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Braided differential structure on affine Weyl groups and nil-Hecke algebras

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This article is based on my joint work with A. N. Kirillov [5]. We construct a model of the affine nil-Hecke algebra as a subalgebra of the Nichols-Woronowicz algebra associated to a Yetter-Drinfeld module over the affine Weyl group. We also discuss the Peterson isomorphism between the homology of the affine Grassmannian and the small quantum cohomology ring of the flag variety in terms of the braided differential calculus.

1 Affine nil-Hecke algebra

Let G be a simply-connected semisimple complex Lie group and W its Weyl group. Denote by Δ the set of the roots. We fix the set Δ_+ of the positive roots by choosing a set of simple roots $\alpha_1, \ldots, \alpha_r$. The Weyl group W acts on the weight lattice P and the coroot lattice $Q^\vee$ of G. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k}$, $\alpha \in \Delta$, $k \in \mathbb{Z}$, with respect to the affine hyperplanes $H_{\alpha,k} := \{\lambda \in P \otimes \mathbb{R} \mid \langle \lambda, \alpha \rangle = k\}$. The affine Weyl group is the semidirect product of W and $Q^\vee$, i.e., $W_{\mathrm{aff}} = W \ltimes Q^\vee$. The affine Weyl group W_{aff} is generated by the simple reflections $s_1 := s_{\alpha_1,0}, \ldots, s_r := s_{\alpha_r,0}$ and $s_0 := s_{\theta,1}$ where $\theta = -\alpha_0$ is the highest root. The affine Weyl group W has the presentation as a Coxeter group as follows:

$$W_{\mathrm{aff}} = \langle s_0, \ldots, s_r \mid s_0^2 = \cdots = s_r^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Definition 1.1. The affine nil-Coxeter algebra $\mathbb{A}_0$ is the associative algebra generated by $\tau_0, \ldots, \tau_r$ subject to the relations

$$\tau_0^2 = \cdots = \tau_r^2 = 0, \quad (\tau_i \tau_j)^{[m_{ij}/2]} \tau_i^{\nu_{ij}} = (\tau_j \tau_i)^{[m_{ij}/2]} \tau_j^{\nu_{ij}},$$

where $\nu_{ij} := m_{ij} - 2[m_{ij}/2]$.

For a reduced expression $x = s_{i_1} \cdots s_{i_l}$ of an element $x \in W_{\mathrm{aff}}$, the element $\tau_x := \tau_{i_1} \cdots \tau_{i_l} \in \mathbb{A}_0$ is independent of the choice of the reduced expression of x. It is known that $\{\tau_x\}_{x \in W_{\mathrm{aff}}}$ form a linear basis of $\mathbb{A}_0$.

The nil-Coxeter algebra $\mathbb{A}_0$ acts on $S := \mathrm{Sym} P_{\mathbb{Q}}$ via

$$\tau_0(f) := \partial_{\alpha_0}(f) = -(f - s_{\theta,0} f)/\theta,$$

$$\tau_i(f) := \partial_{\alpha_i}(f) = (f - s_{\alpha_i,0} f)/\alpha_i, \quad i = 1, \dots, r,$$

for $f \in S$.

Definition 1.2. ([6]) The nil-Hecke algebra $\mathbb{A}$ is defined to be the cross product $\mathbb{A}_0 \ltimes S$, where the cross relation is given by

$$\tau_i f = \partial_{\alpha_i}(f) + s_i(f)\tau_i \quad f \in S, i = 1, \dots, r.$$

The affine Grassmannian $\widehat{\mathrm{Gr}} := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ is homotopic to the loop group ΩK of the maximal compact subgroup $K \subset G$. Let $T \subset G$ be the maximal torus. An associative algebra structure on the T-equivariant homology group $H_*^T(\widehat{\mathrm{Gr}}) \cong H_*^T(\Omega K)$ is induced from the group multiplication

$$\Omega K \times \Omega K \to \Omega K.$$

It is known that the algebra $H_*^T(\widehat{\mathrm{Gr}})$ is commutative. The algebra $H_*^T(\Omega K)$ is called the Pontryagin ring.

We regard the T-equivariant homology $H_*^T(\widehat{\mathrm{Gr}})$ as an S-algebra by identifying $S = H_T^*(pt)$. The diagonal embedding

$$\Omega K \to \Omega K \times \Omega K$$

induces a coproduct on $H_*^T(\widehat{\mathrm{Gr}})$.

Proposition 1.1. ([10]) *The T-equivariant homology $H_*^T(\widehat{\mathrm{Gr}})$ is isomorphic to the centralizer $Z_{\mathbb{A}}(S)$ of S in $\mathbb{A}$ as Hopf algebras.*

2 Nichols-Woronowicz algebra for affine Weyl groups

Let M be a vector space over a field of characteristic zero and $\psi : M^{\otimes 2} \to M^{\otimes 2}$ be a fixed linear endomorphism satisfying the braid relations $\psi_i\psi_{i+1}\psi_i = \psi_{i+1}\psi_i\psi_{i+1}$ where $\psi_i : M^{\otimes n} \to M^{\otimes n}$ is a linear endomorphism obtained by applying ψ to the i-th and $(i+1)$-st components. Denote by s_i the simple transposition $(i, i+1) \in S_n$. For any reduced expression $w = s_{i_1} \cdots s_{i_l} \in S_n$, the endomorphism $\Psi_w = \psi_{i_1} \cdots \psi_{i_l} : M^{\otimes n} \to M^{\otimes n}$ is well-defined. The Woronowicz symmetrizer [11] is given by $\sigma_n := \sum_{w \in S_n} \Psi_w$.

Definition 2.1. ([11]) The Nichols-Woronowicz algebra associated to a braided vector space M is defined by

$$\mathcal{B}(M) := \bigoplus_{n \geq 0} M^{\otimes n}/\mathrm{Ker}(\sigma_n),$$

where $\sigma_n : M^{\otimes n} \to M^{\otimes n}$ is the Woronowicz symmetrizer.

Definition 2.2. A vector space M is called a Yetter-Drinfeld module over a group Γ, if the following conditions are satisfied:
(1) M is a Γ-module,
(2) M is Γ-graded, i.e. $M = \bigoplus_{g \in \Gamma} M_g$, where M_g is a linear subspace of M,
(3) for $h \in \Gamma$ and $v \in M_g$, $h(v) \in M_{hgh^{-1}}$.

The Yetter-Drinfeld module M over a group Γ is naturally braided with the braiding $\psi : M^{\otimes 2} \to M^{\otimes 2}$ defined by $\psi(a \otimes b) = g(b) \otimes a$ for $a \in M_g$ and $b \in M$.

In the following we are interested in the Yetter-Drinfeld module over the affine Weyl groups W_{aff}. Denote by $t_\lambda \in W_{\mathrm{aff}}$ the translation by $\lambda \in Q^\vee$. We define a Yetter-Drinfeld module V_{aff} over W_{aff} by

$$V_{\mathrm{aff}} := \bigoplus_{\alpha \in \Delta, k \in \mathbb{Z}} \mathbb{Q} \cdot [\alpha, k]/([\alpha, k] + [-\alpha, -k]),$$

where the W_{aff} acts on V_{aff} by

$$w[\alpha, k] := [w(\alpha), k], \quad w \in W, \quad t_\lambda[\alpha, k] := [\alpha, k + (\alpha, \lambda)], \quad \lambda \in Q^\vee.$$

The W_{aff}-grading is given by $\deg_{W_{\mathrm{aff}}}([\alpha, k]) := s_{\alpha,k}$. Then it is easy to check the conditions in Definition 2.1. Now we have the Nichols-Woronowicz algebra $\mathcal{B}_{\mathrm{aff}} := \mathcal{B}(V_{\mathrm{aff}})$ associated to the Yetter-Drinfeld module V_{aff}.

Let us define the extension $\mathcal{B}_{\text{aff}}(S) = \mathcal{B}_{\text{aff}} \ltimes S$ by the cross relation

$$[\alpha, k]f = \partial_\alpha f + s_{\alpha,0}(f)[\alpha, k], \quad [\alpha, k] \in V_{\text{aff}}, f \in S.$$

Proposition 2.1. *There exists a homomorphism* $\varphi : \mathbb{A} \longrightarrow \mathcal{B}_{\text{aff}}(S)$ *given by* $\tau_0 \mapsto [\alpha_0, -1]$, $\tau_i \mapsto [\alpha_i, 0]$, $i = 1, \dots, r$, *and* $f \mapsto f$, $f \in S$.

Proof. It is enough to check the Coxeter relations among $\varphi(\tau_0), \dots, \varphi(\tau_r)$ in $\mathcal{B}_{\text{aff}}(S)$ based on the classification of the affine root systems. This is done by the direct computation of the symmetrizer for the subsystems of rank 2 in the similar manner to [1, Section 6].

Example 2.1. Here we list the Coxeter relations in $\mathcal{B}_{\text{aff}}$ involving $[\theta, 1] = -[\alpha_0, -1]$ for the root systems of rank 2. Let $(\varepsilon_1, \dots, \varepsilon_r)$ be an orthonormal basis of the r-dimensional Euclidean space. Put $[ij, k] := [\varepsilon_i - \varepsilon_j, k]$, $[\overline{ij}, k] := [\varepsilon_i + \varepsilon_j, k]$, $[i, k] := [\varepsilon_i, k]$ and $[\alpha] := [\alpha, 0]$.
(i) (Type A_2 case)

$$[13, 1][23][13, 1] + [23][13, 1][23] = 0, \quad [13, 1][12][13, 1] + [12][13, 1][12] = 0$$

(ii) (Type B_2 case)

$$[\overline{12}, 1][2][\overline{12}, 1][2] = [2][\overline{12}, 1][2][\overline{12}, 1]$$

(iii) (Type G_2 case) Let α_1, α_2 be the simple roots for G_2-system. We assume that α_1 is a short root and α_2 is a long one. Then we have $\theta = 3\alpha_1 + 2\alpha_2$.

$$[\theta, 1][\alpha_2][\theta, 1] + [\alpha_2][\theta, 1][\alpha_2] = 0.$$

3 Model of nil-Hecke algebra

The connected components of $P \otimes \mathbb{R} \setminus \cup_{\alpha \in \Delta_+, k \in \mathbb{Z}} H_{\alpha,k}$ are called alcoves. The affine Weyl group W_{aff} acts on the set of the alcoves simply and transitively.

Definition 3.1. ([8]) (1) A sequence $(A_0, \dots, A_l)$ of alcoves A_i is called an alcove path if A_i and A_{i+1} have a common wall and $A_i \neq A_{i+1}$.
(2) An alcove path $(A_0, \dots, A_l)$ is called reduced if the length l of the path is minimal among all alcove paths connecting A_0 and A_l.
(3) We use the symbol $A_i \xrightarrow{\beta,k} A_{i+1}$ when A_i and A_{i+1} have a common wall of the form $H_{\beta,k}$ and the direction of the root β is from A_i to A_{i+1}.

The alcove A° defined by the inequalities $\langle\lambda,\alpha_0\rangle \geq -1$ and $\langle\lambda,\alpha_i\rangle \geq 0$, $i = 1,\dots,r$, is called the fundamental alcove. For a reduced alcove path $\gamma : A_0 = A^\circ \xrightarrow{\beta_1,k_1} \cdots \xrightarrow{\beta_l,k_l} A_l$, we define an element $[\gamma] \in \mathcal{B}_{\mathrm{aff}}$ by

$$[\gamma] := [-\beta_1,-k_1]\cdots[-\beta_l,-k_l].$$

When $A_l = x^{-1}(A^\circ)$ for $x \in W_{\mathrm{aff}}$, we will also use the symbol $[x]$ instead of $[\gamma]$, since $[\gamma]$ depends only on x thanks to the Yang-Baxter relation.

For a braided vector space M, it is known that an element $a \in M$ acts on $\mathcal{B}(M^*)$ as a braided differential operator (see [1], [9]). Let us identify M^* with M via the W_{aff}-invariant inner product (,) given by

$$([\alpha,k],[\beta,l]) = \begin{cases} 1, & \text{if } \alpha=\beta \text{ and } k=l, \\ 0, & \text{otherwise,} \end{cases}$$

for $\alpha,\beta \in \Delta_+$, $k,l \in \mathbb{Z}$. In our case, the differential operator $\overleftarrow{D}_{[\alpha,k]}$, $[\alpha,k] \in V_{\mathrm{aff}}$, acting from the right is determined by the following characterization:
(0) $(c)\overleftarrow{D}_{[\alpha,k]} = 0$, $c \in \mathbb{Q}$,
(1) $([\alpha,k])\overleftarrow{D}_{[\beta,l]} = ([\alpha,k],[\beta,l])$,
(2) $(FG)\overleftarrow{D}_{[\alpha,k]} = F(G\overleftarrow{D}_{[\alpha,k]}) + (F\overleftarrow{D}_{[\alpha,k]})s_{\alpha,k}(G)$,
for $\alpha,\beta \in \Delta$, $k,l \in \mathbb{Z}$, $F,G \in \mathcal{B}_{\mathrm{aff}}$. The operator $\overleftarrow{D}_{[\alpha,k]}$ extends to the one acting on $\mathcal{B}_{\mathrm{aff}}(S)$ by the commutation relation $f \cdot \overleftarrow{D}_{[\alpha,k]} = \overleftarrow{D}_{[\alpha,k]} \cdot s_{\alpha,k}(f)$, $f \in S$.

We use the abbreviation $\overleftarrow{D}_0 := \overleftarrow{D}_{[\alpha_0,-1]}$, $\overleftarrow{D}_i := \overleftarrow{D}_{[\alpha_i,0]}$, $i = 1,\dots,r$. For $x \in W_{\mathrm{aff}}$, fix a reduced decomposition $x = s_{i_1}\cdots s_{i_l}$. We define the corresponding braided differential operator $\overleftarrow{D}_x$ acting on $\mathcal{B}_{\mathrm{aff}}$ by the formula

$$\overleftarrow{D}_x := \overleftarrow{D}_{i_l}\cdots\overleftarrow{D}_{i_1},$$

which is also independent of the choice of the reduced decomposition of x because of the braid relations.

Lemma 3.1. *For $x \in W_{\mathrm{aff}}$, take a reduced alcove path γ from the fundamental alcove A° to $x^{-1}(A^\circ)$. Then, we have $([\gamma])\overleftarrow{D}_x = 1$.*

Proof. Let us take a reduced path

$$\gamma : A_0 = A^\circ \xrightarrow{\beta_1,k_1} A_1 \xrightarrow{\beta_2,k_2} \cdots \xrightarrow{\beta_l,k_l} A_l = x^{-1}(A^\circ).$$

Define a sequence $\sigma_1, \dots, \sigma_l \in W_{\text{aff}}$ inductively by

$$\sigma_1 := s_{\beta_1,k_1}, \quad \sigma_{j+1} := \sigma_j s_{\beta_{j+1},k_{j+1}} \sigma_j.$$

Then it is easy to see that $\sigma_\nu(A_j) \neq A^\circ$, $1 \leq \nu \leq j-1$, $\sigma_j(A_j) = A^\circ$ and the walls $\sigma_j(H_{\beta_{j+1},k_{j+1}})$ are corresponding to simple roots. Hence, $\sigma_1, \dots, \sigma_l$ are simple reflections. This sequence gives a reduced expression $x = \sigma_l \cdots \sigma_1$. Put $\sigma_i = s_{\alpha_{i_j}}$. Since the direction of β_{j+1} is chosen to be from A_j to A_{j+1}, we have

$$[\gamma]\overleftarrow{D}_x = ([\beta_1, k_1])\overleftarrow{D}_{i_1} \cdot (\sigma_1([\beta_2, k_2]))\overleftarrow{D}_{i_2} \cdots (\sigma_{l-1}([\beta_l, k_l]))\overleftarrow{D}_{i_l} = 1.$$

Example 3.1. (1) (A_2-case) The standard realization is given by $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_0 = \varepsilon_3 - \varepsilon_1$. Consider the translation t_{α_1} by the simple root α_1. If we take a reduced path

$$\gamma : A_0 = A^\circ \xrightarrow{-\alpha_2,0} A_1 \xrightarrow{\alpha_1,1} A_2 \xrightarrow{-\alpha_0,1} A_3 \xrightarrow{\alpha_1,2} A_4 = t_{\alpha_1}(A^\circ),$$

then we have $[\gamma] = [23][21,-1][31,-1][21,-2]$. On the other hand, the differential operator corresponding to $t_{-\alpha_1}$ is given by $\overleftarrow{D}_2\overleftarrow{D}_0\overleftarrow{D}_2\overleftarrow{D}_1$, where $\overleftarrow{D}_0 = \overleftarrow{D}_{[31,-1]}$, $\overleftarrow{D}_1 = \overleftarrow{D}_{[12]}$, $\overleftarrow{D}_2 = \overleftarrow{D}_{[23]}$. It is easy to check by direct computation

$$([23][21,-1][31,-1][12,2])\overleftarrow{D}_2\overleftarrow{D}_0\overleftarrow{D}_2\overleftarrow{D}_1 = 1.$$

(2) (B_2-case) The standard realization is given by $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2$, $\alpha_0 = -\varepsilon_1 - \varepsilon_2$. Let us consider the translation $t_{2\varepsilon_1}$ and a reduced path

$$\gamma : A_0 = A^\circ \xrightarrow{[\overline{12},1]} A_1 \xrightarrow{[2,1]} A_2 \xrightarrow{[12,1]} A_3 \xrightarrow{[\overline{12},2]} A_4 \xrightarrow{[1,2]} A_5 \xrightarrow{[12,2]} A_6 = t_{2\varepsilon_1}(A^\circ).$$

Then we have

$$[\gamma] = (-[\overline{12},1])(-[2,1])(-[12,1])(-[\overline{12},2])(-[1,2])(-[12,2])$$

$$= [\overline{12},1][2,1][12,1][\overline{12},2][1,2][12,2].$$

The differential operator corresponding to $t_{-2\varepsilon}$ is given by

$$\overleftarrow{D}_{t_{-2\varepsilon}} = \overleftarrow{D}_0\overleftarrow{D}_2\overleftarrow{D}_0\overleftarrow{D}_1\overleftarrow{D}_2\overleftarrow{D}_1.$$

So we have

$$[\gamma]\overleftarrow{D}_{t_{-2\varepsilon}} = ([\overline{12},1][2,1][12,1][\overline{12},2][1,2][12,2])\overleftarrow{D}_0\overleftarrow{D}_2\overleftarrow{D}_0\overleftarrow{D}_1\overleftarrow{D}_2\overleftarrow{D}_1 = 1.$$

Theorem 3.1. *The algebra homomorphism $\varphi : \mathbb{A} \to \mathcal{B}_{\mathrm{aff}}(S)$ is injective.*

Proof. The nil-Hecke algebra $\mathbb{A}$ is also W_{aff}-graded. Since the homomorphism $\varphi : \mathbb{A} \to \mathcal{B}_{\mathrm{aff}}(S)$ preserves the W_{aff}-grading, it is enough to check $\varphi(\tau_x) \neq 0$, for $x \in W_{\mathrm{aff}}$ in order to show the injectivity of φ. On the other hand, ${\mathcal{B}_{\mathrm{aff}}}^{op}$ acts on $\mathcal{B}_{\mathrm{aff}}$ itself via the braded differential operators. Let γ be a reduced alcove path from A° to $x^{-1}(A^\circ)$. Then we have $([\gamma])\overleftarrow{D}_x = 1$ from Lemma 3.1. This shows $\overleftarrow{D}_x \neq 0$, so $\varphi(\tau_x) \neq 0$.

This theorem implies the following (see Proposition 1.1):

Corollary 3.1. *The T-equivariant Pontryagin ring $H_*^T(\widehat{\mathrm{Gr}})$ is a subalgebra of $\mathcal{B}_{\mathrm{aff}}(S)$.*

By taking the non-equivariant limit, we also have:

Corollary 3.2. *The Pontryagin ring $H_*(\widehat{\mathrm{Gr}})$ is a subalgebra of $\mathcal{B}_{\mathrm{aff}}$.*

4 Affine Bruhat operators

We denote by $x \to y$ the cover relation in the Bruhat ordering of W_{aff}, i.e. $y = xs_{\alpha,k}$ for some $\alpha \in \Delta$ and $k \in \mathbb{Z}$, and $l(y) = l(x) + 1$.

We will use some terminology from [7]. Denote by $\tilde{Q}$ the set of antidominant elements in $Q^\vee$. An element $x \in W_{\mathrm{aff}}$ can be expressed uniquely as a product of form $x = wt_{v\lambda} \in W_{\mathrm{aff}}$ with $v, w \in W$, $\lambda \in \tilde{Q}$. We say that $x = wt_{v\lambda}$ belongs to the "v-chamber". An element $\lambda \in \tilde{Q}$ is called superregular when $|\langle \lambda, \alpha \rangle| > 2(\#W) + 2$ for all $\alpha \in \Delta_+$. If $\lambda \in \tilde{Q}$ is superregular, then $x = wt_{v\lambda}$ is called superregular. The subset of superregular elements in W_{aff} is denoted by ${W_{\mathrm{aff}}}^{\mathrm{sreg}}$. We say that a property holds for sufficiently superregular elements ${W_{\mathrm{aff}}}^{\mathrm{ssreg}} \subset W_{\mathrm{aff}}$ if there is a positive constant $k \in \mathbb{Z}$ such that the property holds for all $x \in {W_{\mathrm{aff}}}^{\mathrm{sreg}}$ satisfying the following condition:

$$y \in W_{\mathrm{aff}},\ y < x, \text{ and } l(x) - l(y) < k \Rightarrow y \in {W_{\mathrm{aff}}}^{\mathrm{sreg}}.$$

The meaning of ${W_{\mathrm{aff}}}^{\mathrm{ssreg}}$ depends on the context, see [7, Section 4] for the details. For $v \in W$, consider the S-submodule M_v^{ssreg} in $\mathcal{B}_{\mathrm{aff}}$ generated by the sufficiently superregular elements $[x]$ where x belongs to the v-chamber.

Lemma 4.1. *Let $x \in W_{\mathrm{aff}}$. For $\alpha \in \Delta$ and $k \in \mathbb{Z}_{>0}$, we have*

$$[x]\overleftarrow{D}_{[\alpha,k]} = \begin{cases} [xs_{\alpha,k}], & \text{if } l(x) = l(xs_{\alpha,k}) + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The fundamental alcove A° is contained in the region $\{\lambda \in P \otimes \mathbb{R} | \langle\lambda, \alpha\rangle < k\}$ for $\alpha \in \Delta$ and $k \in \mathbb{Z}_{>0}$. Let us choose any reduced path $\gamma : A_0 \xrightarrow{\beta_1, k_1} \cdots \xrightarrow{\beta_l, k_l} A_l = x^{-1}(A^\circ)$ with $k_i \geq 0$. If $l(x) > l(xs_{\alpha,k})$, then $(\beta_i, k_i) = (\alpha, k)$ for some i. Take the largest i and consider the path

$$\gamma' : A_0 \xrightarrow{\beta_1, k_1} \cdots \xrightarrow{\beta_{i-1}, k_{i-1}} A_{i-1} \xrightarrow{\beta'_{i+1}, k'_{i+1}} s_{\alpha,k}(A_{i+1}) \xrightarrow{\beta'_{i+2}, k'_{i+2}} \cdots$$

$$\cdots \xrightarrow{\beta'_l, k'_l} s_{\alpha,k}(A_l) = s_{\alpha,k} x^{-1}(A^\circ) = (xs_{\alpha,k})^{-1}(A^\circ),$$

where (β'_j, k'_j) is determined by the condition $s_{\alpha,k}(H_{\beta_j,k_j}) = H_{\beta'_j,k'_j}$. If $l(x) = l(xs_{\alpha,k}) + 1$, then the path γ' is a reduced path. In this case, we have $[x]\overleftarrow{D}_{[\alpha,k]} = [xs_{\alpha,k}]$. If $l(x) > l(xs_{\alpha,k}) + 1$, the above path γ' is not reduced and $[x]\overleftarrow{D}_{[\alpha,k]} = 0$. When $l(x) < l(xs_{\alpha,k})$, the element $[\alpha, k]$ does not appear in the monomial $[\gamma]$, so we have $[x]\overleftarrow{D}_{[\alpha,k]} = 0$.

Proposition 4.1. ([7, Proposition 4.1]) *Let $\lambda \in \tilde{Q}$ be superregular. For $x = wt_{v\lambda}$ and $y = xs_{v\alpha,-n}$ with $v, w \in W$, we have the cover relation $y \to x$ if and only if one of the following conditions holds:*
(1) $l(wv) = l(wvs_\alpha) - 1$ *and* $n = \langle\lambda, \alpha\rangle$, *giving* $y = ws_{v(\alpha)}t_{v(\lambda)}$,
(2) $l(wv) = l(wvs_\alpha) + \langle\alpha^\vee, 2\rho\rangle - 1$ *and* $n = \langle\lambda, \alpha\rangle + 1$, *giving* $y = ws_{v(\alpha)}t_{v(\lambda+\alpha^\vee)}$,
(3) $l(v) = l(vs_\alpha) + 1$ *and* $n = 0$, *giving* $y = ws_{v(\alpha)}t_{vs_\alpha(\lambda)}$,
(4) $l(v) = l(vs_\alpha) - \langle\alpha^\vee, 2\rho\rangle + 1$ *and* $n = -1$, *giving* $y = ws_{v(\alpha)}t_{vs_\alpha(\lambda+\alpha^\vee)}$.

In [7], the first kind of the conditions (1) and (2) are called the near relation because x and y belong to the same chamber. In this paper we denote the near relation by $y \to_{near} x$.

The affine Bruhat operator $B^\mu : S\langle W_{\mathrm{aff}}{}^{\mathrm{ssreg}}\rangle \to S\langle W_{\mathrm{aff}}{}^{\mathrm{sreg}}\rangle$, $\mu \in P$, due to Lam and Shimozono [7, Section 5] is an S-linear map defined by the formula

$$B^\mu(x) = (\mu - wv\mu)x + \sum_{\alpha\in\Delta_+} \sum_{xs_{v(\alpha),k} \to_{near} x} \langle\alpha^\vee, \mu\rangle xs_{v(\alpha),k}$$

for $x = wt_{v\lambda} \in W_{\mathrm{aff}}{}^{\mathrm{ssreg}}$. We also introduce the operator β_v^μ, $\mu \in P$, acting on each M_v^{ssreg} by

$$\beta_v^\mu([x]) := (\mu - wv\mu)[x] + [x] \sum_{\alpha\in\Delta_+, k>1} \langle\alpha^\vee, \mu\rangle \overleftarrow{D}_{[v(\alpha),k]},$$

where $x = wt_{v\lambda} \in W_{\mathrm{aff}}{}^{\mathrm{ssreg}}$. Denote by $W_{\mathrm{aff}}{}^{\mathrm{ssreg}}(v)$ the subset of W_{aff} consisting of the superregular elements belonging to the v-chamber. Fix a left S-module isomorphism

$$\begin{array}{cccc} \iota: & S\langle W_{\mathrm{aff}}{}^{\mathrm{ssreg}}(v)\rangle & \to & M_v^{\mathrm{ssreg}} \\ & x & \mapsto & [x]. \end{array}$$

Proposition 4.2. *For each $v \in W$ and a sufficiently superregular element $x \in W_{\mathrm{aff}}{}^{\mathrm{ssreg}}(v)$,*

$$\beta_v^{\mu}([x]) = \iota(B^{\mu}(x)).$$

Proof. This can be shown by using Lemma 4.1 and Proposition 4.1.

$$\beta_v^{\mu}([x]) = (\mu - wv\mu)[x] + [x] \sum_{\alpha\in\Delta_+, k>1} \langle\alpha^\vee, \mu\rangle \overleftarrow{D}_{[v(\alpha),k]}$$

$$= (\mu - wv\mu)[x] + \sum_{\alpha\in\Delta_+} \sum_{k>1, l(xs_{[v(\alpha),k]})=l(x)-1} \langle\alpha^\vee, \mu\rangle [xs_{v(\alpha),k}]$$

$$= (\mu - wv\mu)[x] + \sum_{\alpha\in\Delta_+} \sum_{xs_{v(\alpha),k} \to_{near} x} \langle\alpha^\vee, \mu\rangle [xs_{v(\alpha),k}] = \iota(B^{\mu}(x)).$$

Remark 4.1. In [4] the authors introduced the quantization operators η_α acting on the model of $H^*(G/B) \otimes \mathbb{C}[q_1, \dots, q_r]$ realized as a subalgebra of $\mathcal{B}_W \otimes \mathbb{C}[q_1, \dots, q_{n-1}]$. For a superregular element $\lambda \in \tilde{Q}$ and $w \in W$, consider a homomorphism θ_w^λ from the λ-small elements (see [7, Section 5]) of $H^*(G/B) \otimes \mathbb{C}[q]$ to $\mathcal{B}_{\mathrm{aff}}$ defined by

$$\theta_w^\lambda(q^\mu \sigma^v) := [vw^{-1} t_{w(\lambda+\mu)}],$$

where σ^v is the Schubert class of G/B corresponding to $v \in W$ and $q^\mu = q_1^{\mu_1} \cdots q_r^{\mu_r}$ for $\mu = \sum_{i=1}^r \mu_i \alpha_i^\vee$. The following is an interpretation of the formula of [7, Proposition 5.1] in our setting:

$$\theta_w^\lambda(\eta_\alpha(\sigma)) = \beta_w^{\overline{w}\alpha}(\theta_w^\lambda(\sigma)).$$

In [5, Section 5], the comparison between the operators β_v^μ and the quantum Bruhat representation of the quantized Fomin-Kirillov quadratic algebra $\mathcal{E}_n^q$ is discussed.

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