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双曲型方程式の超局所解とその例
Microlocal Solutions of Hyperbolic Equations and Their Examples

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§ 1. Introduction

In the development of microlocal analysis, boundary values of Sato's hyperfunctions are defined in a few manner. Kataoka introduces a concept of mildness to determine boundary values ([5]). Concretely speaking, he shows boundary values to use defining functions of hyperfunctions. He also applies it to propagations of singularities in diffraction problems. Extending his theory, Oaku defines F-mild hyperfunctions ([7], [8]). He shows a local solvability of boundary value problems for a weakly hyperbolic equation of Fuchs type.

In [10], Yamane shows some results about branching of singularities of solutions of boundary value problems. In particular, he studies some examples of hyperbolic equations of second and third order and essentially argues the Jordan–Pochhammer equations. Furthermore, the author shows a construction of solutions whose singularities are only just one of the characteristic roots of the operator for a weakly hyperbolic equation ([3]). He also shows how boundary values are obtained.

In this paper, we review its construction and study the equations and the solutions in [3] through some examples.

§ 2. Boundary values of Sato's hyperfunctions

For studying boundary value problems, it is useful to review definitions of mildness of hyperfunctions. Let $M = \mathbb{R}_t \times \mathbb{R}^n_x$ and X be a complexification of M with $X = \mathbb{C}_t \times \mathbb{C}^n_z$. We set their submanifolds M_+, N and Y as $M_+ = \{(t, x) \in M : t \geq 0\}$, $N = \{(t, x) \in M : t = 0\}$ and $Y = \{(t, z) \in X : t = 0\}$. Then a sheaf $\mathscr{B}_{N|M_+} = (\iota_* \iota^{-1} \mathscr{B}_M)|_N$ can be

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defined on the positive side of N, where $\iota : \mathrm{Int}M_+ \to M$ and $\mathscr{B}_M$ stands for a sheaf of hyperfunctions on M.

Let $u(t,x)$ be a germ of $\mathscr{B}_{N|M_+}$ at a point $(0,\mathring{x}) \in N$. Then we call $u(t,x)$ mild at $(0,\mathring{x})$ if and only if u has an expression as

$$u(t,x) = \sum_{j=1}^{J} F_j(t, x + \sqrt{-1}0\Gamma_j)$$

in a domain $\{(t,x) \in \mathrm{Int}\, M : |t|+|x-\mathring{x}| < r\}$, where J is a positive integer, r is a positive number, Γ_j $(j = 1, 2, \cdots, J)$ are open convex cones and each $F_j(t,z)$ is holomorphic on a domain

$$D(\mathring{x}, \Gamma_j, \epsilon) = \{(t,z) \in X : |t| + |z - \mathring{x}| < \epsilon, \mathrm{Im}\ z \in \Gamma_j, |\mathrm{Im}\ t| + (-\mathrm{Re}\ t)_+ < \epsilon|\mathrm{Im}\ z|\}.$$

Here ϵ is a small positive integer and $(\cdot)_+$ stands for

$$(t)_+ = \begin{cases} t & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all mild hyperfunctions is denoted by $\mathring{\mathscr{B}}_{N|M_+}$, which is a sheaf.

Oaku further introduces F-mild hyperfunction by altering $D(\mathring{x}, \Gamma_j, \epsilon)$ above into

$$\text{(2.1)} \qquad D'(\mathring{x}, \Gamma_j, \epsilon) = \{(t,z) \in X : |t| + |z - \mathring{x}| < \epsilon, \mathrm{Re}\ t \geq 0, \mathrm{Im}\ t = 0, \mathrm{Im}\ z \in \Gamma_j\}.$$

A sheaf of F-mild hyperfunctions is denoted by $\mathring{\mathscr{B}}^F_{N|M_+}$.

§ 3. Solutions for weakly hyperbolic equations of general order

From now on, let $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$. In [3], for a partial differential operator $P(t, \partial_t, \partial_x) = \partial_t^m + \sum_{j=1}^m a_j(t, \partial_x)\partial_t^{m-j}$ on $\mathbb{R}_t \times \mathbb{R}_x$ with its principal symbol

$$\sigma(P)(t, \tau, \xi) = \prod_{j=1}^{m} (\tau - t^\lambda \alpha_j(t)\xi)$$

at the origin $(\lambda = 1, 2, 3, \cdots)$, the author studies a boundary value problem with the equation $P(t, \partial_t, \partial_x)u(t,x) = 0$. Here $a_j(t, \partial_x) = \sum_{|k|=0}^{j} a_{jk}(t)\partial_x^k$ and each $a_{jk}(t)$ $(k = 0, 1, \cdots, j;\ j = 1, \cdots, m)$ is analytic in a neighborhood of $t = 0$.

Furthermore, hyperbolicity and the Levi condition are supposed for the operator P:

(i) each $\alpha_j(t)$ $(j = 1, 2, \cdots, m)$ is a real-valued function and $\alpha_j(0)$ are mutually distinct.

(ii) for $0 \leq s < k(\lambda + 1) - j$, we assume $\partial_x^s a_{jk}(0) = 0$, where $k = 0, 1, \cdots, j$ and $j = 1, 2, \cdots, m$.

Theorem 3.1 ([3]). *For any $j = 1, \cdots, m$ and any microfunction $u_0(x)$ at a point $\mathring{p} = (0, 0; \pm\sqrt{-1}) \in \mathbb{R}_t \times \sqrt{-1}T^*\mathbb{R}_x$, we have a unique mild microfunction solution*

$u(t,x) \in \overset{\circ}{\mathscr{C}}_{\{t=0\}|\{t\geq 0\}}$ *of a microlocal boundary value problem at* $\overset{\circ}{p}$*:*

$$\text{(3.1)} \quad \begin{cases} P(t,\partial_t,\partial_x)u(t,x) = 0, \quad t > 0 \ (\textit{in the sense of } \overset{\circ}{\mathscr{C}}_{\{t=0\}|\{t\geq 0\}}), \\ u(+0,x) = u_0(x), \\ \mathrm{supp}(\mathrm{ext}(u)(t,x)) \cap \{t>0\} \subset \{(t,x;\sqrt{-1}(\tau,\xi)); \tau - \sqrt{-1}t^{\lambda}\alpha_j(t)\xi = 0\}. \end{cases}$$

Further, we have the equations

$$\partial_t^k u(+0,x) = R_{jk}(\partial_x)u_0(x)$$

($j = 1,2,\cdots,m; k = 0,1,2,\cdots,m-1$*), where* $R_{jk}(\partial_x)$ *is a microdifferential operator with fractional order at most* $k/(\lambda+1)$*.*

Here $\overset{\circ}{\mathscr{C}}_{\{t=0\}|\{t\geq 0\}}$ is a sheaf on $\{t=0\} \times \sqrt{-1}T^*\mathbb{R}_x$ of mild microfunctions ([5]) and $\mathrm{ext} : \overset{\circ}{\mathscr{C}}_{\{t=0\}|\{t\geq 0\}} \ni u(t,x) \mapsto u(t,x)Y(t) \in \mathscr{C}_{\mathbb{R}_t \times \mathbb{R}_x}$ is the canonical extension to $t \leq 0$.

Sketch of the Proof. We firstly multiply the operator $P(t,\partial_t,\partial_x)$ by t^m. Secondly we use a fractional coordinate transform $\widetilde{t} = t^{\lambda+1}/(\lambda+1)$. Then we have a partial differential equation $Q(\widetilde{t},\partial_{\widetilde{t}},\partial_x)v(\widetilde{t},x) = 0$.

By the quantized Legendre transform

$$\begin{aligned} &\beta \circ \partial_{\widetilde{t}} \circ \beta^{-1} = -\sqrt{-1}w\partial_x, \\ &\beta \circ \partial_x \circ \beta^{-1} = \partial_x, \\ &\beta \circ \widetilde{t} \circ \beta^{-1} = -\sqrt{-1}\partial_w\partial_x^{-1}, \\ &\beta \circ x \circ \beta^{-1} = x + w\partial_w\partial_x^{-1}, \end{aligned}$$

where ∂_x^{-1} is a pseudodifferential operator, we obtain the transformed equation

$$\text{(3.2)} \qquad (\beta \circ Q \circ \beta^{-1})\beta[v] = 0.$$

Here the operator $\beta \circ Q \circ \beta^{-1}$ becomes a microdifferential operator with fractional power.

We divide $\beta \circ Q \circ \beta^{-1}$ into $L(w,\partial_w) + R(w,\partial_w,\partial_x)$, where L is a dominant part of the operator. We note that L is an ordinary differential operator with regular singular points at $w = -\sqrt{-1}\alpha_j(0)$ $(j = 1,2,\cdots,m)$ and $w = \infty$.

By the iteration scheme

$$\text{(3.3)} \qquad \begin{aligned} &LU_0 = 0, \\ &LU_{k+1} = -R \circ U_k \mod \mathscr{E}^{\mathbb{R}}_{\mathbb{R}_w \times \mathbb{R}_x} \cdot \partial_w \qquad (k = 0,1,2,\cdots) \end{aligned}$$

for a formal symbol $U(w,\xi) = \sum_{j=0}^{\infty} U_j(w)\xi^{-j/(\lambda+1)}$, we construct $U(w,\partial_x)f(x)$ as a solution of the equation (3.2) for any microfunction $f(x)$.

Lastly, we can show the convergence of this scheme. See the details in [3]. □

Here we note that derivatives of fractional order appear in the equation after the fractional coordinate transform. In this case, we introduce a derivation of fractional order as the Riemann–Liouville integral.

Let $f(w)$ be a holomorphic function in a domain $\{w \in \mathbb{C} : \mathrm{Re}\, w > c\}$ for a real number $c \in \mathbb{R}$ and $0 < \alpha < 1$. If $\lim_{\mathrm{Re}\, w \to \infty} f(w)$ is finite, we define $\partial_w^\alpha f(w)$ by

$$\partial_w^\alpha f(w) := \frac{\Gamma(1+\alpha)}{2\pi\sqrt{-1}} \int_\gamma \frac{f(s)}{(s-w)^{1+\alpha}}\, ds,$$

where $\Gamma(\cdot)$ stands for a gamma function and γ is a proper contour from $w = \infty$ to $w = 0$.

We further remark that the solution $u(t,x)$ once becomes $u(t^{1/(\lambda+1)}, x)$ with fractional singularities at $\widetilde{t} = 0$ by the fractional coordinate transform above. Generally speaking, we cannot substitute $\widetilde{t} = t^{\lambda+1}/(\lambda+1)$ into a hyperfunction $u(t,x)$. Then we define F-mild hyperfunctions with fractional order.

Definition 3.2. Let $u(t,x)$ be a germ of $\mathscr{B}_{N|M_+}$ at a point $(0, \mathring{x}) \in N$. We call $u(t,x)$ $1/\ell$-F-mild at $(0,\mathring{x})$ if and only if u has an expression as

$$u(t,x) = \sum_{j=1}^{J} F_j(t^{1/\ell}, x + \sqrt{-1}0\Gamma_j)$$

in a domain $\{(t,x) \in \mathrm{Int}\, M : |t| + |x - \mathring{x}| < r\}$, where J is a positive integer, r is a positive number, Γ_j $(j = 1, 2, \cdots, J)$ are open convex cones and each $F_j(t,z)$ is holomorphic on a domain (2.1).

This definition yields a correspondence between a solution $u(t,x)$ of the equation $Pu = 0$ and $u(\widetilde{t}, x)$ of $Qu = 0$.

§ 4. Some examples of the second order case

At the last of this paper, we give some examples of such equations we consider in the previous section. In [1], Alinhac studies the hyperbolic operator $P(t, \partial_t, \partial_x) = \partial_t^2 - t^2\partial_x^2 + \pi(t, \partial_x)$, where π is a classical pseudodifferential operator of order 1. Yamane in [10] treats operators $(\partial_t - \alpha_1 t\partial_x)(\partial_t - \alpha_2 t\partial_x) +$ (lower order) of second order and $(\partial_t - t\partial_x)\partial_t(\partial_t + t\partial_x) +$ (lower order) of third order. Taniguchi and Tozaki in [9] treat $\partial_t^2 - t^{2\ell}\partial_x^2 + \sqrt{-1}at^{\ell-1}\partial_x$, where $2 \le \ell \in \mathbb{N}$ and $a \in \mathbb{R}$.

On the other hand, we construct solutions for the equations of general order. After the fractional coordinate transform and the quantized Legendre transform, we introduce a scheme (3.3) for formal symbols. Before the quantized Legendre transform, it corresponds to an iteration scheme

$$Q_0 V_0 = 0, \qquad Q_0 V_{k+1} = -Q_1 V_k \qquad (k = 0, 1, 2, \cdots)$$

for a solution $V = \sum_{k=0}^{\infty} V_k(\widetilde{t}, \xi)$, where $Q(\widetilde{t}, \partial_{\widetilde{t}}, \xi) = Q_0 + Q_1$ with a dominant part Q_0. Here we regard ∂_x as a large parameter ξ.

In the case of $n = 1$, that is, that the variable we consider is t and x, we have some typical examples of the Jordan–Pochhammer operators which are the generalization of the Gauss hypergeometric operators (for instance, refer [4] about the Jordan–Pochhammer operator). We remark again that the principal part of the ordinary differential operator L has regular singular points at $w = -\alpha_j(0)$ $(j = 1, 2, \cdots, m)$ and $w = \infty$.

Example 4.1. (1) When $\sigma(P) = (\tau - t\xi)(\tau + t\xi)$ of the Weber operator, the transformed operator $\beta \circ Q \circ \beta^{-1}$ becomes $(w - \sqrt{-1})(w + \sqrt{-1})\partial_w^2 + (7/2)w\partial_w + 3/2$ of the Jordan–Pochhammer operator (the generalized Gauss hypergeometric operator).

(2) When $\sigma(P) = (\tau - t\xi)\tau$, we have an operator $w(1-w)\partial_w^2 + (-2 + (7/2)w)\partial_w - 3/2$ of the Gauss hypergeometric operator after a suitable coordinate transform.

(3) When $\sigma(P) = (\tau - t^{1/2}\xi)(\tau + t^{1/2}\xi)$ of the Airy type, we obtain the Jordan–Pochhammer operator $\beta \circ Q \circ \beta^{-1} = (w - \sqrt{-1})(w + \sqrt{-1})\partial_w^2 + (11/3)w\partial_w + 5/3$.

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