with* $A, M, N \in \boldsymbol{K}$*:*

$$\begin{tikzcd} & N \\ A \arrow[ur] \arrow[dr] & \\ & M \end{tikzcd}$$

there exists an $N_1 \in \boldsymbol{K}$ *such that*

[3] Naturally we require that both $\boldsymbol{K}$ and $\prec_{\boldsymbol{K}}$ are closed under isomorphism.
[4] The arrows denote strong embeddings.

$$\begin{tikzcd} & N \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & N_1 \\ & M \arrow[ur] & \end{tikzcd}$$

2. *The class* $\boldsymbol{K}$ *satisfies the* set amalgamation property *if in the situation of item 1) we allow any subset* A *of* $N \cap M$ *instead of requiring* A *to a submodel of each.*

The standard notion of sequence homogeneity depends on the syntactic notion of a type in first order logic.

Definition 1.3 *1. A structure* M *is* κ*-sequence homogeneous if for any* $\mathbf{a}, \mathbf{b} \in M$ *of length less than* κ*, if* $(M, \mathbf{a}) \equiv (M, \mathbf{b})$ *then for every* c*, there exists* d *such that* $(M, \mathbf{a}c) \equiv (M, \mathbf{b}d)$*. Usually, the 'sequence' is omitted and one just says* κ*-homogeneous.*

2. M *is* strongly μ-sequence homogeneous *if any two sequences of length less than* μ *that realize the same first order type are automorphic in* M*.*

For a complete first order theory, as Morley [Mor65] showed, we can expand the language to make every definable set quantifier free definable and both notions of amalgamation hold. This implies that by iterating the amalgamation we obtain models M which (subject to cardinality limitations) are $|M|$-sequence homogenous, the usual monster model. (See Chapter 8 of [Bal09] for a careful analysis of the set theoretic conditions.) But this strong amalgamation condition is easily seen to fail in $L_{\omega_1,\omega}$ and more importantly to fail even for $\aleph_1$-categorical sentences that satisfy model amalgamation. In contrast the 'monster model' of an AEC with amalgamation will be only model homogeneous.

Definition 1.4 *1.* M *is* μ-model homogeneous *if for every* $N \prec_{\boldsymbol{K}} M$ *and every* $N' \in \boldsymbol{K}$ *with* $|N'| < \mu$ *and* $N \prec_{\boldsymbol{K}} N'$ *there is a* $\boldsymbol{K}$*-embedding of* N' *into* M *over* N*.*

2. M *is* strongly μ-model homogeneous *if it is* μ*-model homogeneous and for any* $N, N' \prec_{\boldsymbol{K}} M$ *and* $|N|, |N'| < \mu$*, every isomorphism* f *from* N *to* N' *extends to an automorphism of* M*.*

3. M *is* strongly model homogeneous *if it is strongly* $|M|$*-model homogeneous.*

Given a monster model $\mathcal{M}$ one can define the *Galois type* of an element a over an $M \prec_{\boldsymbol{K}} \mathcal{M}$ as the orbit of a under automorphisms of $\mathcal{M}$ fixing M pointwise. But only rarely (except in first order) does this correspond to the natural syntactic notion (even when the AEC is the class of models of an $L_{\omega_1,\omega}$-sentence). Although we only mention Galois types here, they are the proper notion of study for AEC.

Zilber's examples of quasiminimal excellent classes have amalgamation over models but the interesting algebraic examples discussed in Subsection 2.3 do *not* have set amalgamation (see Chapter 3 of [Bal09].)

We give a simple model theoretic example to show the difficulties in obtaining sequence homogeneity for models of sentences of $L_{\omega_1,\omega}$.

Example 1.5 There is a first order theory T with a prime model M such that M has no proper elementary submodel but M contains an infinite set of indiscernibles[Kni78]. The vocabulary contains three unary predicates, W, F, I which partition the universe. Let W and I be countably infinite sets and fix an isomorphism f_0 between them. F is the collection of all bijections between W and I that differ from f_0 on only finitely many points. Add also a successor function on W so that (W, S) is isomorphic to ω under successor and the evaluation predicate $E(n, f, i)$ which holds if and only if $n \in W, f \in F, i \in I$ and $f(n) = i$.

The resulting structure M is atomic and minimal. Since every permutation of I with finite support extends to an automorphism of M, I is a set of indiscernibles. But it is not ω-stable.

Now consider a two sorted structure with this model in one sort and a pure infinite set in the other. This class is easily axiomatized in $L_{\omega_1,\omega}$ (is the atomic models of a first order theory). It is categorical in all uncountable powers (and the class of atomic models is ω-stable in the sense described below). But no model is $\aleph_1$-sequence homogeneous.

2 Canonicity of Fundamental Structures

The structure $(\mathbb{C}, +, \cdot, e^x, 0, 1)$ is Godelian: the ring of integers is defined as a translate of $\{a : e^a = 1\}$. Therefore the first order theory is undecidable and 'wild'; there are definable sets of arbitrary quantifier complexity and there is no reasonable notion of dimension. Zilber conjectured that Z is the source of all the difficulty. Fix Z by adding the axiom:

$$(\forall x)e^x = 1 \to \bigvee_{n \in Z} x = 2n\pi.$$

Here we are taking π as a constant and this axiom asserts that it is a generator of the kernel of exponentiation. We will say something later about the connection of this generator with the historical π. It turns out that (even conjecturally) some further non-elementary input is required; see below.

As we'll see there is a sentence of $L_{\omega_1,\omega}(Q)$ which is categorical in all uncountable powers and axiomatizes some expansion of the complex numbers by a homomorphism from $(\mathbb{C}, +)$ to $(\mathbb{C}, \cdot)$. But we will reach this axiomatization by considering a specific kind of AEC.

2.1 Quasiminimal excellence

Definition 2.1 *A* closure system *is a set G together with a dependence relation*

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following three axioms.

- **A1.** $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$

- **A2.** $X \subseteq cl(X)$
- **A3.** $cl(cl(X)) = cl(X)$

  (G, cl) *is* pregeometry *if in addition:*

  A4. *If* $a \in cl(Xb)$ *and* $a \notin cl(X)$, *then* $b \in cl(Xa)$.

If a closure system satisfies that the closure of countable set is countable we say it has the *countable closure property (ccp)*. If points are closed the structure is called a *geometry*. Such combinatorial geometries arise in model theory in a number of ways; we consider here notions of minimality which yield geometries.

Definition 2.2 *The structure* M *is* strongly minimal *if every first order definable subset of* any elementary extension M' *of* M *is finite or cofinite.*

We say a is in the *algebraic closure* of B and write $a \in \mathrm{acl}(B)$ if for some $\mathbf{b} \in B$ and some $\phi(x, \mathbf{y})$: $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only **finitely many** solutions. It is an easy but instructive exercise to show:

Exercise 2.3 *If* f *mapping* X *to* Y *is an elementary isomorphism,* f *extends to an elementary isomorphism from* $\mathrm{acl}(X)$ *to* $\mathrm{acl}(Y)$.

One can reformulate the notion of strong minimality in geometric terms.

Lemma 2.4 *A complete theory* T *is strongly minimal if and only if it has infinite models and*

1. *algebraic closure induces a pregeometry on models of* T;
2. *any bijection between acl-bases for models of* T *extends to an isomorphism of the models*

It is now immediate that any first order theory that is strongly minimal is categorical in any uncountable power. We turn to a weaker notion that will be useful in organizing the study of categoricity for infinitary logic.

Definition 2.5 *A structure* M *is* 'quasiminimal' *if every first order* ($L_{\omega_1,\omega}$?) *definable subset of* M *is countable or cocountable.*

The parenthetical ($L_{\omega_1,\omega}$) appears because under natural homogeneity conditions, the stronger condition holds, but for definitional purposes we require only the solution sets of first order formulas be countable or cocountable. Note that we have not required that quasiminimality is preserved under elementary extension. It is easy to see that a quasiminimal structure whose first order theory is $\aleph_1$-categorical is strongly minimal. However, the notion is much weaker than categoricity. Note in particular that a strongly minimal structure M is $|M|$-sequence homogenous. But such a conclusion easily fails for quasiminality. Consider a structure M in a language with one binary relation symbol which is required to be an equivalence relation with two classes. Every countable model is strongly ω-homogenous. But the model in $\aleph_1$ with one countable and one uncountable equivalence class is quasiminimal but not strongly $\aleph_0$-homogeneous.

Consider the following notion: We say a is in the *quasi-algebraic closure of* X and write $a \in \mathrm{acl}'(X)$ if there is a first order formula over X with **countably many** solutions which is satisfied by a. In general acl' is not closure operator.

Example 2.6 Let the vocabulary have two binary relations E_1, E_2 and let T assert these are crosscutting equivalence relations such that E_1 has two classes and E_2 has three. Consider a model M such that one E_1-class is split into three classes of size $\aleph_0$ by E_2 while the other is split into two classes of size $\aleph_1$ and one countable class. Now if a is in an uncountable E_2-class it is easy to check that $\mathrm{acl}'(a) \subsetneq \mathrm{acl}'(\mathrm{acl}'(a))$.

To remedy this, define $\mathrm{ccl}^n(X)$ by induction: $\mathrm{ccl}^0(X) = X$; $\mathrm{ccl}^{n+1}(X) = \mathrm{acl}'(\mathrm{ccl}^n(X))$. And let $\mathrm{ccl}(X) = \bigcup_{n<\omega} \mathrm{ccl}^n(X)$ Zilber observed in the early 90's if M is quasiminimal then (M, ccl) is a closure system.

But in general the closure system is *not* a geometry. Pillay and Tanovic [PT], building on Itai, Tsuboi and Wakai[ITW04], study structures that are quasiminimal; they consider when acl' is a closure operator and when these closure systems are pregeometries in terms of symmetry properties of the generic type (containing all those formulas with uncountably many solutions). Pillay and Tanovic generalize Itai, Tsuboi, and Wakai by replacing an assumption of strong ω-homogeneity by a weaker condition that the generic type not split (see Definition 2.20) over the empty set.

Varying an example from [ITW04] emphasizes the distinction between the various notions of homogeneity in studying quasiminimality. Itai, Tsuboi, and Wakai consider the first order theory of infinitely many crosscutting equivalence relations with infinite splitting . Let E_∞ denote the intersection of these equivalence relations. It is straightforward to axiomatize in $L_{\omega_1,\omega}(Q)$ the requirement that there be exactly one E_∞-class with uncountably many elements and that only countably many (fixed with respect to the big class) E_∞-classes are realized (each countably many times). This sentence is quasiminimal excellent. But no uncountable model is strongly ω-homogeneous. And no model of the underlying first order theory with cardinality strictly greater than $\aleph_1$ is both quasiminimal and strongly ω-homogeneous (for first-order types).

To emphasize the distinction in the various notions of homogeneity and quasiminimality we note the following lemma.

Lemma 2.7 *Suppose there is a structure M of cardinality $\aleph_1$ which is quasiminimal and is the only model in $\aleph_1$ of a sentence ψ in $L_{\omega_1,\omega}(Q)$. Then acl' is a geometry on M.*

Proof. By Shelah [She75, She87, Bal09], M has an $L_{\omega_1,\omega}(Q)$ elementary extension N of cardinality $\aleph_2$. Let A be a finite subset of N and suppose $b \notin \mathrm{acl}'(A)$, $b \in \mathrm{acl}'(cA)$ but $c \notin \mathrm{acl}'(bA)$. Let $p(x, y)$ be the $L_{\omega_1,\omega}(Q)$ type of bc over A. Let $\langle b_i : i < \aleph_1\rangle$ be $\aleph_1$ members of $N - \mathrm{acl}'(A)$. Since $|N| = \aleph_2$, the meaning of the Q-quantifier gives there is a c' realizing all the $p(b_i, x)$. Then each $b_i \in \mathrm{acl}'(c'A)$ contradicting the definition of acl'. This gives exchange on N and *a fortiori* on M. $\square_{2.7}$

[ITW04] obtains a similar result but with the stronger hypothesis that N is strongly ω-homogenous (but not assuming $\aleph_1$-categoricity). And Lemma 2.7 follows trivially from a Lemma in [PT] if we know ψ is ω-stable. ω-stability can be derived from $\aleph_1$-categoricity using CH but it open whether this holds in ZFC.

The second clause of Lemma 2.4 can be split into two properties. It first asserts than (M, cl) is geometrically homogeneous in the following sense (adapted from [PT]).

Definition 2.8 *An infinite dimensional pregeometry* (M, cl) *is* Geometrically Homogenous *if for each finite* $B \subset M$, $\{a \in M : a \notin \mathrm{cl}(B)\}$ *are the realizations in* M *of a unique complete type in* $S(B)$.

And further it asserts the following extension property analogous to Exercise 2.3: If f takes X to Y is an elementary isomorphism, and X and Y are independent f extends to an elementary isomorphism from $\mathrm{cl}(X)$ to $\mathrm{cl}(Y)$.

We will see below that a substantial additional hypothesis is apparently needed to extend this last condition to acl' for a quasiminimal structure. In order to consider the notion of categoricity in power, Zilber moved to considering a class of structures satisfying certain abstract properties. He strengthened the notion of quasiminimality by requiring abstractly that a combinatorial geometry is defined. With certain weak homogeneity conditions he recovers that the closure is in fact acl'. The classes defined are a special case of the same move made by Shelah[She87] a few years earlier in introducing the concept of *Abstract Elementary Classes*. In particular, a class $\boldsymbol{K}$ as in Definition 2.9 with $M \prec_{\boldsymbol{K}} N$ if M is a closed substructure of N (in the sense of the closure system) forms an AEC.

Definition 2.9 (Basic Conditions for Quasiminimal Excellence) *Let* $\boldsymbol{K}$ *be a class of L-structures.*

1. *Suppose each* $M \in \boldsymbol{K}$ *admits a closure relation* cl_M *mapping* $X \subseteq M$ *to* $\mathrm{cl}_M(X) \subseteq M$ *that satisfies the following properties.*

   (a) *Each* cl_M *defines a pregeometry on* M.

   (b) *For each* $X \subseteq M$, $\mathrm{cl}_M(X) \in \boldsymbol{K}$.

   (c) *If* f *is a partial monomorphism from* $H \in \boldsymbol{K}$ *to* $H' \in \boldsymbol{K}$ *taking* $X \cup \{y\}$ *to* $X' \cup \{y'\}$ *then* $y \in \mathrm{cl}_H(X)$ *iff* $y' \in \mathrm{cl}_{H'}(X')$.

2. $\aleph_0$*-homogeneity over models*

   Let $G \subseteq H, H' \in \boldsymbol{K}$ *with* G *empty or a countable member of* $\boldsymbol{K}$ *that is closed in* H, H'.

   (a) *If* f *is a partial* G*-monomorphism from* H *to* H' *with finite domain* X *then for any* $y \in \mathrm{cl}_H(X)$ *there is* $y' \in H'$ *such that* $f \cup \{\langle y, y'\rangle\}$ *extends* f *to a partial* G*-monomorphism.*

   (b) *If* f *is a bijection between* $X \subset H \in \boldsymbol{K}$ *and* $X' \subset H' \in \boldsymbol{K}$ *which are* separately cl*-independent (over* G*) subsets of* H *and* H' *then* f *is a* G*-partial monomorphism.*

Suppose $(\boldsymbol{K}, \mathrm{cl})$ satisfies the Basic Conditions. It is shown in Chapter 2 of [Bal09] that for any finite set $X \subset M$, if $a, b \in M - \mathrm{cl}_M(X)$, a, b realize the same $L_{\omega_1,\omega}$-type over X. Thus, the model is quasiminimal and even for formulas in $L_{\omega_1,\omega}$ not just first order logic. Moreover, (M, cl) is geometrically homogeneous. Furthermore there is one and only one model M in $\aleph_1$. This raises the following problem.

Question 2.10 *Find a class of quasiminimal structures that satisfies the Basic Conditions but is* not *categorical in some $\kappa > \aleph_1$. (Such an example will necessarily have a model in $\aleph_2$. But perhaps there would be no larger model.)*

We discus below examples of categoricity of a sentence ϕ_n of $L_{\omega_1,\omega}$ first failing at $\aleph_n$. But no quasiminimal (rank 1) example is known. Thus, this problem is analogous to the question of whether there is a first order strongly minimal theory that is finitely axiomatizable theory. $\aleph_1$-categorical examples are known [Per80]. But no strongly minimal or even rank one example is known.

In order to get categoricity in $\aleph_2$ and to construct larger models we need the notion of excellence. In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition 2.11 (Independent systems) *1. For any Y, $\mathrm{cl}^-(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.*

2. *We call C (the union of)* an n-dimensional cl-independent system *if $C = \mathrm{cl}^-(Z)$ and Z is an independent set of cardinality n.*

Here, by independent we mean the notion of independence in the combinatorial geometry. We employ the standard first order notion of a primary model.

Definition 2.12 *Given any sequence $\langle e_i : i < \lambda \rangle$, we write $E_{<j}$ for $\langle e_i : i < j \rangle$. If M can be written as $A \cup \langle e_i : i < \lambda \rangle$ such that $\mathrm{tp}(e_j/AE_{<j})$ is isolated for each j we say M is* primary *over A.*

Definition 2.13 *A class (of atomic models) $(\boldsymbol{K}, \mathrm{cl})$ is* quasiminimal excellent *if each model in $\boldsymbol{K}$ admits a combinatorial geometry which satisfies the Basic Conditions and there is a primary model over any finite independent system of countable models.*

Equivalently, the last clause (excellence) can be replaced by requiring that types over countable independent systems are dense in the following sense.

Let $C \subseteq H \in \boldsymbol{K}$ and let X be a finite subset of H. We say $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is *defined over* the finite C_0 contained in C if it is determined by its restriction to C_0.

Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^-(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is defined over C_0.

Now we sketch the proof that Excellence implies by a direct limit argument:

Lemma 2.14 *If $(\boldsymbol{K}, \mathrm{cl})$ is quasiminimal excellent an isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.*

For this version of the proof see [Bal09, Kir10b]; the argument is adapted from that in [Zil05b].

Lemma 2.15 *Suppose $H, H' \in \boldsymbol{K}$ satisfy the countable closure property. Let $\mathcal{A}, \mathcal{A}'$ be* cl*-independent subsets of H, H' with $\mathrm{cl}_H(\mathcal{A}) = H$, $\mathrm{cl}_{H'}(\mathcal{A}') = H'$, respectively, and ψ a bijection between $\mathcal{A}$ and $\mathcal{A}'$. Then ψ extends to an isomorphism of H and H'.*

Proof Sketch: We have the obvious directed union $\{\mathrm{cl}(X) : X \subseteq \mathcal{A}; |X| < \aleph_0\}$ with respect to the partial order of finite subsets of X by inclusion. And $H = \bigcup_{X \subset \mathcal{A}; |X| < \aleph_0} \mathrm{cl}(X)$. So the theorem follows immediately if for each finite $X \subset \mathcal{A}$ we can choose $\psi_X : \mathrm{cl}_H(X) \to H'$ so that $X \subset Y$ implies $\psi_X \subset \psi_Y$.

We prove this by induction on $|X|$. If $|X| = 1$, the condition is immediate from $\aleph_0$-homogeneity and the countable closure property. Suppose $|Y| = n+1$ and we have appropriate ψ_X for $|X| < n + 1$. We will prove two statements.

1. $\psi_Y^- : \mathrm{cl}^-(Y) \to H'$ defined by $\psi_Y^- = \bigcup_{X \subset Y} \psi_X$ is a monomorphism.

2. ψ_Y^- extends to ψ_Y defined on $\mathrm{cl}(Y)$.

2) is immediate from excellence; 1) requires a one-page argument [Bal09, Kir10b]. $\square_{2.15}$

And now we can conclude.

Theorem 2.16 *Suppose the quasiminimal excellent (I-IV) class $\boldsymbol{K}$ is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable. Then, for any infinite κ there is a unique structure in $\boldsymbol{K}$ of cardinality κ which satisfies the countable closure property.*

The proof of existence of large models is inductive using categoricity in κ to obtain a model in κ^+.

2.2 Excellence for $L_{\omega_1,\omega}$

Zilber's argument is for a very specific kind of sentence in $L_{\omega_1,\omega}(Q)$; Quasiminimal is analogous to strongly minimal. In fact, Shelah had proved a more general result for $L_{\omega_1,\omega}$ much earlier ([She83a, She83b]). We will describe Shelah's result and then very briefly discuss the role of the quantifier Q. In addition to Shelah's papers these arguments are expounded in [Bal09].

Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory in an expanded language (Subsection 1.2).

Assumption 2.17 *In this subsection $\boldsymbol{K}$ is the class of atomic models of first order theory T. $\prec_{\boldsymbol{K}}$ is elementary submodel.*

Thus, we have switched to a first order context and consider types in the normal first order sense of the word. But we restrict the Stone space to types satisfying the following condition.

Definition 2.18 *Let A be an atomic subset of a model of first order theory T. We define $S_{at}(A)$.*

1. $p \in S_{at}(A)$ *if* $a \models p$ *implies* Aa *is atomic.*

2. $\boldsymbol{K}$ *is* ω*-stable if for every countable* model M, $S_{at}(M)$ *is countable.*

Theorem 2.19 (Keisler/Shelah) *(*$2^{\aleph_0} < 2^{\aleph_1}$*) If* $\boldsymbol{K}$ *has* $< 2^{\aleph_1}$ *models of cardinality* $\aleph_1$, *then* $\boldsymbol{K}$ *is* ω*-stable.*

This argument uses the continuum hypothesis twice. If the class $\boldsymbol{K}$ is assumed to have arbitrarily large models then ω-stability is provable from $\aleph_1$-categoricity in ZFC.

Now we turn to the more general notion of excellence for an arbitrary complete sentence of $L_{\omega_1,\omega}$. As in the quasiminimal excellent case, we will demand the existence of a unique amalgam of finite independent systems of countable models. Just as in passing from the rank one case to arbitrary first order theories, there is no longer a geometry on the universe of each model. But there is an independence relation satisfying many of the properties of first order forking. Independence is now defined in terms of splitting.

Definition 2.20 *A complete type* p *over* A splits *over* $B \subset A$ *if there are* $\mathbf{b}, \mathbf{c} \in A$ *which realize the same type over* B *and a formula* $\phi(\mathbf{x}, \mathbf{y})$ *with* $\phi(\mathbf{x}, \mathbf{b}) \in p$ *and* $\neg\phi(\mathbf{x}, \mathbf{c}) \in p$.

Definition 2.21 *Let* ABC *be atomic. We write* $A \underset{C}{\perp} B$ *and say* A *is free or independent from* B *over* C *if for any finite sequence* $\boldsymbol{a}$ *from* A, $\mathrm{tp}(\boldsymbol{a}/B)$ *does not split over some finite subset of* C.

It is relatively straightforward ([She83a, Bal09]) to show this notion has the basic properties of an abstract dependence relation: monotonicity, transitivity of independence and with somewhat more effort symmetry. However this dependence relation is well-behaved only over models or (assuming excellence) independent systems of models.

Definition 2.22 *A set* A *is* good *if the isolated types are dense in* $S_{at}(A)$.

For countable A, this is the same as $|S_{at}(A)| = \aleph_0$. But there may not be prime models over good sets. There are in $\aleph_0$ and $\aleph_1$, but not generally above $\aleph_1$ [Kni78, Kue78, LS93].

Definition 2.23 1. $\boldsymbol{K}$ *is* (λ, n)-good *if for any independent* n*-system* $\mathcal{S}$ *(of models of size* λ*), the union of the nodes is good.*

$\boldsymbol{K}$ *has* (λ, n)-existence *if for any independent* n*-system* $\mathcal{S}$ *(of models of size* λ*), their is a model that is primary over the union of the nodes.*

2. $\boldsymbol{K}$ *is* excellent *if it is* $(\aleph_0, n)$*-good for every* $n < \omega$. *That is, there is a prime model over any countable independent* n*-system.*

Note that excellence does not imply by definition the property: (λ, n)*-existence* for uncountable λ, i.e. that there is a primary model over an independent system of models of size λ.

Definition 2.24 *Let K be the class of models of a sentence of $L_{\omega_1,\omega}$. K is excellent if K is ω-stable* and *any of the following equivalent conditions hold. For any finite independent system of countable models with union C:*

1. *$S_{at}(C)$ is countable.*

2. *There is a unique primary model over C.*

3. *The isolated types are dense in $S_{at}(C)$.*

Shelah proved the following three theorem in [She83a, She83b].

Theorem 2.25 (ZFC) *Let λ be infinite and $n < \omega$. Suppose K has $(< \lambda, \leq n+1)$-existence and is $(\aleph_0, n)$-good. Then K has (λ, n)-existence.*

Theorem 2.26 (ZFC) *If an atomic class K is excellent and has an uncountable model then*

1. *it has models of arbitrarily large cardinality;*

2. *if it is categorical in one uncountable power it is categorical in all uncountable powers.*

By the very weak generalized continuum hypothesis (VWGCH) we mean the condition:

$$2^{\aleph_n} < 2^{\aleph_{n+1}} \text{for } n < \omega.$$

$2^{\aleph_n} < 2^{\aleph_{n+1}}$ is equivalent to a combinatorial principle on principle on $\aleph_n$ called weak diamond. 'Very few models in $\aleph_n$' means at most less than $2^{\aleph_{n-1}}$ while 'few models in $\aleph_n$' means less than $2^{\aleph_n}$.

Theorem 2.27 (VWGCH) *An atomic class K that has at least one uncountable model and very few models in $\aleph_n$ for each $n < \omega$ is excellent.*

The argument proceeds by a difficult induction using weak diamond a number of time to show: Very few models in $\aleph_n$ implies $(\aleph_0, n-2)$-goodness. It is open whether VWGCH suffices to delete 'very' from 'very few'. The next examples show that categoricity up to $\aleph_\omega$ is essential to conclude excellence. They also show the divergence between the natural 'syntactic' notion of type for sentences in $L_{\omega_1,\omega}$ and the natural 'semantic' notion of Galois type. Hart-Shelah extended by Baldwin-Kolesnikov [HS90, BK09] prove the following.

Theorem. For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;

2. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

3. ϕ_k has the disjoint amalgamation property;

4. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;

5. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.

6. ϕ_k is not $\aleph_{k-2}$-Galois stable;

7. But for $m \leq k-3$, ϕ_k is $\aleph_m$-Galois stable;

The arguments we have just discussed rely heavily on the properties of countable models of sentences of $L_{\omega_1,\omega}$. Shelah [She09, She10] attacks the more general and much harder situation where categoricity may begin above $\aleph_0$. He works in AEC which admit a 'frame'; an axiomatic framework for independence in a fixed cardinal. The goal is to propogate this behavior to larger cardinals.

As Zilber's quasiminimal excellence deals with categoricity of certain sentences in $L_{\omega_1,\omega}(Q)$, (i.e. those which axiomatize quasiminimal excellent classes) it is somewhat orthogonal to the theory for arbitrary $L_{\omega_1,\omega}$-sentences that satisfy categoricity conditions that I have just sketched. Variants on Shelah's analysis [She09] may clarify this but I don't know a clear reference.

2.3 Examples

We have discussed the categoricity transfer for abstract classes defined in $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$. We now consider some concrete examples.

2.3.1 Covers of Algebraic Groups

Definition 2.28 *1. A $\mathbb{Z}$-cover of a commutative algebraic group $\mathbb{A}(\mathbb{C})$ is a short exact sequence*

$$0 \to Z^N \to V \overset{\exp}{\to} \mathbb{A}(\mathbb{C}) \to 1. \tag{1}$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathbb{C}, +, \cdot)$ and so interdefinable with the field.

2. An $\mathbb{E}$-cover is the same sequence but viewed as a sequence of $End(\mathbb{A})$-modules.

Covers can be axiomatized by some simple first order axioms plus the requirement that the kernel is standard. Let $\mathbb{A}$ be a commutative algebraic group over an algebraically closed field F. Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q\in\mathbb{Q}}$ is a $\mathbb{Q}$-vector space.

2. The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each k_0-definable variety (where k_0 is the field of definition of $\mathbb{A}$).

3. exp is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

Let $T_A + \Lambda = \mathbb{Z}^N$ result from adding $\Lambda = \mathbb{Z}^N$ asserting the kernel of exp is standard.

$$(\exists \mathbf{x} \in (\exp^{-1}(1))^N)(\forall y)[\exp(y) = 1 \rightarrow \bigvee_{\mathbf{m} \in \mathbb{Z}^N} \Sigma_{i<N} m_i x_i = y]$$

Zilber[Zil03] raised the problem, ' Is $T_A + \Lambda = \mathbb{Z}^N$ categorical in uncountable powers?' Paraphrasing Zilber:

> Categoricity would mean the short exact sequence is a reasonable 'algebraic' substitute for the classical complex universal cover.

Zilber was aware in [Zil03] that the formulation in terms of $\mathbb{Z}$-covers was inadequate in general although sufficient when $\mathbb{A} = (\mathbb{C}, \cdot)$. After a number of years this project has been brought to a successful conclusion by work of Zilber, Gavrilovich and Bays [Bay09, Bay, Gav08, Gav06, BZ00, Zil06, Zil03] using $\mathbb{E}$-covers. Here is a quick list of their results.

Theorem 2.29 *1. Viewed as a $\mathbb{Z}$-cover if $\mathbb{A}$ is*

(a) $(\mathbb{C}, \cdot)$ then $T_A + \Lambda = \mathbb{Z}^N$ is quasiminimal excellent

(b) $(\tilde{F}_p, \cdot)$ then $T_A + \Lambda = \mathbb{Z}^N$ is not small. *Each completion is quasiminimal excellent.*

(c) elliptic curve w/o cm then $T_A + \Lambda = \mathbb{Z}^N$ is ω-stable.

(d) elliptic curve w cm then $T_A + \Lambda = \mathbb{Z}^N$ is not ω-stable as $\mathbb{Z}$-module

2. As an $\mathbb{E}$-module any simple abelian variety is quasiminimal excellent.

The case where the quotient is the multiplicative group of the algebraic closure of a $\mathbb{Z}_p$ is actually of mixed characteristic because the cover is still a $\mathbb{Q}$-vector space. 'cm' abbreviates that the elliptic curve admits complex multiplication.

Bays and Pillay [BP, Bay] have provided an exciting and different perspective on this problem. They show that the first order theory of an $\mathbb{E}$-cover (of a simple algebraic group) is classifiable (superstable, ndop, shallow, notop). Moreover each model is determined by two invariants, the isomorphism type of the kernel and the transcendence degree of the field coordinatizing the image variety. This recovers Zilber's categoricity in a more first order context at the cost of the formal axiomatization. It raises the question of which classifiable first order theories lead to quasiminimal excellent classes. In one way this result is unsurprising; Shelah was led to the notion of notop ('not the omitting types order property') by his study of excellence in the infinitary situation.

The key algebraic point in the study of covers is the selection of roots.

Definition 2.30 *A* multiplicatively closed divisible subgroup *associated with $a \in \mathbb{C}^*$, denoted $a^{\mathbb{Q}}$, is a* **choice** *of a multiplicative subgroup isomorphic to $\mathbb{Q}$ containing a .*

Definition 2.31 $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ *over* F *if given subgroups of the form* $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ *and* ϕ_m *such that*

$$\phi_m : F(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \rightarrow F(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : F(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \rightarrow F(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

Theorem 2.32 (thumbtack lemma)
For any $b_1, \ldots b_\ell \subset \mathbb{C}^*$, *there exists an* m *such that* $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ *over* F.

The Thumbtack Lemma (over finite independent systems of fields) is the key algebraic step for proving the quasiminimal excellence (both basic conditions and excellence) of Theorem 2.29.1.a. Zilber [Zil03] showed *equivalence* between certain 'arithmetic' statements about Abelian varieties (algebraic translations of excellence) and categoricity below $\aleph_\omega$ of the associated $L_{\omega_1,\omega}$-sentence. The equivalence depends on weak extensions of set theory and Shelah's categoricity transfer theorem. Recent work of Bays shows derives these conditions algebraically.

2.3.2 Pseudo-exponentiation

We first sketch the idea of the Hrushovski construction. There are many accounts of this construction and the many applications of the idea (See [Bal] for an annotated bibliography.) Similarly to the account of quasiminimal excellence we define a geometry. In this case, the construction begins with a class $\boldsymbol{K}$ of models with a dimension function

$$d : \{X : X \subseteq_{fin} G\} \rightarrow \mathbf{N}$$

which satisfies the axioms:

D1. $d(XY) + d(X \cap Y) \leq d(X) + d(Y)$

D2. $X \subseteq Y \Rightarrow d(X) \leq d(Y)$.

Each such dimension function gives rise to a geometry as follows.

Definition 2.33 *For* A, b *contained* M, $b \in \mathrm{cl}(A)$ *if* $d_M(bA) = d_M(A)$.

Naturally we can extend to closures of infinite sets by imposing finite character. If d satisfies:

D3 $d(X) \leq |X|$.

we get a full combinatorial (pre)-geometry with exchange.

Note that given a class of models with a dimension such as D1 and D2, we can derive an abstract elementary class by defining $A \leq B$ if for every finite sequence **b**, $d(\mathbf{b}/A) \geq 0$. This insight drove the connection of the Hrushovski construction with the analysis of probability on finite models [BS97]. In the language of quasiminimal excellence, $A \leq B$ means A is closed under the geometric closure operator.

The goal of Zilber's program is to realize $(\mathbb{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}$-sentence discovered by the Hrushovski construction. This program has two more specific objectives

Objective A Expand $(\mathbb{C}, +, \cdot)$ by a unary function f which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}$-sentence Σ satisfied by $(\mathbb{C}, +, \cdot, f)$ is categorical and has 'quantifier elimination' (in a suitable expansion of the language by some $L_{\omega_1,\omega}$-definable predicates).

Objective B Prove $(\mathbb{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

The notion of 'quantifier elimination' in Objective A is weaker than usual. It means that types are determined by quantifier free types. Since the class of models is not first order one cannot conclude each first order formula is equivalent to a quantifier free first order formula; indeed [Kir09] shows the Zilber exponential fields are not model complete in the usual first order sense; in [Kir10a] he shows the expansion of the base language to get infinitary quantifier elimination can be done using only first-order existentially definable predicates.

We say a homomorphism E from the additive to the multiplicative group of a field satisfies the *Schanuel Property* if for any n linearly independent elements over $\mathbb{Q}$, $\{z_1, \ldots z_n\}$

$$d_f(z_1, \ldots z_n, E(z_1), \ldots E(z_n)) \geq n.$$

Schanuel conjectured that complex exponentiation satisfies this equation.

Objective A has been realized in [Zil04] where the following axioms Σ are proved to define a quasiminimal excellent class. Let $L = \{+, R_i, E, 0, 1\}$ where the R_i are names for the all the irreducible varieties defined over $\mathbb{Q}$ in $(\mathbb{C}, +, \cdot)$. $(K, +, \cdot, E) \models \Sigma$ if

1. K is an algebraically closed field of characteristic 0.
2. E is a homomorphism from $(K, +)$ onto $(K^x, \cdot)$ and there is $\nu \in K$ transcendental over $\mathbb{Q}$ with $\ker E = \nu Z$.
3. E satisfies the Schanuel Property.
4. K is strongly exponentially algebraically closed.
5. $\ker E \approx (\mathbb{Z}, +)$.

We won't write out the details of strong exponential closure here [Zil04, Kir10a]. Nor do we discuss in detail the conjecture on intersection of tori (CIT). This is a strong conjecture of Zilber. Certain special (proved) cases of it (called the weak CIT) are used in the construction of the green and bad fields and permit formulating the axiom of strong existential closure for pseudo-exponential fields in first order logic. The last observation is not essential for proving Objective A since other axioms require $L_{\omega_1,\omega}(Q)$.

The axioms Σ are both consistent and categorical in all uncountable powers This result was first established using the Hrushovski construction with the following dimension function. For a finite subset X of an algebraically closed field k with a partial exponential function E. Let

$$\delta(X) = d_f(X \cup E(X)) - ld(X).$$

Apply the Hrushovski construction to the collection of such (k, E) with $\delta(X) \geq 0$ for all finite X *and with standard kernel*. The result is a quasiminimal excellent class.

Nominally we are left with objective B: establish the axioms Σ hold for complex exponentiation. But that is a major project. Zilber took one step in that direction.

Theorem 2.34 (Zilber) $(\mathbb{C}, +, \cdot, \exp)$ *has the countable closure property.*

Note that this does *not* suffice to obtain quasiminimality. However, establishing the Basic Conditions of Definition 2.9, which are at least *a priori* weaker than full excellence, would yield quasiminimality.

Recall Schanuel's conjecture for complex exponentiation: If $x_1, \ldots x_n$ are $\mathbb{Q}$-linearly independent complex numbers then $x_1, \ldots x_n, e^{x_1}, \ldots e^{x_n}$ has transcendence degree at least n over $\mathbb{Q}$. Assuming Schanuel, Marker [Mar06], extended by Günaydin and Martin-Pizarro (forthcoming), have verified existential closure axioms for $(\mathbb{C},, \cdot, \exp)$ for irreducible polynomials $p(X, Y) \in \mathbb{C}[X, Y]$. This requires fairly serious complex analysis (Hadamard factorization) plus the Schanuel conjecture, which has remained open for 50 years. And it provides only the most basic case of the strong exponential closure axioms.

A further natural question emerges, what are the properties of the unique pseudo-exponential field with cardinality $2^{\aleph_0}$? Does it play a role as a universal domain for exponential fields as the complex numbers do for fields? The following definition simplifies the following statements.

Definition 2.35 *An exponential field* $(F, +, \cdot, E, 0, 1)$ *satisfies the* Schanuel Nullstellensatz *if every exponential polynomial* $p(\mathbf{x})$ *over* F *which does not have a root in* F *is of the form* $p(\mathbf{x}) = E(g(\mathbf{x}))$ *for some exponential polynomial* g.

Theorem 2.36 $(F, +, \cdot, E, 0, 1)$ *satisfies the Schanuel Nullstellensatz if*

1. $(F, +, \cdot, E, 0, 1)$ *is the standard complex exponential field [HL84];*
2. *or* $(F, +, \cdot, E, 0, 1)$ *satisfies the axioms* Σ *[DMT09, Shk].*

Statement 1) uses Nevalinna theory; statement 2) is proved using exponential algebra. Shkop continues this theme by proving several consequences of Σ in [Sho10]. Kirby-Macintyre-Onshus (forthcoming) have made some intriguing observations about models of Σ. There is no obvious way to distinguish $\pm i$. But since the definitions of sin and cos are symmetric in $\pm i$, one can define each of those functions. $2\pi i$ is then definable over the emptyset (as the generator x of the kernel with $\sin(x/2) = 1$). This leads to developing an analog of complex conjugation on a very small subfield but with no good idea of how to extend it to the entire field.

As noted there are several frameworks in which this investigation could take place. Zilber simultaneously described the situation in a semantically defined class of Quasiminimal Excellence and provided axioms in $L_{\omega_1,\omega}(Q)$. Kirby clarified this situation by proving: The class of models of a quasiminimal excellent class is necessarily axiomatizable in $L_{\omega_1,\omega}(Q)$ [Kir10b]. The universal covers are axiomatized in $L_{\omega_1,\omega}$ and can be considered from a first order standpoint [BP]. And earlier Shelah [She87] had been led to a similar semantic approach of generalizing infinitary logics to AEC.

3 Challenge and Response

We return to study the response to the challenge of the Hrushovski construction. The conjectures specifying Zilber's thesis arose from the study of a model theoretic problem: is there a totally categorical first order theory which has only infinite models and is finitely axiomatized. The solution of that problem [Zil80, Zil84a, Zil84b, CHL85] depended on the analysis of the geometry on strongly minimal sets. Zilber identified three sorts of geometry: disintegrated (the lattice of closed subsets is distributive), (locally) modular (over a parameter) the lattice of closed subsets is modular and otherwise. The conjecture asserted that all other strongly minimal structures could be interpreted in an algebraically closed field. As noted this conjecture failed.

3.1 Zariski Structures

The new strongly minimal set constructed by Hrushoski [Hru93], provides a non-locally modular (i.e. not group-like) strongly minimal set that cannot be interpreted in a field; this refute's Zilber's conjecture. One response is to strengthen the hypothesis of the conjecture by imposing a further condition on the strongly minimal set. The way to do this is suggested by a basic problem with the project of formalizing algebraic geometry through the study of algebraically closed fields: equations (and conjunctions of equations) occupy a special role in algebraic geometry. But from a model theoretic standpoint there is no semantic way to distinguish them among the definable sets. The solution is to consider strongly minimal sets equipped with a topology.

Generalizing the algebraic geometric notion of Zariski geometry, Hrushovski and Zilber in [HZ93] describe axioms on the set of definable subsets of a set X and the powers X^n specifying a topology on each X^n and relations between the powers to formalize the notion of 'smooth algebraic variety'. They are able to establish that every *very* ample Noetherian Zariski structure is interpretable in an algebraically closed field.

The following result (e.g. in [Zil10]) establishes that an ample (non-linear) Zariski structure is a finite cover of an algebraic variety.

Theorem 3.1 (Hrushovski-Zilber) *If M is an ample Noetherian Zariski structure then there is an algebraically closed field K, a quasi-projective algebraic curve* $C_M = C_M(K)$ *and a surjective map*

$$p : M \mapsto C_M$$

of finite degree such that for every closed $S \subseteq M^n$*, the image* $p(S)$ *is Zariski closed in* C_M^n *(in the sense of algebraic geometry); if* $\hat{S} \subseteq C_M^n$ *is Zariski closed, then* $p^{-1}(\hat{S})$ *is a closed subset of* M^n *(in the sense of the Zariski structure* M*).*

But such a structure may be a finite cover of the affine line, $P^1(\boldsymbol{K})$, or of an elliptic curve that can *not* be interpreted in an algebraically closed field[Zil10]. The additional requirement that the Zariski structure be 'very ample' implies that the structure on the fibers can also be recovered in the field. In an attempt to understand these structures Zilber turns to non-commutative geometry. In particular, these n-covers induce an nth root quantum torus[Zil10].

3.2 Analytic Structures

As another response to the Hrushovski construction, Zilber suggested weakening the conclusion. Instead of interpreting the models in algebraically closed fields by first order formulas, find an analytic model. A crucial issue is to specify what is meant by 'analytic'. The choices include: real analytic, complex analytic, and 'Zilber analytic'. The last mean satisfy an axiomatic definition akin to the notion of a Zariski geometry (but now non-Noetherian). (See [PZ03, Zil10] for a more precise definition.) There can be no complex analytic structure for the finite rank case but there are real analytic models for some closely related structures that we now examine.

Poizat produced a variant on the Hrushovski construction to expand an algebraically closed field by a unary predicate for a proper subgroup of the multiplicative group. For this structure he used the dimension function:

$$\delta(X) = 2d_f(X) - \mathrm{ld}(X \cap G).$$

This yields an ω-stable theory of rank $\omega \times 2$. Poizat calls this the green field; since it was a step towards the construction of a bad field, we call it a naughty field.

Zilber proposed to find an analytic model for this theory[CZ08]. For reasons we explain below, a near relative is also important. The basic notion is to take as the subgroup the spiral given by: $G = \{\exp(\epsilon t) : t \in \mathbb{R}\}$. This structure is real analytic but is not a model of Poizat's theory; a result of Marker (extending his argument in [Mar90]) shows the resulting structure is unstable. But taking copies of the spiral indexed by $\mathbb{Q}$ or $\mathbb{Z}$ gives interesting model theoretic results at the cost of losing the real analyticity of the model. The concept of Zilber-analyticity would try to axiomatize a notion including some of these examples.

Theorem 3.2 (Caycedo-Zilber) *Assume Schanuel's conjecture and weak CIT; let* $\epsilon = 1 + i$.

The naughty/green field

$$(\mathbb{C}, +, \cdot, \mathbb{G})$$

where $G = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$ *and* exp *is complex exponentiation satisfies Poizat's axioms and so is an* ω*-stable field of rank* $\omega \times 2$.

The emerald field

$$(\mathbb{C}, +, \cdot, \mathbb{G}')$$

where $G' = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Z}\}$ *and* exp *is complex exponentiation is a superstable structure.*

For the first result see [Zil05a, CZ08]; for the second see [Cay08]. Caycedo has similar results for elliptic curves.

In attempting to understand these examples Zilber introduced still another meaning of geometry into model theory. Connes' theory of non-commutative geometry provides three different descriptions of a 'quantum torus'.

1. A structure
2. A finitely generated non-commutative C^*-algebra
3. A foliation

Zilber [Zil10] attempts to use Connes theory to understand the non-algebraic examples in Theorem 3.1 and examples in Theorem 3.2; he attaches non-commutative C^*-algebras and or foliations to the structures. Baldwin and Gendron noticed that only one of these is 'the' quantum torus.

Lemma 3.3 *Assume Schanuel's conjecture; let* $\epsilon = 1 + i$. *The concrete superstable (emerald field) version*
$\mathbb{R}^2/\mathbb{G}'$ *where*

$$\mathbb{G}' = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Z}\}$$

and exp *is complex exponentiation is the leaf space of the Kronecker foliation, the quantum torus.*

The concrete naughty (green) field is a 'near' Quantum Torus. That is, $\mathbb{R}^2/\mathbb{G}$ *where*

$$\mathbb{G} = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$$

and exp *is complex exponentiation. This is apparently a 'new' structure to topologists, a quotient of the quantum torus by* $\mathbb{Q}$.

Just as Zariski structures are defined axiomatically to generalize algebraic geometry, analytic Zariski structures [Zil10] are defined axiomatically to generalize the properties of analytic subsets of the complex numbers. A key distinction from Zariski structures is the loss of the Noetherian property.

4 The visions of Zilber and Shelah

We have described Zilber's thesis in terms of the connection to fundamental structures of mathematics. This thesis has led to many fascinating results and conjectures in logic, number theory, and complex analysis. It has produced a deeper understanding of complex exponentiation and suggested the notion of a new universal structure for exponential algebra. The attempt to understand complex exponentiation has forced investigation of logics extending first order. And the attempt to understand counterexamples to various conjectures has led to connections with non-commutative geometry. Since the entire project developed from attempts to understand the geometries associated with strongly minimal sets, it is perhaps natural that geometries that are unusual from the model theoretic standpoint are also unusual from other mathematical viewpoints.

In a sense, Shelah proceeds from the other direction (See the introduction to [She09]. His goal is to classify classes of structures. We consider three themes of his analysis. A key theme is finding dividing lines: a property P of a class of structures such that both P and $\neg P$ have strong consequences. For example, in the first order case, a stable theory admits a nice notion of independence. On the other hand an unstable theory has the maximal number of models in every uncountable power, which, Shelah argues, makes the models unclassifiable. In trying to develop the model theory of infinitary logic, he used the same motif: model amalgamation is a key dividing line. It is a powerful tool for obtaining structure results. The notion of excellence arises as a condition of n-dimensional on countable models which is sufficient to guarantee the existence of arbitrarily large models and to guarantee amalgamation in all cardinalities. And Shelah shows (using weak-diamond) that failure of amalgamation in a categoricity cardinal λ leads to many models in λ^+. Variants of this argument are crucial 9in proving Theorem 2.27. A second significant component of Shelah's work on infinitary logic is to discover algebraic/model theoretic notions which are cardinal dependent. One striking example is that $(\aleph_0, n)$ goodness propagates to $(\aleph_n, n-1)$ goodness and so $(\aleph_0, n)$ goodness for all n propagates to the existence of arbitrarily large models. In fact, this need to build the existence of a model in larger powers from below in some ways motivates the semantic approach of abstract elementary classes. An apparently key observation concerning AEC's is that they can be defined as class of first order models omitting a family of types ([She87] and Chapter 4 of [Bal09]). This in turns allows the use of Ehrenfeucht-Mostowski models and the calculation of the Hanf number for AEC's. But these tools are not central to the development in [She09] That work is 'algebraic'; i.e. the work is with structures, not logics.

The main gap is most easily stated as every first order theory either has the maximal number of models in sufficiently large cardinals or the number of models of T in $\aleph_\alpha$ is bounded by $\beth_\beta(\alpha)$ for a β depending on T. But the real point is that the every model of a theory with few models can be decomposed as a tree of submodels and each of these models is determined by a family of geometries. The connections between these geometries (regularity, orthogonality, hereditary orthogonality) were key tools along with the study of the (modularity, triviality etc.) of the individual geometries (geometric stability theory) both in obtaining the fine structure of spectrum of models [HEL00] and in such crucial applications as Hrushovski's work [Hru96] on Manin-

Mumford for function fields.

References

[Bal] J.T. Baldwin. A field guide to Hrushovski constructions. Report: http://www.math.uic.edu/~jbaldwin/pub/hrutrav.pdf.

[Bal09] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, 2009. www.math.uic.edu/~jbaldwin.

[Bay] M. Bays. Model theory of exponential maps of algebraic varieties; slides march 2010. http://people.maths.ox.ac.uk/~bays/misc/borisfest/notes/bays.pdf.

[Bay09] M. Bays. *Categoricity results for exponential maps of 1-dimensional algebraic groups & Schanuel Conjectures for Powers and the CIT*. PhD thesis, Oxford, 2009. http://people.maths.ox.ac.uk/~bays/dist/thesis/.

[BF85] J. Barwise and S. Feferman, editors. *Model-Theoretic Logics*. Springer-Verlag, 1985.

[BHMPW07] A. Baudisch, M. Hils, A. Martin-Pizarro, and F.O. Wagner. Die böse Farbe. 2007.

[BK09] John T. Baldwin and Alexei Kolesnikov. Categoricity, amalgamation, and tameness. *Israel Journal of Mathematics*, 170, 2009. also at www.math.uic.edu/~\jbaldwin.

[BL71] J.T. Baldwin and A.H. Lachlan. On strongly minimal sets. *Journal of Symbolic Logic*, 36:79–96, 1971.

[BP] M. Bays and A. Pillay. title not determined. paper in preparation on the classifiability of the first order theory of covers.

[BS97] J.T. Baldwin and S. Shelah. Randomness and semigenericity. *Transactions of the American Mathematical Society*, 349:1359–1376, 1997.

[BZ00] M. Bays and B.I. Zilber. Covers of multiplicative groups of an algebraically closed field of arbitrary characteristic. preprint: arXive math.AC/0401301, 200?

[Cay08] J.D. Caycedo. The first-order theory of emerald points. preprint, 2008.

[CHL85] G.L. Cherlin, L. Harrington, and A.H. Lachlan. $\aleph_0$-categorical, $\aleph_0$-stable structures. *Annals of Pure and Applied Logic*, 28:103–135, 1985.

[CZ08] J.D. Caycedo and B. Zilber. Green points in the complex numbers. preprint, 2008.

[DMT09] P. DAquino, A. Macintyre, and G. Terzo. Schanuel nullstellensatz for the zilber field. preprint, 2009.

[Gav06] Misha Gavrilovich. *Model Theory of universal covering spaces of complex analytic varieties.* PhD thesis, Balliol college Oxford, 2006. http://people.maths.ox.ac.uk/bays/misha-thesis.pdf.

[Gav08] Misha Gavrilovich. A remark on transitivity of galois action on the set of uniquely divisible abelian extensions of $E(\overline{Q})$ by z^2. *Journal of K-theory*, 38:135–152, 2008.

[GV06] R. Grossberg and M. VanDieren. Shelah's categoricity conjecture from a successor for tame abstract elementary classes. *The Journal of Symbolic Logic*, 71:553–568, 2006.

[HEL00] B. Hart, Hrushovski E., and C. Laskowski. The uncountable spectra of countable theories. *Annals of Mathematics*, 152:207–257, 2000.

[HK06] T. Hyttinen and M. Kesälä. Independence in finitary abstract elementary classes. *Ann. Pure Appl. Logic*, 143(1-3):103–138, 2006.

[HL84] W. Henson and L.Rubel. Some applications of Nevalinna theory to mathematical logic: identities of exponential functions. *Transactions of the American Mathematical Society*, 282:1–32, 1984.

[Hru93] E. Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62:147–166, 1993.

[Hru96] E. Hrushovski. The Mordell-Lang conjecture over function fields. *Journal of the American Mathematical Society*, 9:667–690, 1996.

[HS90] Bradd Hart and Saharon Shelah. Categoricity over P for first order T or categoricity for $\phi \in l_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$. *Israel Journal of Mathematics*, 70:219–235, 1990.

[HZ93] E. Hrushovski and B. Zilber. Zariski geometries. *Bulletin of the American Mathematical Society*, 28:315–324, 1993.

[ITW04] M. Itai, A. Tsuboi, and K. Wakai. Construction of saturated quasiminimal structures. *Journal of Symbolic Logic*, 69:9–22, 2004.

[Kir09] Jonathan Kirby. Finitely presented exponential fields. http://arxiv.org/abs/0912.4019, 2009.

[Kir10a] Jonathan Kirby. A note on the axioms for zilber's pseudo-exponential fields. http://arxiv.org/abs/0912.4019, 2010.

[Kir10b] Jonathan Kirby. On quasiminimal excellent classes. *Journal of Symbolic Logic*, 75:551–564, 2010. http://arxiv.org/PS_cache/arxiv/pdf/0707/0707.4496v3.pdf.

[Kni78] J.F. Knight. Prime and atomic models. *Journal of Symbolic Logic*, 43:385–393, 1978.

[Kue78] D. W. Kueker. Uniform theorem in infinitary logic. In A. Macintyre, L. Pacholski, and J. Paris, editors, *Logic Colloquium 77*. North Holland, 1978.

[LS93] Michael C. Laskowski and Saharon Shelah. On the existence of atomic models. *J. Symbolic Logic*, 58:1189–1194, 1993.

[Mar66] W. Marsh. *On ω_1-categorical and not ω-categorical theories*. PhD thesis, Dartmouth College, 1966.

[Mar90] D. Marker. Semialgebraic expansions of **c**. *Trans. Amer. Math. Soc.*, 320:581–592, 1990.

[Mar06] D. Marker. A remark on Zilber's pseudo-exponentiation. *The Journal of Symbolic Logic*, 71:791–798, 2006.

[Mor65] M. Morley. Categoricity in power. *Transactions of the American Mathematical Society*, 114:514–538, 1965.

[Per80] M. G. Peretyatkin. An example of an ω_1-categorical finitely axiomatizable theory. *Algebra and Logic*, 19:202–229, 1980. translated from Algebra y Logica.

[PT] A. Pillay and P. Tanović. Generic stability, regularity, and quasiminimality. preprint.

[PZ03] N. Peatfield and B. Zilber. Analytic Zariski structures and the Hrushovski construction. preprint, 2003.

[She75] S. Shelah. Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$. *Israel Journal of Mathematics*, 20:127–148, 1975. paper 48.

[She83a] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part A. *Israel Journal of Mathematics*, 46:3:212–240, 1983. paper 87a.

[She83b] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part B. *Israel Journal of Mathematics*, 46;3:241–271, 1983. paper 87b.

[She87] Saharon Shelah. Classification of nonelementary classes II, abstract elementary classes. In J.T. Baldwin, editor, *Classification theory (Chicago, IL, 1985)*, pages 419–497. Springer, Berlin, 1987. paper 88: Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; volume 1292 of *Lecture Notes in Mathematics*.

[She09] S. Shelah. *Classification Theory for Abstract Elementary Classes*. Studies in Logic. College Publications `www.collegepublications.co.uk`, 2009. Binds together papers 88r, 600, 705, 734 with introduction E53.

[She10] S. Shelah. *Classification Theory for Abstract Elementary Classes: II.* Studies in Logic. College Publications `<www.collegepublications.co.uk>`, 2010. Binds together papers 300 A-G, E46, 838.

[Shk] A. Shkop. Henson and rubel's theorem for zilber's pseudo-exponentiation. preprint.

[Sho10] A. Shokp. *On pseudoexponential fields and Schanuel's conjecture*. PhD thesis, University of Illinois at Chicago, 2010.

[Zil80] B.I. Zil'ber. Strongly minimal countably categorical theories. *Siberian Mathematics Journal*, 24:219–230, 1980.

[Zil84a] B.I. Zil'ber. Strongly minimal countably categorical theories II. *Siberian Mathematics Journal*, 25:396–412, 1984.

[Zil84b] B.I. Zil'ber. Strongly minimal countably categorical theories III. *Siberian Mathematics Journal*, 25:559–571, 1984.

[Zil91] B.I. Zil'ber. *Uncountably Categorical Theories*. Translations of the American Mathematical Society. American Mathematical Society, 1991. summary of earlier work.

[Zil03] B.I. Zilber. Model theory, geometry and arithmetic of universal covers of a semi-abelian variety. In *Model Theory and Applications*, Quaterna di matematica, pages 427–458, 2003.

[Zil04] B.I. Zilber. Pseudo-exponentiation on algebraically closed fields of characteristic 0. *Annals of Pure and Applied Logic*, 132:67–95, 2004.

[Zil05a] B.I. Zilber. Bi-coloured fields on the complex numbers. *Journal of Symbolic Logic*, 69:1171–1186, 2005.

[Zil05b] B.I. Zilber. A categoricity theorem for quasiminimal excellent classes. In *Logic and its Applications*, Contemporary Mathematics, pages 297–306. AMS, 2005.

[Zil06] B.I. Zilber. Covers of the multiplicative group of an algebraically closed field of characteristic 0. *J. London Math. Soc.*, pages 41–58, 2006.

[Zil10] B.I. Zilber. *Zariski Geometries, Geometry from the Logician's Point of View*. London Mathematical Society; Cambridge University Press, 2010.