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無限次元固有値問題に対する固有値の非存在証明

Some eigenvalue excluding methods for infinite dimensional operators

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1 Introduction

Consider the following eigenvalue problem

$$(A+Q)u = \lambda Bu, \tag{1}$$

where $A : D(A) \to Y$, $Q : X \to Y$ and $B : X \to Y$ are linear operators for the complex Hilbert spaces $D(A) \subset X \subset Y$. The inner products and norms of X and Y are denoted $(u,v)_X$, $\langle u,v\rangle_X$, $\|u\|_X = \sqrt{\langle u,u\rangle_X}$, $(u,v)_Y$ and $\|u\|_Y = \sqrt{(u,u)_Y}$, respectively. Here, note that it is possible that we use two inner products $(u,v)_X$ and $\langle u,v\rangle_X$.

Assumption 1

A1 For all $\phi \in Y$, $A\psi = \phi$ has the unique solution $\psi \in D(A) \subset X$. Denote this mapping by $A^{-1} : Y \to X$.

A2 The operator A satisfies

$$(u,v)_X = (Au,v)_Y, \qquad \forall u \in D(A), \quad \forall v \in X. \tag{2}$$

A3 There exists a constant $C_p > 0$ such that

$$\|Bu\|_Y \le C_p\|u\|_X, \qquad \forall u \in X. \tag{3}$$

A4 There exists a constant $C_b > 0$ such that

$$\|A^{-1}Bu\|_X \le C_b\|u\|_X, \qquad \forall u \in X. \tag{4}$$

In actual validated computation, explicit values of C_p and C_b have to be evaluated. If the imbedding $D(A) \hookrightarrow X$ is compact, A^{-1} is also compact.

1.1 Example 1

In the case of linear elliptic problem for a bounded domain Ω in $\mathbb{R}^n (n = 1, 2, 3)$, where $b \in L^\infty(\Omega)^n$, $c \in L^\infty(\Omega)$,

$$\left\{\begin{array}{rcll} -\Delta u + b \cdot \nabla u + cu &=& \lambda u & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{5}$$

$$A = -\Delta, \quad Q = b \cdot \nabla + c, \quad B = I_{X \to Y},$$

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad X = H_0^1(\Omega), \quad Y = L^2(\Omega),$$

$$(u, v)_X = \langle u, v \rangle_X = (\nabla u, \nabla v)_{L^2(\Omega)}, \quad (u, v)_Y = (u, v)_{L^2(\Omega)} =: \int_\Omega uv\, dx.$$

The constant C_p is the Poincaré or Rayleigh-Ritz constant. For example, if $\Omega = (0, 1) \times (0, 1)$, $C_p = 1/(\pi\sqrt{2})$ and $C_b = C_p^2$.

1.2 Example 2

In the case of Orr-Sommerfeld equation

$$\left\{\begin{array}{r} (-D^2 + a^2)^2 u + iaR[V(-D^2 + a^2) + V'']u = \lambda(-D^2 + a^2)u, \\ u(x_1) = u(x_2) = u'(x_1) = u'(x_2) = 0, \end{array}\right. \tag{6}$$

$$A = (-D^2 + a^2)^2, \quad Q = iaR[V(-D^2 + a^2) + V''], \quad B = -D^2 + a^2,$$

$$\Omega = [x_1, x_2], \quad D(A) = H^4(\Omega) \cap H_0^2(\Omega), \quad X = H_0^2(\Omega), \quad Y = L^2(\Omega),$$

$$(u, v)_X = \langle u, v \rangle_X = ((-D^2 + a^2)u, (-D^2 + a^2)v)_{L^2(\Omega)}, \quad (u, v)_Y = (u, v)_{L^2(\Omega)} =: \int_\Omega uv\, dx.$$

If $\Omega = (-1, 1)$, $C_p = 1$, and C_b can be taken as $C_b = 1/(\pi^2/4 + a^2)$ [9].

1.3 Example 3

In the case of a linearized problem of the Kolmogorov equation [5]

$$\Delta^2\psi + R(J(\phi_N, \Delta\psi) + J(\psi, \Delta\phi_N)) = \lambda R \Delta\psi \quad \text{in } \Omega, \tag{7}$$

where $\Omega = (-\pi/\alpha, \pi/\alpha) \times (-\pi, \pi)$, $J(u, v) = u_x v_y - u_y v_x$, $R > 0$, $0 < \alpha < 1$ and $\psi \in X^3$ and $\phi_N \in X^4$ are in function spaces such that

$$X^k := X_0^k \oplus X_1^k \oplus X_2^k \oplus \cdots,$$

$$X_0^k := \left\{ \sum_{n=1}^\infty a_n \cos(ny) \;\middle|\; a_n \in \mathbb{C}, \sum_{n=1}^\infty n^{2k} a_n^2 < \infty \right\},$$

$$X_m^k := \left\{ \sum_{n=-\infty}^\infty a_n \cos(m\alpha x + ny) \;\middle|\; a_n \in \mathbb{C}, \sum_{n=-\infty}^\infty ((\alpha m)^{2k} + n^{2k}) a_n^2 < \infty \right\}, \quad m \geq 1.$$

We can set

$$A = \Delta^2, \quad Q = R(J(\phi_N, \Delta\cdot) + J(\cdot, \Delta\phi_N)), \quad B = R\Delta,$$

$$D(A) = X^4, \quad X = X^3, \quad Y = X^0,$$

$$(u, v)_X = (\Delta u, \Delta v)_{L^2(\Omega)}, \quad (u, v)_Y = (u, v)_{L^2(\Omega)} =: \int_\Omega uv\,dx.$$

Here, for this problem (7), we introduce one more inner product

$$\langle u, v \rangle_X = (u_{xxx} + 3u_{xxy} + 3u_{xyy} + u_{yyy}, v_{xxx} + 3v_{xxy} + 3v_{xyy} + v_{yyy})_{L^2(\Omega)}$$

which implies H^3-seminorm $|u|_{H^3(\Omega)}$ because Q has the third order differential term. Under these definitions, we can take $C_p = R\alpha^{-1}$ and $C_b = R\alpha^{-2}$.

2 Eigenvalue excludings

Our concept of excluding method is due to the idea by Lahmann-Plum [2](pp.192). Let $\mu \in \mathbb{C}$ be a *candidate* point which is suspected that no eigenvalue is close to μ. We consider equivalently shifted eigenvalue problem of Eq.(1) by

$$\hat{L}u = (\lambda - \mu)Bu, \tag{8}$$

where

$$\hat{L}u := Au - f(u) : \quad D(A) \to Y, \tag{9}$$

and

$$f(u) := -(Q - \mu B)u : \quad X \to Y. \tag{10}$$

Then the following excluding result is obtained.

Lemma 1 If the operator $\hat{L}$ has the inverse $\hat{L}^{-1} : Y \to D(A)$, and there exists $\hat{M} > 0$ such that

$$\|\hat{L}^{-1}\phi\|_X \le \hat{M}\|\phi\|_Y, \quad \forall \phi \in Y, \tag{11}$$

then there is *no* eigenvalue $\tilde{\lambda}$ of Eq.(1) in the area such that

$$|\tilde{\lambda} - \mu| < \frac{1}{C_p\hat{M}}. \tag{12}$$

Proof.
For any eigenpair $(\tilde{\lambda}, \tilde{u}) \in \mathbb{C} \times D(A)$ such that

$$\hat{L}\tilde{u} = (\tilde{\lambda} - \mu)B\tilde{u}, \quad \tilde{u} \ne 0,$$

since $\hat{L}\tilde{u} \in Y$, substituting $\hat{L}\tilde{u}$ into the condition (11) as ϕ and using **A3**, we have

$$\begin{aligned} \|\tilde{u}\|_X &\le \hat{M}\|\hat{L}\tilde{u}\|_Y \\ &\le \hat{M}C_p\,|\tilde{\lambda} - \mu|\,\|\tilde{u}\|_X, \end{aligned}$$

therefore

$$|\tilde{\lambda} - \mu| \ge \frac{1}{C_p\hat{M}},$$

then the result is derived. □

Next, we show an another excluding criterion using an operator on X.

From the assumption **A2**, the weak problem of Eq.(1) is

$$(u, v)_X = ((\lambda B - Q)u, v)_Y. \qquad \forall v \in X. \tag{13}$$

Using $A^{-1} : Y \to X$, the weak problem (13) can be rewritten equivalently in the form

$$u = A^{-1}(\lambda B - Q)u$$

on X, hence

$$u + A^{-1}Qu = \lambda A^{-1}Bu.$$

Then we have shifted eigenvalue problem for $(u, \lambda) \in X \times \mathbb{C}$ such that

$$Lu = (\lambda - \mu)A^{-1}Bu, \tag{14}$$

where

$$Lu := u - A^{-1}f(u) : \quad X \to X. \tag{15}$$

Then, an another excluding lemma is obtained as follows.

Lemma 2 If the operator L has the inverse $L^{-1} : X \to X$, and there exists $M > 0$ such that

$$\|L^{-1}\phi\|_X \le M\|\phi\|_X, \quad \forall \phi \in X, \tag{16}$$

then there is *no* eigenvalue $\tilde{\lambda}$ of Eq.(13) in the area such that

$$|\tilde{\lambda} - \mu| < \frac{1}{C_b M}. \tag{17}$$

Proof.

For any eigenpair $(\tilde{\lambda}, \tilde{u}) \in \mathbb{C} \times X$ of Eq.(13) which satisfies

$$L\tilde{u} = (\tilde{\lambda} - \mu)A^{-1}B\tilde{u}, \quad \tilde{u} \ne 0,$$

taking $\phi \in X$ as $L\tilde{u}$ in (16), we have

$$\begin{aligned} \|\tilde{u}\|_X &\le M\|L\tilde{u}\|_X \\ &= |\tilde{\lambda} - \mu| \; M\|A^{-1}B\tilde{u}\|_X \\ &\le |\tilde{\lambda} - \mu| \; C_b M\|u\|_X, \end{aligned}$$

by **A4**, therefore

$$|\tilde{\lambda} - \mu| \ge \frac{1}{C_b M}.\square$$

Now we will show the relation between the invertibility of L and $\hat{L}$.

Lemma 3 If L is invertible, then $\hat{L}$ is also invertible.

Proof. Assume $L : X \to X$ has the inverse. For any $\phi \in Y$, there exists $v \in X$ such that $v = A^{-1}\phi$ by **A1**, and there exists $u \in X$ such that $Lu = v$, namely,

$$u - A^{-1}f(u) = A^{-1}\phi \quad \Rightarrow \quad u = A^{-1}(f(u) + \phi).$$

Then by the definition of A^{-1}, $u \in D(A)$ and

$$Au = f(u) + \phi \quad \Rightarrow \quad \hat{L}u = \phi.$$

□

Combining Lemmata, the following excluding theorem is obtained.

Theorem 1 Assume the operator L has the inverse $L^{-1} : X \to X$, and there exists $M > 0$ such that

$$\|L^{-1}\phi\|_X \le M\|\phi\|_X, \quad \forall \phi \in X, \tag{18}$$

then there is *no* eigenvalue $\tilde{\lambda}$ of Eq.(13) in the area such that

$$|\tilde{\lambda} - \mu| < \frac{1}{C_b M}. \tag{19}$$

Moreover if there exists $\hat{M} > 0$ such that

$$\|\hat{L}^{-1}\phi\|_X \le \hat{M}\|\phi\|_Y, \quad \forall \phi \in Y, \tag{20}$$

then also there is *no* eigenvalue $\tilde{\lambda}$ of Eq.(1) in the area such that

$$|\tilde{\lambda} - \mu| < \frac{1}{C_p \hat{M}}. \tag{21}$$

3 Invertibility condition of L

This section describes a computable condition assuring the invertibility of the linear operator L such that

$$Lu = u - A^{-1}f(u).$$

Basically, this verification method is an extension of the one for solutions of second-order elliptic boundary value problems introduced by a part of the authors [6, 7]. From now on, the identity maps on X are denoted by the symbol I.

3.1 Finite dimensional subspace and projection error

First we introduce a finite dimensional approximation subspace $S_h \subset X$, and let $P_h : X \to S_h$ be the orthogonal projection defined by

$$(v - P_h v, v_h)_X = 0, \qquad \forall v_h \in S_h. \tag{22}$$

Since S_h is the closed subspace of X, any element $u \in X$ can be uniquely decomposed into

$$u = u_h + u_*, \qquad u_h \in S_h,\ u_* \in S_*,$$

where

$$S_* := \{u_* \in X \mid u_* = (I - P_h)u, \ \ u \in X\}.$$

We assume P_h has the following properties.

Assumption 2

A5 There exists $C(h) > 0$ such that

$$\|(I - P_h)u\|_X \le C(h)\|Au\|_Y, \qquad \forall u \in D(A). \tag{23}$$

A6 There exists $\nu_1 > 0$ such that

$$\|P_h A^{-1} f(u_*)\|_X \le \nu_1 \|u_*\|_X, \qquad \forall u_* \in S_*. \tag{24}$$

A7 There exist $\nu_2 > 0$ and $\nu_3 > 0$ such that

$$\|f(u)\|_Y \le \nu_2 \|P_h u\|_X + \nu_3 \|(I - P_h)u\|_X, \qquad \forall u \in X. \tag{25}$$

For the case of Example 1 (5), P_h is the usual H_0^1-projection, and it can be taken as $C(h) = h/\pi$ and $h/(2\pi)$ for bilinear and biquadratic element, respectively, for the rectangular mesh on the square domain [3]. And $C(h) = 0.493h$ for the linear and uniform triangular mesh of the convex polygonal domain [1]. Here, $h > 0$ stands for the maximum mesh size for given finite elements.

For $u_* \in S_*$, since $P_h(-\Delta)^{-1}(b \cdot \nabla u_* + cu_* - \mu u_*) \in S_h$,

$$\begin{aligned}
&\|\nabla P_h(-\Delta)^{-1}(b \cdot \nabla u_* + cu_* - \mu u_*)\|_{L^2(\Omega)}^2 \\
&= (\nabla P_h(-\Delta)^{-1}(b \cdot \nabla u_* + cu_* - \mu u_*), \nabla P_h(-\Delta)^{-1}(b \cdot \nabla u_* + cu_* - \mu u_*))_{L^2(\Omega)} \\
&= (b \cdot \nabla u_* + cu_* - \mu u_*, P_h(-\Delta)^{-1}(b \cdot \nabla u_* + cu_* - \mu u_*))_{L^2(\Omega)} \\
&\le \|b \cdot \nabla u_* + cu_* - \mu u_*\|_{L^2(\Omega)}\ C_p \|P_h \nabla(-\Delta)^{-1}(b \cdot \nabla u_* + cu_* - \mu u_*)\|_{L^2(\Omega)},
\end{aligned}$$

we have

$$\begin{aligned}
\|P_h \nabla(-\Delta)^{-1}(b \cdot \nabla u_* + cu_* - \mu u_*)\|_{L^2(\Omega)} &\le C_p\ \|b \cdot \nabla u_* + cu_* - \mu u_*\|_{L^2(\Omega)} \\
&\le C_p\ (\|b \cdot \nabla u_*\|_{L^2(\Omega)} + \|(c - \mu)u_*\|_{L^2(\Omega)}) \\
&= C_p\ (\||b|_E\|_{L^\infty(\Omega)} + C_p\|c - \mu\|_{L^\infty(\Omega)})\|\nabla u_*\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore we can take

$$\begin{aligned}
\nu_1 &= C_p\ (\||b|_E\|_{L^\infty(\Omega)} + C_p\|c - \mu\|_{L^\infty(\Omega)}), \\
\nu_2 = \nu_3 &= \||b|_E\|_{L^\infty(\Omega)} + C_p\|c - \mu\|_{L^\infty(\Omega)}.
\end{aligned}$$

3.2 Newton-like method

We will show that the problem $Lu = 0$ has only unique solution $u = 0$. Defining $F : X \to X$ by

$$Fu = A^{-1} f(u), \tag{26}$$

the problem $Lu = 0$ can be rewritten equivalently in the fixed-point form

$$u = Fu.$$

In order to prove the uniqueness ($u = 0$) of the fixed-point of F on X, for a nonempty, bounded, convex and closed set $U \subset X$ centered zero, we will check

$$\overline{FU} \subset \mathrm{int}(U).$$

From uniqueness of decomposition for S_h and S_*, the fixed-point equation $u = Fu$ on X is equivalently rewritten as

$$\begin{cases} P_h u &= P_h F u, \\ (I - P_h)u &= (I - P_h)Fu. \end{cases} \tag{27}$$

Now, let us define the Newton-like operator $\mathcal{N}_h : X \longrightarrow S_h$ by

$$\mathcal{N}_h u := P_h u - [I - F]_h^{-1} P_h (I - F)u.$$

Here $[I - F]_h^{-1} : S_h \longrightarrow S_h$ means the inverse of the restriction of the operator $P_h(I - F) : X \longrightarrow S_h$ to S_h. Note that the existence of $[I - F]_h^{-1}$ is equivalent to the invertibility of a matrix, which is numerically checked in the actual verified computations. Since $P_h u = P_h \mathcal{N}_h u \Leftrightarrow P_h u = P_h F u$, using a map T on X defined by

$$Tu = \mathcal{N}_h u + (I - P_h)Fu,$$

we find that the two fixed-point problems: $u = Fu$ and $u = Tu$ are equivalent.

Next, for positive constants $\hat{\gamma}$ and $\hat{\alpha}$, set

$$U_h := \{u_h \in S_h \mid \|u_h\|_X \le \hat{\gamma}, \} \subset S_h,$$

$$U_* := \{u_* \in S_* \mid \|u_*\|_X \le \hat{\alpha}, \} \subset S_*,$$

and define a candidate set $U \subset X$ by

$$U := U_h + U_*.$$

Then a sufficient condition for the invertibility result is as follows [4].

Lemma 4 When an inclusion

$$\overline{TU} \subset \mathrm{int}(U) \tag{28}$$

holds, then L is invertible.

Proof. If there exists $u \in X$ such that $Lu = 0$ and $u \neq 0$, u also satisfies $u = Tu$. Since T is linear operator, for any $t \in \mathbb{R}$, we have

$$\begin{aligned} T(tu) &= tT(u) \\ &= tu. \end{aligned}$$

Then, we can choose $\hat{t} \in \mathbb{R}$ satisfying

$$\hat{t}u \in \partial U.$$

However, this contradicts with $\overline{TU} \subset \mathrm{int}(U)$ and $T(tu) = tu$. Therefore $u = 0$. That is, $u = 0$ is a unique solution of $Lu = 0$. □

We now describe a procedure to construct the candidate set U of X which is expected to satisfy the inclusion (28). From the unique decomposeness of $u \in U$, we will check for finite and infinite part separately.

The finite dimensional part of the inclusion, $\overline{\mathcal{N}_h U} \subset \mathrm{int}(U_h)$, can be written as

$$\sup_{u \in U} \|\mathcal{N}_h u\|_X < \hat{\gamma}.$$

On the other hand, the infinite dimensional part of the inclusion, $\overline{(I-P_h)FU} \subset \mathrm{int}(U_*)$, means

$$(I-P_h)A^{-1}f(u) \in \mathrm{int}(U_*)$$

for any $u \in U$. Therefore, from the assumption **A5** (23), the condition

$$C(h) \sup_{u \in U} \|f(u)\|_Y < \hat{\alpha}$$

is sufficient. From this we can derive the following lemma.

Lemma 5 If one can check the conditions:

$$\sup_{u \in U} \|\mathcal{N}_h u\|_X < \hat{\gamma},$$
$$C(h) \sup_{u \in U} \|f(u)\|_Y < \hat{\alpha},$$

then L is invertible.

3.3 Criterion for the invertibility of L

In order to confirm the verification Lemma 5, for given positive parameters $\hat{\alpha}$ and $\hat{\gamma}$, we have to compute

$$\gamma := \sup_{\hat{u} \in U} \|\mathcal{N}_h \hat{u}\|_X, \qquad \alpha := C(h) \sup_{\hat{u} \in U} \|f(u)\|_Y,$$

and confirm

$$\gamma < \hat{\gamma}, \qquad \alpha < \hat{\alpha}.$$

In the actual computation, the candidate set U contains the infinite dimensional term U_*. Moreover, it is impossible to avoid the effect of rounding error of floating point arithmetic. However, by norm estimates, and interval arithmetic software taking into account effects of rounding error, we can obtain mathematically rigorous upper bounds for γ and α and with possible over-estimates. Let us describe these computations in more detail.

For any $u \in U$ such that $u = u_h + u_*$, $u_h \in U_h$, $u_* \in U_*$, we obtain

$$\begin{aligned}\mathcal{N}_h u =& P_h u - [I-F]_h^{-1} P_h (I-F) u \\ =& [I-F]_h^{-1} P_h A^{-1} f(u_*)\end{aligned}$$

from the linearity of f.

Here setting

$$s_h := P_h A^{-1} f(u_*) = \sum_{n=1}^{N} s_{h,n} \hat{\phi}_n \in S_h, \quad \boldsymbol{s} := [s_{h,n}] \in \mathbb{C}^N,$$
$$\mathcal{N}_h u := \sum_{n=1}^{N} t_{h,n} \hat{\phi}_n \in S_h, \quad \boldsymbol{t} := [t_{h,n}] \in \mathbb{C}^N,$$

where $\{\hat{\phi}_n\}_1^N$ is basis of S_h with $N := \dim S_h$, the definition of $[I-F]_h^{-1}$ implies

$$(\,(I-F)\mathcal{N}_h u, v_h\,)_X = (\,s_h, v_h\,)_X, \qquad \forall v_h \in S_h. \tag{29}$$

From **A2** (2), the eq.(29) is equivalent as

$$\sum_{n=1}^{N} t_{h,n}\Big((\hat{\phi}_n,\hat{\phi}_m)_X - (f(\phi_n),\hat{\phi}_m)_Y\Big) = \sum_{n=1}^{N} s_{h,n}(\hat{\phi}_n,\hat{\phi}_m)_X, \qquad 1 \le m \le N.$$

Therefore defining

$$\begin{aligned}
[A_1]_{mn} &:= (\hat{\phi}_n,\hat{\phi}_m)_X,\\
[A_2]_{mn} &:= -(f(\phi_n),\hat{\phi}_m)_Y,\\
G &:= A_1 + A_2,\\
[A_3]_{mn} &:= \langle \hat{\phi}_n,\hat{\phi}_m\rangle_X,
\end{aligned}$$

and L_3 is the matrix decomposed factor of A_3 such that $A_3 = L_3L_3^T$ we have

$$Gt = A_1 s,$$

and

$$\|\mathcal{N}_h u\|_X = \|L_3^T t\|_E \le \rho\, \|s_h\|_X, \tag{30}$$

where $\rho > 0$ is an upper bound satisfying

$$\boxed{\|L_3^T G^{-1} A_1 L_3^{-T}\|_E \le \rho} \tag{31}$$

for the matrix 2-norm $\|\cdot\|_E$. Evaluations of ρ can be reduced to the computation of the maximum singular value of a matrix.

Here note that when $(u,v)_X = \langle u,v\rangle_X$, $A_1 = L_3L_3^T$ then ρ is estimated by

$$\boxed{\|L_3^T G^{-1} L_3\|_E \le \rho}.$$

From assumption **A6**

$$\|s_h\|_X \le \nu_1\|u_*\|_X \le \nu_1\hat{\alpha}, \tag{32}$$

hencefore

$$\|\mathcal{N}_h u\|_X \le \rho\nu_1\hat{\alpha}, \qquad \forall u \in U.$$

Moreover, from assumption **A7**,

$$\begin{aligned}
\|f(u)\|_Y &\le \nu_2\|u_h\|_X + \nu_3\|u_*\|_X & (33)\\
&\le \nu_2\hat{\gamma} + \nu_3\hat{\alpha}. & (34)
\end{aligned}$$

Therefore, the following criterion for verification holds.

Theorem 2 If

$$\kappa := C(h)(\rho\nu_1\nu_2 + \nu_3) < 1 \tag{35}$$

holds, then the operator L has the inverse.

4 Upper bound of M

We can get an upper bound $M > 0$ satisfies

$$\|L^{-1}\phi\|_X \leq M\|\phi\|_X, \quad \forall\phi \in X.$$

under the invertibility criterion for L by the following theorem.

Theorem 3 Under the same assumption (35), an upper bound $M > 0$ for (16) can be taken as

$$M = \|\mathcal{M}\|_E,$$

where

$$\mathcal{M} = \begin{bmatrix} \rho\left(1 + \dfrac{\nu_1 C(h)\nu_2\rho}{1-\kappa}\right) & \dfrac{\rho\nu_1}{1-\kappa} \\ \dfrac{C(h)\nu_2\rho}{1-\kappa} & \dfrac{1}{1-\kappa} \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{36}$$

5 Upper bound for $\hat{M}$

We can obtain an upper bound $\hat{M} > 0$ satisfies

$$\|\hat{L}^{-1}\phi\|_X \leq \hat{M}\|\phi\|_Y, \quad \forall\phi \in Y.$$

under the invertibility criterion for $\hat{L}$ by the following theorem.

Defining

$$[A_4]_{nm} := (\hat{\phi}_m, \hat{\phi}_n)_Y, \tag{37}$$

L_4 is the matrix decomposed factor of A_4: $A_4 = L_4 L_4^T$, and $\hat{\rho} > 0$ is an upper bound satisfying

$$\boxed{\|L_3^T G^{-1} L_4\|_E \leq \hat{\rho}.} \tag{38}$$

Then the following can be shown.

Theorem 4 Under the assumption (35), and

$$\hat{\kappa} := C(h)\nu_3(1 + \hat{\rho}\nu_2) < 1,$$

then,

$$\hat{M} = \frac{\sqrt{\hat{\rho}^2 + C(h)^2(1+\nu_2\hat{\rho})^2}}{1-\hat{\kappa}}. \tag{39}$$

6 Verified examples

Consider the two-dimensional self-adjoint eigenvalue problem:

$$\left\{ \begin{array}{rcll} -\Delta u + \nu(3u_h^2 - 2(a+1)u_h + a)u & = & \lambda u & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{40}$$

where $\Omega = (0,1) \times (0,1)$, ν and a are positive constants, and u_h is an approximate solution of the so-called Allen-Cahn equation:

$$\begin{cases} -\Delta u &= \nu u(u-a)(1-u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{41}$$

It is known that this equation has two solution branches with respect to the parameter $\nu > 0$ [8]. We considered both case in which u_h are lower and upper branch finite element solutions for $\nu = 150$ and $a = 0.01$ in linear and uniform triangular mesh of the Ω. We can take $C(h) = 0.493h$ for the uniform partition size $h > 0$.

Setting $w_h = \nu(3u_h^2 - 2(a+1)u_h + a)$, $A = -\Delta$, $Q = c = w_h$, $B = I_{X\to Y}$, $C_p = 1/(\pi\sqrt{2})$, $C_b = 1/(2\pi)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $X = H_0^1(\Omega)$, $Y = L^2(\Omega)$, For $u_* \in S_*$, since

$$\begin{aligned} \|\nabla P_h(-\Delta)^{-1}(c-\mu)u_*\|_{L^2(\Omega)}^2 &\le \|\nabla(-\Delta)^{-1}(c-\mu)u_*\|_{L^2(\Omega)}^2 \\ &= ((c-\mu)u_*, (-\Delta)^{-1}(c-\mu)u_*)_{L^2(\Omega)} \\ &\le \|(c-\mu)u_*\|_{L^2(\Omega)}\, C_p\|\nabla(-\Delta)^{-1}(c-\mu)u_*\|_{L^2(\Omega)}, \end{aligned}$$

we have

$$\|\nabla(-\Delta)^{-1}(cu-\mu)u_*\|_{L^2(\Omega)} \le C_p\ \|c-\mu\|_{L^\infty(\Omega)}\|u_*\|_{L^2(\Omega)}$$

Therefore using

$$\|u_*\|_{L^2(\Omega)} \le C(h)\|u_*\|_{H_0^1(\Omega)},$$

we can take

$$\nu_1 = C_pC(h)\|c-\mu\|_{L^\infty(\Omega)}, \quad \nu_2 = C_p\|c-\mu\|_{L^\infty(\Omega)}, \quad \nu_3 = C(h)\|c-\mu\|_{L^\infty(\Omega)}.$$

The norm $\|c-\mu\|_{L^\infty(\Omega)}$ can be estimated as

$$\|c-\mu\|_{L^\infty(\Omega)} = \left\| 3\nu\left(u_h - \frac{a+1}{3}\right)^2 - \frac{\nu(a+1)^2}{3} + a\nu - \mu \right\|_{L^\infty(\Omega)}$$

Since $0 \le u_h(x) \le \|u_h\|_{L^\infty(\Omega)}$ for any $x \in \Omega$, $\|c-\mu\|_{L^\infty(\Omega)}$ can be attained when $u_h(x) = 0$ or $u_h(x) = \|u_h\|_{L^\infty(\Omega)}$ or $u_h(x) = \dfrac{a+1}{3}$.

6.1 Excluding result for $h = 1/50$ (lower solution)

The approximate absolute minimum eigenvalue is 16.616847.

μ	ρ	$\hat{\rho}$	κ	M	R_1	$\hat{\kappa}$	$\hat{M}$	R_2
-16.4	99.9171	21.9878	0.5426	229.5885	0.0859	0.5306	46.9618	0.0946
-16	35.0991	7.7296	0.1974	46.9723	0.4293	0.1932	9.6071	0.4624
-14	8.2357	1.8222	0.0547	9.1809	2.1500	0.0538	1.9324	2.2991
-11	3.8255	0.8493	0.0317	4.1686	4.7352	0.0313	0.8808	5.0444
0	2.7078	0.3780	0.0375	2.9916	6.5981	0.0251	0.3913	11.3567
11	6.0905	0.8556	0.1158	7.3341	2.6915	0.0745	0.9342	4.7560
15	11.2572	1.5841	0.2370	15.7184	1.2558	0.1506	1.8855	2.3564
17	19.5897	2.7588	0.4333	36.8329	0.5359	0.2735	3.8403	1.1569
18	31.1216	4.3843	0.7051	112.4621	0.1755	0.4438	7.9725	0.5572
18.3	37.8005	5.3258	0.8626	293.1021	0.0673	0.5424	11.7725	0.2773

R_1 and R_2 stand for the excluding radius $1/(C_b M)$ and $1/(Cp\hat{M})$, respectively.

6.2 Excluding result for $h = 1/50$ (upper solution)

The approximate absolute minimum eigenvalue is 47.107986.

μ	ρ	$\hat{\rho}$	κ	M	R_1	$\hat{\kappa}$	$\hat{M}$	R_2
-23	1.1915	0.1140	0.1485	1.8421	10.7161	0.0716	0.1333	33.3452
0	1.5426	0.1645	0.1369	2.1532	9.1674	0.0714	0.1873	23.7298
23	2.5973	0.3126	0.1517	3.4311	5.7532	0.0856	0.3537	12.5644
35	4.7528	0.6161	0.2114	6.5881	2.9966	0.1256	0.7218	6.1561
40	7.8043	1.0465	0.3167	12.3947	1.5926	0.1922	1.3232	3.3579
43	13.2038	1.8087	0.5655	32.9839	0.5984	0.3477	2.8337	1.5679
44	17.3209	2.3901	0.7550	76.7302	0.2572	0.4664	4.5781	0.9704
44.5	20.5637	2.8481	0.9042	232.8762	0.0847	0.5599	6.6147	0.6716

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