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シュレディンガー方程式に対する経路積分
— ベクトル値の経路積分を考える—
Path Integrals for Schrödinger Equation
(A Kind of Operator–Valued Integration)

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Abstract

In this paper we shall introduce a *generalized equi-continuity of a family of semigroups* and prove a new type of Trotter-Kato Theorem, applicable to the weak convergence of semigroups. In [13], we prove the existence of non-unitary solutions to formally self-adjoint Schrödinger equations. In that paper, we need the Trotter-Kato Theorem for the weak convergence. However, various versions of the Trotter-Kato Theorem in locally convex spaces already published are not applicable to the weak convergence as far as the authors knows. Therefore we shall give a generalized form of the Trotter-Kato Theorem in Yosida [18].

Next we shall define a kind of operator-valued integration and define the Feynman path integrals of Riemann integral type. It seems that it is one of the best possible conditions of the existence of the path integrals of Riemann integral type for Schrödinger equation with singular potentials. Our class of potentials is wide enough:the real measurable potential U should be continuous except a closed set of measure zero.

§1. そもそもの問題意識

与えられた空間 (2 乗可積分関数全体の空間, ソボレフ空間等) の中での解の存在の研究は多いが, その空間で解が存在しない方程式は研究の対象となりにくかった. 我々は逆に方程式が『適切』となる空間を構成したい. 研究が進めば方程式に応じた空間をかなり自由に選べるようになる事が期待される.

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その様な方程式としてシュレディンガー方程式の解を数学を立場から厳密に定義づけた経路積分により積分表示をし，その性質を研究し始めた．これまでに具体的に計算のできるものの研究はされている．また，数学の理論としては，振動積分を用いる方法があり，藤原大輔，谷島賢二，中村周，熊ノ郷尚人を始め，多くの数学者により精密な理論が発展している．G. W. Johnsdon-M. L. Lapidus『The Feynman Integral and Feynman's Operator Calculus』(2000年) は現在までに知られている重要な結果の集大成とも言えるもので，E. Nelson，加藤敏夫の方法を発展させたものも示されているが，ポテンシャルに不自然な条件が付いている (Cf. E. Nelson, J. Mathematical Phys. **5** (1964)).

目標:

1. 『測度』即ちベクトル値の無限次元空間上の一般化された測度という概念を導入し，シュレディンガー方程式を経路積分によって『測度』を用いて表現する．
2. 『測度』で積分可能になるシュレディンガー方程式のポテンシャルの性質を研究する．
3. 非線型半群 の理論を応用し特異点を持つ場合 などの具体的な条件を求める．『弱収束に関するトロッター・加藤の定理』を用いて シュレディンガー方程式の解となる縮小半群の生成作用素の性質を研究する．

意義: 確率論で用いられるウィナー測度は経路の全体の空間という無限次元の空間上の数学的に確立された測度であるが，ファインマンの経路積分は無限次元空間での条件収束はするが絶対収束はしない広義積分の一種であるため，測度では表現できないことはよく知られている．しかし，現実には物理の世界，特に量子力学の分野でファインマンの経路積分は重要な位置を占めている．従って『測度』によるファインマンの経路積分の数学的な定式化は，数学者のみならず物理学者にとっても意義があるであろう．弱収束に関する理論は物理学への応用上重要であるが シュレディンガー方程式についてはあまり研究されていないように思われる．弱収束に関するトロッター・加藤の定理を利用した，方程式の解となる縮小半群を用いた 応用が期待される

参考: 物理学の立場から次のような意見が寄せられた:

『物理学者は (数理物理学者と呼ばれる一部の数学に強い人を除いて) 弱収束や弱位相については，ほとんど知らないと思います (私自身がそうです)．量子論では微分よりも積分が多用されますが，こうした積分計算をひたすら行っているうちに，多くの物理学者は，いつの間にか「測度が無限小の事象は物理的に意味を持たない (したがって無視して良い)」という経験則を身につけてしまいます (私自身も)．数学者は眉をひそめるかもしれませんが，その差が積分に効いてこなければ，閉包も開核も違いはないのです．ヒルベルト空間の議論でも，状態ベクトルそのものではなく内積を取った結果が物理的に重要なので，点列の収束を考えるときには，弱収束に相当する議論しかしません (弱収束と強収束の違いなど意に介さないということです)．

それどころか，内積を取る相手が完全系を構成するかどうかについても，あまり厳密に考えていません(実際に，そうした論文を読んだことがあります)．証明が厳密でなくても，物理的な直観に照らして正当(と感じられる)ならば充分であり，そのうち数学の得意な人がちゃんとした証明をしてくれると期待している訳です.』

§ 2. Trotter-Kato Theorem for Weak Covergence

We shall introduce a *generalized equicontinuity of a family of semigroups* and prove a new type of Trotter-Kato Theorem, applicable to the weak convergence of semigroups. We begin by introducing some terminology and notation and present those aspects of the basic theory which are required in subsequent subsections.

§ 2.1. Filter

Definition 2.1. Given a set E, a partial ordering $\subset$ can be defined on the powerset $\mathcal{P}(E)$ by subset inclusion. Define a filter $\mathcal{F}$ on E as a subset of $\mathcal{P}(E)$ with the following properties:

i) $\emptyset \notin \mathcal{F}$ (the empty set is not in $\mathcal{F}$);
ii) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \bigcap B \in \mathcal{F}$ ($\mathcal{F}$ is closed under finite meets);
iii) If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$ (therefore $E \in \mathcal{F}$).

Definition 2.2. Let $\mathcal{B}$ is a subset of $\mathcal{P}(E)$. $\mathcal{B}$ is called filter base on E if

i) The intersection of any two sets of $\mathcal{B}$ contains a set of $\mathcal{B}$,
ii) $\mathcal{B}$ is non-empty and the empty set is not in $\mathcal{B}$.

Let X be a topological space.

Definition 2.3. $\mathcal{U}(x)$ is called the neighborhood filter at point x for X if $\mathcal{U}(x)$ is the set of all topological neighborhoods of the point x.

Definition 2.4. We say that filter base $\mathcal{B}$ converges to x, denoted by $\mathcal{B} \to x$, if for every neighborhood U of x, there is a $B \in \mathcal{B}$ such that $B \subset U$. In this case, x is called a limit of $\mathcal{B}$ and $\mathcal{B}$ is called a convergent filter base.

Lemma 2.5. *For every neighbourhood base $\mathcal{U}(x)$ of x, it follows that $\mathcal{U}(x) \to x$.*

Lemma 2.6. *X is a Hausdorff space if and only if every filter base on X has at most one limit.*

For details concerning the filter, we refer to Bourbaki [1].

§ 2.2. Locally Convex Topologies

Definition 2.7. A linear topological space X over the complex number field $\mathbb{C}$ is called a locally convex linear topological space, or, in short, a locally convex space, if and only if its open sets $\ni 0$ contains a convex, balanced and absorbing open set. Let $M \subset X$. Then:

1. M is said to be balanced if $x \in M$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ imply $\alpha x \in M$.
2. M is said to be absorbing if for any $x \in X$, there exists $0 < \alpha \in \mathbb{R}$ such that $\alpha^{-1} x \in M$.

§ 2.3. Mackey Topology

Let X, X' be two linear spaces over the complex number field $\mathbb{C}$ and a scalar product $\langle x, x' \rangle \in \mathbb{C}$ $(x \in X, x' \in X')$ be defined. We say $\langle X, X' \rangle$ is a dual pair. Let τ be a locally convex topology on a linear space X and $\mathcal{U}_\tau = \{U_\gamma\}$ be a fundamental system of τ-neighbourhoods of zero. We denote by X_τ the space X equipped with the topology τ.

Definition 2.8. Let X be topological vector space. The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology such that all elements of X' remains continuous.

Definition 2.9. Let X be topological vector space. The Macky topology on X, denoted by $\tau_M(X, X')$, is the strongest topology such that all elements of X' remains continuous.

The weak topology $\sigma(X, X')$ is the weakest locally convex topology in all locally convex topologies $\{\tau_\gamma\}$ such that $X'_{\tau_\gamma} = X'$ and the Mackey topology $\tau_M = \tau_M(X, X')$ is the strongest one in $\{\tau_\gamma\}$ such that $X'_{\tau_\gamma} = X'$.

§ 2.4. Compact Open Topology

Definition 2.10. The strong topology β of X' is the topology of uniform convergence on every $\sigma(X, X')$-bounded set in X. We denote by X'_β the space $(X')_\beta$.

Definition 2.11. We denote by τ_0 the locally convex topology on X defined by the seminorm system $\mathcal{P} = \{p_\gamma \mid p_\gamma(f) = \sup_{g \in C_\gamma} |\langle f, g \rangle|, C_\gamma \in \mathcal{C}\}$, where $\mathcal{C} = \{C_\gamma\}$ denotes the family of the compact subsets of X'_β. Equivalently, $\mathcal{U}_{\tau_0} = \{U_p\}_{p \in \mathcal{P}}$, where $U_p = \{x \in X \mid p(x) < 1\}$ is a fundamental system of τ_0-neighbourhoods of zero. τ_0 is called the compact open topology.

In the case of Banach space, J. Dieudonné has proved the following theorem.

Theorem 2.12 (Dieudonné [3]). *The bounded weak* topology in a Banach space is identical with the compact open tpoplogy.*

We denote by X'^* the space of linear functionals bounded on every bounded set in X'_β.

Proposition 2.13. *Let $\overline{X}_{\tau_0}$ be the completion of the space X_{τ_0}. Then $(X'_\beta)' \subset \overline{X}_{\tau_0} \subset X'^*$.*

Corollary 2.14. *If X is a Banach space, then $(X'_\beta)' = \overline{X}_{\tau_0}$.*

§ 2.5. Locally Convex Topologies

Definition 2.15. Let X be a locally convex linear topological space, and $\{T_t \mid t \geq 0\}$ a one-parameter family of continuous linear operators in the algebra $\mathcal{L}(X, X)$ of all continuous linear operators defined on X into X. If for any continuous seminorm p on X, there exists a continuous seminorm q on X such that

$$p(T_t x) \leq q(x), \quad \text{for any } t \geq 0 \text{ and } x \in X, \tag{2.1}$$

then $\{T_t\}$ is said to be equicontinuous.

Definition 2.16. Let X be a locally convex linear topological space, and $\{T_t \mid t \geq 0\}$ be a one-parameter family of continuous linear operators in $\mathcal{L}(X,X)$ satisfying the following conditions:

(2.2) $$T_t T_s = T_{t+s}, \quad T_0 = I,$$

(2.3) $$\lim_{t \to t_0} T_t x = T_{t_0} x \quad \text{for any } t_0 \geq 0 \text{ and } x \in X,$$

(2.4) the family of mappings $\{T_t\}$ is equicontinuous in t.

Then such a family $\{T_t\}$ is called an equicontinuous semigroup of class (C_0).

Theorem 2.17 ([18, p. 233 Theorem]). *Assume that a family* $\{T_t \mid t \geq 0\}$ *of operators in* $\mathcal{L}(X,X)$ *satisfy* (2.2). *Then condition* (2.3) *is equivalent to the condition*

(2.5) $$w\text{-}\lim_{t \downarrow 0} T_t x = x \quad \textit{for every } x \in X.$$

§ 2.6. Generalization of Equi-Continuity of Semigroups

Let X be a locally convex linear topological space and X' its dual, and τ_0 *the compact-open topology* of X.

Remark. Note that τ_0 is equal to the weak topology $\sigma(X,X')$ on any $\sigma(X,X')$-compact set; that is, a sequence $\{x_k\}$ is weakly convergent if and only if it is τ_0-convergent. However, a bounded C_0-semigroup $\{T_t\}$ is not necessarily equicontinuous with respect to the weak topology but equicontinuous with respect to the topology τ_0. In order to apply Hille-Yosida or Trotter-Kato Theorem, the equicontinuity of semigroups is necessary.

Let $(X,\tau) = X_\tau$ be a linear space X equipped with a locally convex topology τ. Denote by τ_M the Mackey topolpgy of (X,τ). Their duals are equal : $(X,\tau)' = (X,\tau_M)'$ by definition and $\sigma \prec \tau \prec \tau_M$. We consider an infinite semi-orderd index set $\mathcal{A} = \{\alpha\}$ and a family of semigroups $\{T_t^\alpha\}_{\alpha \in \mathcal{A}}$. From Definition 2.15 the condition of equi-continuity of the family is: *for any continuous seminorm p on X, there exists continuous seminorm q on X such that*

(2.6) $$p(T_t^\alpha x) \leq q(x), \quad \text{for all } t \geq 0, \quad x \in X, \quad \alpha \in \mathcal{A}.$$

The relation (2.6) is written as $\bigcup_{\alpha \in \mathcal{A}} \bigcup_{t \geq 0} T_t^\alpha V \subset U$ for $U = \{x \in X \mid p(x) < 1\}$ and $V = \{x \in X \mid q(x) < 1\}$. This is the equicontinuity of the family $\{T_t^\alpha \colon X_\tau \to X_\tau\}_{\alpha \in \mathcal{A}}$. We shall define the equicontinuity of the family $\{T_t^\alpha \colon X_{\tau_M} \to X_\tau\}_{\alpha \in \mathcal{A}}$, a modified form of (2.1).

Definition 2.18. The family $\{T_t^\alpha\}$ is said to be (τ,τ_M)-equicontinuous if for any τ-continuous seminorm p on X_τ there exists a τ_M-continuous seminorm q_M on X_{τ_M} such that $p(T_t^\alpha x) \leq q_M(x)$ $(t \geq 0, x \in X, \alpha \in \mathcal{A})$.

Remark. We may define the (τ,τ_1)-equicontinuity for a locally convex topology τ_1 satisfying $(X,\tau)' = (X,\tau_1)'$. However, (τ,τ_1)-equicontinuity implies (τ,τ_M)-equicontinuity.

The Hille-Yosida Theorem for (τ,τ_M)-equicontinuous semigroups is:

Theorem 2.19. *Suppose that A is a linear operator with dense domain* D(A) *in X and the resolvent $R(n;A) = (nI - A)^{-1} \in \mathcal{L}(X,X)$ exists for $n \in \mathbb{N}$. Then A is the generator of an τ-equicontinuous semigroup if and only if the family $\{(I - n^{-1}A)^m\} = \{nR(n;A)^m\}$ is (τ,τ_M)-equicontinuous in $m \in \mathbb{N}$ and $n \in \mathbb{N}$.*

§ 2.7. Trotter-Kato Theorem

Now we shall give a generalized form of the Trotter-Kato Theorem.

Theorem 2.20. *Suppose the following conditions:*

1) *for any $\alpha \in \mathcal{A}$, a semigroup $\{T_t^\alpha\}$ is τ-equicontinuous and C_0 type with respect to τ.*
2) *the family $\{T_t^\alpha\}_{\alpha\in\mathcal{A}}$ is (τ,τ_M)-equicontinuous; that is, for any τ-neighbourhood U of zero, there exists τ_M-neighbourhood V of zero such that $\bigcup_{\alpha\in\mathcal{A}}\bigcup_{t\geq 0} T_t^\alpha V \subset U$.*
3) *there exists some filter Φ of subsets of $\mathcal{A}$ and some complex number λ_0 with* $\mathrm{Re}\,\lambda_0 > 0$, *such that the following holds: there exists pseudo-resolvent $J(\lambda_l)x$ in X such that for any $f \in X$, there exists $\varphi_\Phi = \tau\text{-}\lim_{\alpha\in\varphi\in\Phi}(I - \lambda_l A_\alpha)^{-1} f$, where $\{\lambda_l\}_{l\in\mathbb{N}}$ is a sequence of distinct points in $\mathbb{C}$ and $\lambda_l \to \lambda_0$ as $l \to \infty$ in such a way that the range* $\mathrm{R}(J(\lambda_l))$ *is dense in X.*

Thus the operator $(I-\lambda_0 A_\Phi)^{-1}$ can be defined. If the range $\mathrm{R}((I-A_\Phi)^{-1})$ *is dense in X, then A_Φ is a densely defined closed operator and generates a semigroup $\{T_t^\Phi\}$, which is a C_0-semigroup with respect to the topology τ and $\tau\text{-}\lim_{\alpha\in\phi\in\Phi} T_t^\alpha x = T_t^\Phi x$ for all $x \in X$.*

Lemma 2.21. *The family $\{(I - n^{-1}A)^m\} = \{nR(n;A)^m\}$ is (τ,τ_M)-equicontinuous in $m \in \mathbb{N}$ and $n \in \mathbb{N}$.*

By Theorem 2.19, we have

Lemma 2.22. *A_Φ generates a semigroup $\{T_t^\Phi\}$.*

§ 2.8. Weak Convergence of Semigroups

We consider a family of contraction C_0-semigroups $\{T_t^\alpha\}_{\alpha\in\mathcal{A}}$ in a reflexive Banach space X.

Theorem 2.23. *Suppose that for some filter Φ, for all $f \in X$, there exists $\varphi_\Phi = w\text{-}\lim_{\alpha\in\varphi\in\Phi}(I - A_\alpha)^{-1} f$.*
Thus the operator $(I-A_\Phi)^{-1}$ is defined. If the range $R((I-A_\Phi)^{-1})$ is dense in X, A_Φ is a densely defined closed operator and generates a semigroup $\{T_t^\Phi\}$: $w\text{-}\lim_{\alpha\in\phi\in\Phi} T_t^\alpha x = T_t^\Phi x, \quad \forall x \in X$. Moreover, we have $\{T_t^\Phi\}$ is a contraction C_0-semigroup in X.

Proof. By Corollary 2.14, X_{τ_0} is complete. The family $\{T_t^\alpha\}$ is norm-equi-continuous, since each semigroup T_t^α is a contraction : $\|T_t^\alpha\| \leq 1$. For a contraction semigroup, we have $\|(I - A_\alpha)^{-1}\| \leq 1$. Hence $\varphi_\Phi = w\text{-}\lim_{\alpha\in\phi\in\Phi}(I - A_\alpha)^{-1} f$ implies $\varphi_\Phi = \tau_0\text{-}\lim_{\alpha\in\phi\in\Phi}(I - A_\alpha)^{-1} f$. Since

$R((I - A_\Phi)^{-1})$ is dense in X, Theorem 2.20 implies τ_0- $\lim_{\alpha \in \phi \in \Phi} T_t^\alpha x = T_t^\Phi x$ for some semigroup T_t^Φ of C_0-type with respect to τ_0. Hence the C_0-semigroup T_t^Φ in X_{τ_0}, $T_t^\Phi \varphi = \tau_0$- $\lim_{\alpha \in \phi \in \Phi} T_t^\alpha \varphi$, exists. Since we have

$$\|T_t^\Phi \varphi\| = \|w\text{-} \lim_{\alpha \in \phi \in \Phi} T_t^\alpha \varphi\| \leq \lim_{\alpha \in \phi \in \Phi} \|T_t^\alpha \varphi\| = \|\varphi\|, \tag{2.7}$$

T_t^Φ is a contraction. It suffices to show the strong continuity of $T_t^\Phi \varphi$. This follows from Theorem 2.17 which says *a weakly continuous semigroup in a Banach space is strongly continuous.* Therefore the proof is complete. □

In the case of Hilbert space we can give more simple proof.

§ 3. Trotter-Kato Theorem for Weak Convergence on Hilbert Space Cases

Here we study this theorem in Hilbert space. In the case of Hilbert spaces we can give more simple proof. We consider a family of contraction C_0-semigroups $\{T_t^n\}_{n \in \mathbb{N}}$ in a separable Hilbert space H, with inner product denoted by $\langle \cdot, \cdot \rangle$ and crresponding norm $\|\cdot\|$. In this paper we prove the weak convergence of $\{T_t^n\}_{n \in \mathbb{N}}$. Our main theorem is as follows:

Theorem 3.1. *Let A_n be the infinitegimal generator of unitary C_0-semigroups $\{T_t^n\}_{n \in \mathbb{N}}$. Suppose that, for some λ_0 in $\mathbb{C}$ with* $\mathrm{Re}\,\lambda_0 > 0$ *there exists $J(\lambda_l)x$ in H such that*

$$J(\lambda_l)x = w\text{-} \lim_{n \to \infty} (\lambda_l I - A_n)^{-1} x$$

for any $x \in H$, where $\{\lambda_l\}_{l \in \mathbb{N}}$ is a sequence of distinct points in $\mathbb{C}$ and $\lambda_l \to \lambda_0$ as $l \to \infty$ in such a way that the range $\mathcal{R}(J(\lambda_l))$ is dense in H. Then $J(\lambda_0)$ is the resolvent of the densely defined closed operator A_∞, which generates a contraction semigroup T_t^∞ of class (C_0) in H and

$$w\text{-} \lim_{n \to \infty} T_t^n x = T_t^\infty x \tag{3.1}$$

for all x in H.

§ 3.1. Basic Theory of Hilbert Spaces

In this subsection we present those aspects of the basic theory of Hilbert spaces which are required in subsequent sections. Let H be a Hilbert space with inner product denoted by $< \cdot, \cdot >$ and crresponding norm $\|\cdot\|$.

Definition 3.2. A subset $S \subset H$ is said to be *fundamental* if the closed span of S is H (in other wards, if the span of S is everywhere dense).

Definition 3.3. H is *separable if H contain a countable subset which is dense in H.*

Lemma 3.4. *For the separability of H, it sufficient that H contains a countable subset S which is fundamental. A subset of a separable set is separable.*

Example. $C(\Omega)$ is separable, where Ω is a compact space. $L^2(\mathbb{R}^n)$ is also separable. Sobolev space $H^l(\mathbb{R}^n)$ is also separable for l in $\mathbb{N}$.

Definition 3.5. A sequence u_n in H is said to *converge weakly if* $\langle f,u_n\rangle$ *converges for every* f *in* H. *If this limit is equal to* $\langle f,u\rangle$ *for some* u *in* H *for every* f, *then* $\{u_n\}$ *is said to converge weakly to* u *or have weak limit* u. *We denote this by the symbol* $u = w\text{-}\lim_{n\to\infty} u_n$.

Lemma 3.6. (1) *A sequence can have at most one weak limit.*

(2) $\|u\| \le \liminf_{n\to\infty} \|u_n\|$ *for* $u = w\text{-}\lim_{n\to\infty} u_n$

(3) *A weakly convergent sequence is bounded.*

Lemma 3.7. (1) *If* u_n *in* H *is a bounded sequence, then there is a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that* $w\text{-}\lim_{n_k\to\infty} u_{n_k} = u$ *for some* u *in* H.

(2) *Let* u_n *in* H *be a bounded sequence. In order that* u_n *converge weakly to* u, *it suffices that* $\langle f,u_n\rangle$ *converge to* $\langle f,u\rangle$ *for all* f *of a fundamental subset* S *of* H.

Lemma 3.8. *H is weakly complete (i.e. every weakly convergent sequence has a weak limit).*

Definition 3.9. Let Ω be open domain of $\mathbb{C}$. $f:\Omega\to H$ is called weakly holomorphic for λ in $\mathbb{C}$ if, for each x in H, the numerical function $\langle f(\lambda),x\rangle$ of λ is holomorphic in Ω.

Lemma 3.10. *Let* Ω *be open domain of* $\mathbb{C}$ *and* $f\colon \Omega\to H$. *If* f *is weakly holomorphic on* Ω, *then* f *is holomorphic on* Ω.

Using this lemma we obtain that Hilbert space valued holomorphic function has the same character as the usual holomorphic function of a complex number value. The result in the case of a complex number value is extended to the holomorphic function of a Hiibert space value. Thus we have Cauchy's integral theorem, Taylor's and Laurent's expantion, and so on,

§ 3.2. Proof of Theorem 3.1

3.2.1. A_∞ is the infinitesimal generator of a semigroup

We first prove that if there exists the operator A_∞ it is the infinitesimal generator of a contraction semigroup T_t^∞ of class (C_0).

Lemma 3.11. *Let* A_n *be the infinitegimal generator of unitary* C_0*-semigroups* $\{T_t^n\}_{n\in\mathbb{N}}$. *Suppose that for any* x *in* H $(\lambda_0 I - A_\infty)^{-1}x = w\text{-}\lim_{n\to\infty}(\lambda_0 I - A_n)^{-1}x$, *in such a way that the range* $\mathcal{R}((\lambda_0 I + A_\infty)^{-1})$ *is dence in* H. *Then* A_∞ *generates a contraction semigroup* T_t^∞ *of class* (C_0) *in* H.

Proof. Note that ${}^tA_n = -A_n$. We obtain that

$$\langle(\lambda_0 I - A_\infty)^{-1}x,y\rangle = \lim_{n\to\infty}\langle(\lambda_0 I - A_n)^{-1}x,y\rangle = \lim_{n\to\infty}\langle x,(\lambda_0 I + A_n)^{-1}y\rangle = \langle x,(\lambda_0 I + A_\infty)^{-1}y\rangle.$$

Assume that $(\lambda_0 I - A_\infty)^{-1}$ is not one to one mapping, that is to say, there exists $x_0\in H$ such that $x_0\neq 0$ and $(\lambda_0 I - A_\infty)^{-1}x_0 = 0$. Therefore $\langle x_0,(\lambda_0 I + A_\infty)^{-1}y\rangle = 0$ for any $y\in H$. It follows that $\mathcal{R}((\lambda_0 I + A_\infty)^{-1})\subset\{x_0\}^\perp$, where $\{x_0\}^\perp$ is the orthogonal complement of x_0. At the same time for any $y\in\mathcal{R}((\lambda_0 I - A_\infty)^{-1})$ there exists $x\in H$ such that $y = (\lambda_0 I - A_\infty)^{-1}x$. Then we have

$\lambda_0 y - A_\infty y = x$ and $\lambda_0 y + A_\infty y = -x + 2\lambda_0 y$ which implies $y = (\lambda_0 I + A_\infty)^{-1}(-x + 2\lambda_0 y)$. It means that $\mathcal{R}((\lambda_0 I + A_\infty)^{-1}) \supset (\lambda_0 I - A_\infty)^{-1} \cdot H = \mathcal{R}((\lambda_0 I - A_\infty)^{-1})$.

Since $\mathcal{R}((\lambda_0 I - A_\infty)^{-1})$ is dense in H, $\mathcal{R}((\lambda_0 I + A_\infty)^{-1})$ is also dense in H. It is contradiction. Then we obtain that $(\lambda_0 I - A_\infty)^{-1}$ is one to one mapping and A_∞ is a closed operator. By $\|(\lambda_0 I - A_\infty)^{-1}\| \leq 1$ Hille-Yosida Theorem implies that A_∞ is the infinitesimal generator of a contraction semigroup $\{T_t^\infty\}$ of class (C_0). □

3.2.2. The properties of resolvent equations

Lemma 3.12. *$(\lambda I - A_n)^{-1}x$ converge weakly to a holomorphic function $J(\lambda)x$ as $n \to \infty$ for* $\mathrm{Re}\,\lambda > 0$: $w\text{-}\lim_{n\to\infty}(\lambda I - A_n)^{-1}x = J(\lambda)x$.

Lemma 3.13. *For $\lambda \in \Lambda$ and $m \in \mathbb{N}$,*

$$w\text{-}\lim_{n\to\infty}((\lambda I - A_n)^{-1})^m x = ((\lambda I - A_\infty)^{-1})^m x.$$

3.2.3. $\{T_t^n\}_{n\in\mathbb{N}}$ Converges $\{T_t^\infty\}$

We show in section 4,1 that A_∞ is the infinitesimal generator of a contraction semigroup T_t^∞ of class (C_0). Now we show (3.1) in Theorem 3.1. A fundamental system of neighborhoods of x_0 in H of weak topology $\sigma\langle H,H\rangle$ is $V(x_0; y_1, \cdots, y_n : \varepsilon) = \{x \in H; |\langle x - x_0, y_j\rangle| < \varepsilon, j = 1, \cdots, n\}$, where $y_1, \cdots, y_n$ are an arbitrary finite system of element of H.

Lemma 3.14. *We fixed $x_0, y_1, \cdots, y_k$ in H and $t > 0$. Then we obtain that*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : |\langle T_t^n x_0 - T_t^\infty x_0, y_j\rangle| < \varepsilon, \qquad \forall n > n_0.$$

Lemma 3.15. $w\text{-}\lim_{n\to\infty} T_t^n x = T_t^\infty x, \quad$ *for all* $x \in H$.

If $\{T_n\}$ is weakly convergent, it is uniformly bounded, that is, $\{\|T_n\|\}$ is bounded. To see this we recall that by lemma 3.6 $\{\|T_n x\|\}$ is bounded for each $x \in H$ since $\{T_n x\}$ is weakly convergent. The assertion then follows by the principle of uniformness. Finally since by lemma 3.6 we have $\|T_t^\infty x\| = \|w\text{-}\lim_{n\to\infty} T_t^n x\| \leq \lim_{n\to\infty}\|T_t^n x\| = \|x\|$,it follows that T_t^∞ is a contraction. Then the proof of Theorem 1 is complete.

Remark. (1) For simplicity we assume that H is separable. But this assumption is not necessary condition.

(2) $s\text{-}\lim_{n\to\infty} T_t^n x = T_t^\infty x$ if and only if $\lim_{n\to\infty}\|T_t^n x\| = \|T_t^\infty x\|$.

(3) Strong convergence implies weak convergence. The converse is not true unless H is finite-dimensional.

§ 4. Schrödinger Equation

In this section we make an attempt to apply our results to the Schrödinger equation. For details concerning this equation, we refer to Kōmura [13]. We shall construct a family of unique solutions to the Schrödinger equation in $\mathbb{R}^N$

$$h\frac{\partial u(t,x)}{\partial t} = \frac{ih^2}{2m}\triangle u(t,x) - iU(x)u(t,x), \quad u(0,x) = \varphi(x), \tag{4.1}$$

for $U \in L^{\infty}_{loc}(\mathbb{R}^N \backslash \mathcal{N}, \mathbb{R})$ where $\mathcal{N}$ is a closed set of measure 0 (for further information, see (4.3)). Here h and m are positive constants.

For simplicity we consider the following normalized equation :

$$\frac{\partial u(t,x)}{\partial t} = i\triangle u(t,x) - iU(x)u(t,x), \quad u(0,x) = \varphi(x), \ \varphi \in H^{(2)}(\mathbb{R}^N;\mathbb{C}), \tag{4.2}$$

where $H^{(2)}(\mathbb{R}^N;\mathbb{C})$ denote the Sobolev space of L^2-functions with first and second distributional derivatives also in L^2 on $\mathbb{R}^n$ to $\mathbb{C}$.

If $\triangle - U$ is essentially self-adjoint, the operator family $\{T_t\}$ defined by $T_t\varphi = u(t)$ is uniquely extended to a group of unitary operators from $L^2(\mathbb{R}^N;\mathbb{C})$ to $L^2(\mathbb{R}^N;\mathbb{R})$.

Let $\mathcal{N} =$ *a fixed closed subset of* $\mathbb{R}^N$ *of measure* 0.

Let $\mathcal{D} = \{D\}$ be the maximum family such that each element $D \subset \overline{D} \subset \mathbb{R}^N \backslash \mathcal{N}$ is a finite union of connected bounded open sets. The family $\mathcal{D} = \{D\}$ satisfies $\bigcup_{D\in\mathcal{D}} D = \mathbb{R}^N \backslash \mathcal{N}$. We denote the restriction of f to D by $f|_D$. We use the following notation

$$L^{\infty}_{loc}(\mathbb{R}^N \backslash \mathcal{N}, \mathbb{R}) = \left\{ f \;\middle|\; f(x) \in \mathbb{R},\ x \in \mathbb{R}^N,\ f|_D \in L^{\infty}(D), \forall D \in \mathcal{D} \right\}. \tag{4.3}$$

Let $U \in L^{\infty}_{loc}(\mathbb{R}^N \backslash \mathcal{N}, \mathbb{R})$. We assume for any neighbourhood of any point of $\mathcal{N}$, U is not essentially bounded. By this assumption, U uniquely determines $\mathcal{N}$ in the following sense : $\mathcal{N} = \bigcap_{\nu} \left\{ \mathcal{N}_\nu \;\middle|\; U \in L^{\infty}_{loc}(\mathbb{R}^N \backslash \mathcal{N}_\nu, \mathbb{R}) \right\}$.

Let $B_n = \left\{ x \in \mathbb{R}^N \,|\, -n < U(x) < n \right\}, n \in \mathbb{N}$. Then we have $B_m \supset B_n$ for $m > n$ and

$$\forall D \in \mathcal{D}, \quad \exists B_n \,:\, D \subset \overline{D} \subset B_n. \tag{4.4}$$

(Strictly speaking, $\overline{D} \backslash B_n$ is not necessarily empty, but a null set.) We denote

$$U_n(x) = \min\left\{ n, \max\{-n, U(x)\} \right\}.$$

Thus U_n in $L^{\infty}(\mathbb{R}^N;\mathbb{R})$.

For U in $L^{\infty}_{loc}(\mathbb{R}^N \backslash \mathcal{N}, \mathbb{R})$ we consider the approximative equation

$$\frac{d}{dt}u_n(t) = A_n u_n(t), \quad \text{where } A_n = i(\triangle - U_n). \tag{4.5}$$

In this case the operator $-iA_n$ is essentially self-adjoint. We obtain that the semigroup $\{T^n_t\}$ generated by $-iA_n$ is the family of solutions to (4.5) and is a group of unitary operators : $||T^n_t\varphi|| = ||\varphi||, \quad -\infty < t < \infty, \forall \varphi \in L^2(\mathbb{R}^N;\mathbb{C})$.

Theorem 4.1. *For any U in $L^{\infty}_{loc}(\mathbb{R}^N \backslash \mathcal{N};\mathbb{R})$, there exists a closed extension of the operator $(i\triangle - iU)|_{C^{\infty}_0(R^N \backslash \mathcal{N})}$ in $L^2(\mathbb{R}^N;\mathbb{C}) \to L^2(\mathbb{R}^N;\mathbb{C})$ which generates a contraction C_0-semigroup $\{T_t \mid t \geq 0\}$ such that $T_t\varphi = w\text{-}\lim_{n\to\infty} T^n_t\varphi$, $\forall \varphi \in L^2(\mathbb{R}^N;\mathbb{C})$, where $T^n_t\varphi$ is the solution to* (4.5) *and w-*lim *means the weak convergence.*

For the proof of existence of $\{T_t\}$, we use Theorem 2.23. For details, see Kōmura [13].

§ 5. Feynman Path Integral of Riemann Type

Now we shall define a kind of operator-valued integration and define the Feynman path integrals of Riemann integral type. It seems that it is one of the best possible conditions of the existence of the path integrals of Riemann integral type for Schrödinger equation with singular potentials. Our class of potentials is wide enough:the real measurable potential U should be continuous except a closed set of measure zero.

Heuristic Feynman path integrals have played a remarkable role in various aspects of quantum physics. But rigorous mathematical treatment of this integral is not enough. It is well known that Feynman path integrals for Schrödinger equations are not represented by scalar-valued measure(see E.Nelson [16]).

In this paper, we discuss a kind of operator-valued integration and define the path integral of Riemann type, analogically to Riemann integration of scalar functions. So our integration is different from the one of Nelson (see T. Ichinose [9], E. Nelson [16] and F. Takeo [17]). We shall show that the solution to the Schrödinger equation in $\mathbb{R}^N (N \geq 2)$

$$\frac{\partial}{\partial t}u(t,x) = i\triangle u(t,x) - iU(t,x)u(t,x), \quad u(0,x) = \varphi(x), \quad \varphi \in L^2(\mathbb{R}^N;\mathbb{C}) \tag{5.1}$$

is written as the path integral

$$u(t,x) = \int_{\Omega_{[0,t]}} e^{-i\int_0^t U(\tau,\gamma(\tau))d\tau}\varphi(\gamma(0))d\mu(\gamma), \quad \varphi \in L^2(\mathbb{R}^N;\mathbb{C}) \tag{5.2}$$

of Riemann type. Here we denote by γ a path on $\mathbb{R}^N$, that is, $\gamma \in \Omega_{[0,t]} \equiv \prod_{\alpha\in[0,t]} \mathbb{R}^N_\alpha$ ($\mathbb{R}^N_\alpha =$ a copy of $\mathbb{R}^N$): $\gamma = (x_\alpha \in \mathbb{R}^N)_{\alpha\in[0,t]}$ (or $\gamma(\alpha) = x_\alpha$).

We study the conditions to define the path integrals of Riemann integral type for Schrödinger equation with singular potentials. The paper of Nelson [16] is concerned with the Schrödinger operator $i[(1/2m)\Delta - V(x)]$, except for a set N of m with measure 0 and he assume that V is continuous on the complement of a closed set F of capacity 0 . In this paper Nelson mentions that "*The restoriction to almost every real value of the mass parameter is an unsatisfactory feature of the theory*" ([16, p. 335]). As G. W. Johnson and M. L. Lapidus point out that it is a serious weakness ([11, p. 295]). Notice that we have no restriction of this type.

§ 6. Abstract Evolution Equation

Definition 6.1. The space of functions f in $L^\infty(\mathbb{R}^N;\mathbb{C})$ such that f is uniformly continuous on $\mathbb{R}^N$ will be denoted $C_\infty(\mathbb{R}^N;\mathbb{C})$ where $L^\infty(\mathbb{R}^N;\mathbb{C})$ consisting of all essentially bounded functions on $\mathbb{R}^N$.

The equation (5.1) is written as an evolution equation

$$\frac{d}{dt}u(t) = (A + V(t))u(t), \quad u(0) = \varphi, \tag{6.1}$$

where $A = i\triangle$ and $V(t) = -iU(t,\cdot)$ is an $C_\infty(\mathbb{R}^N;\mathbb{C})$-valued function. The associate semigroup with $V \equiv 0$ is written as $\{S_t\}$. More precisely, $\{S_t | -\infty < t < \infty\} \subset L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C}))$ is a group of unitary operators, where $L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C}))$ is the space of all bounded linear operators from $L^2(\mathbb{R}^N;\mathbb{C})$ to $L^2(\mathbb{R}^N;\mathbb{C})$.
Let m be a natural number and $\theta = t/m,\ s_0 = 0,\ s_{j+1} = s_j + \theta,\ s_m = t$ for $j = 0, \cdots, m-1$. The subject of this section is that the solution $u(t)$ to the equation (6.1) is approximated as

$$u(t,x) \sim \Big(\prod_{j=1}^{m} S_\theta e^{V(\tau_j,x)\theta}\Big)\varphi(x), \quad s_{j-1} \le \tau_j < s_j \quad j = 1, \cdots, m. \tag{6.2}$$

We wish to provide some back ground in abstract evolution equation theory.
Let $H = (H, \|\cdot\|)$ be a Hilbert space. Here $\|\cdot\|$ is a norm of H. We consider the following abstract evolution equation in H.

$$\frac{d}{dt}u(t) = (A + B(t))u(t), \quad u(0) = \varphi \in H, \tag{6.3}$$

where A is the generator of a semigroup of unitary operators and $B(t)$ is a bounded linear operator for any $t > 0$.

Definition 6.2. A function u which is differentiable almost everywhere on [0,T] such that $\frac{du}{dt} \in L^1(0,T;H)$ is called a strong solution of the initial value problem (6.3) if $u(0) = \varphi$ and $\frac{d}{dt}u(t) = (A + B(t))u(t)$ a.e. on $[0,T]$.

Lemma 6.3. *The strong solution to*

$$\frac{d}{dt}u(t) = (A + B(t))u(t), \quad u(0) = \varphi \in D(A), \tag{6.4}$$

is given by

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A}B(s)u(s)ds, \tag{6.5}$$

if $B(t)$ is an $L(D(A),D(A))$-valued continuous function. Here $D(A)$ is the domain of A equipped with the graph norm $\|f\|_{D(A)} = (\|f\|^2 + \|Af\|^2)^{1/2}$.

Definition 6.4. The solution to the integral equation (6.5) is called the mild solution to the evolution equation (6.4), if it uniquely exists.

Lemma 6.5. *The mild solution to* (6.4) *uniquely exists if $B(t)$ is an $L(H,H)$-valued continuous function.*

From equation (6.5) we have

$$u(t+\theta) = e^{\theta A}u(t) + \int_0^\theta e^{(\theta-s)A}B(t+s)u(t+s)ds. \tag{6.6}$$

In general $e^{A+B} \neq e^A e^B$. This is becouse A and B need not commute.
Therefore we shall approximate $u(t+\theta)$ by $e^{\theta A}e^{\theta B(t)}u(t)$:

Lemma 6.6. *Let $T > 0$ and $B(t)$ a $L(H,H)$-valued continuous function. Then we have for each $\varepsilon > 0$, there exists $\theta_0 > 0$ such that*

$$\|e^{\theta A}e^{\theta B(t)}u(t) - u(t+\theta)\| < \theta\varepsilon \quad \text{for} \quad 0 < \theta \leq \theta_0,\ 0 \leq t \leq T.$$

We turn now to the solution $u(t)$ to equation (6.1)

Lemma 6.7. *Let $u(t)$ be the solution to the equation (6.1). If $V(t)$ is an $L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C}))$ -valued continuous function, then it holds that*

$$u(t,x) = \lim_{\theta \to 0} \Big(\prod_{j=1}^{m} S_\theta e^{\theta V(\tau_j)} \Big) \varphi(x) \quad \text{for} \quad s_{j-1} \leq \tau_j < s_j. \tag{6.7}$$

§ 7. Path Integral of Riemann Type

§ 7.1. Operator-Valued Integral of Riemann Type

In this subsection we express the operator $S_t e^V : \varphi \longmapsto S_t(e^V \varphi)$ as the integral of $e^V \varphi$ by dS_t. We denote by $\mathbb{Z}$ the set of integers. We consider a division of $\mathbb{R}^N$:

$$\bigcup_{k \in \mathbb{Z}^N} I_k^h = \mathbb{R}^N, \quad I_k^h = [hk_1, hk_1 + h) \times \cdots \times [hk_N, hk_N + h),\ k = (k_1, \cdots, k_N),\ k_j \in \mathbb{Z}.$$

A function e^V in $L^\infty(\mathbb{R}^N;\mathbb{C})$ is considered as an operator in $L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C}))$:

$$e^V : L^2(\mathbb{R}^N;\mathbb{C}) \ni \varphi \longmapsto e^V \varphi \in L^2(\mathbb{R}^N;\mathbb{C}).$$

For simplicity we denote $L^\infty = L^\infty(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C}))$.
The characteristic function $\chi(I_k^h)$ of I_k^h is in the same time an operator in

$$L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})):\ (\chi(I_k^h) \cdot \varphi)(x) \equiv \chi(I_k^h)(x) \cdot \varphi(x) = \begin{cases} \varphi(x) & \text{for } x \in I_k^h, \\ 0 & \text{for } x \notin I_k^h. \end{cases}$$

Note that $\varphi(x) = \sum_{k \in \mathbb{Z}^N} \chi(I_k^h)(x)\varphi(x)$. We denote

$$\Delta_k^h S_t = S_t \chi(I_k^h) \in L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})) : \varphi \to S_t(\chi(I_k^h)\varphi). \tag{7.1}$$

Lemma 7.1. *If e^V in $C_\infty(\mathbb{R}^N;\mathbb{C})$ then the sum $S_t e^V = \sum_{k \in \mathbb{Z}^N} \Delta_k^h S_t e^V$ is unconditionally strongly convergent. That is, for any φ in $L^2(\mathbb{R}^N;\mathbb{C})$, $\sum_{k \in \mathbb{Z}^N} \Delta_k^h S_t e^V \varphi$ strongly converges independent of the order of the sum.*

Proof. The lemma follows from the unconditional strong convergence of $\varphi = \sum_{k \in \mathbb{Z}^N} \chi(I_k^h)\varphi$ or $e^V \varphi = \sum_{k \in \mathbb{Z}^N} \chi(I_k^h)(e^V \varphi)$. In fact we get that if $\chi(I_k^h)\varphi \perp \chi(I_{k'}^h)\varphi$ for $k \neq k'$, then $S_t\chi(I_k^h)\varphi \perp S_t\chi(I_{k'}^h)\varphi$ for $k \neq k'$, since S_t is unitary. Therefore if $\mathbb{Z}_1 \subset \mathbb{Z}_2 \subset \mathbb{Z}^N$, then $\|S_t e^V - \sum_{k \in \mathbb{Z}_1} \Delta_k^h S_t e^V\| \geq \|S_t e^V - \sum_{k \in \mathbb{Z}_2} \Delta_k^h S_t e^V\|$. □

Definition 7.2. For $h > 0$ and $k \in \mathbb{Z}^N$, let an element $x_h^k \in I_k^h$ be fixed. $\sum_k \triangle_k^h S_t e^{V(x_h^k)}$ is called the Riemann sum. $\lim_{h\to 0}\sum_k \triangle_k^h S_t e^{V(x_h^k)}$ is called the Riemann integral of $e^{V(x)}$ by $dS_t(x)$ and denoted by

$$(7.2)\quad R\text{-}\int_{\mathbb{R}^N} dS_t(x)e^{V(x)} = \int_{\mathbb{R}^N} S_t(dx)e^{V(x)} = \lim_{h\to 0}\sum_{k\in\mathbb{Z}^N} \triangle_k^h S_t e^{V(x_h^k)} \in L(L^2(\mathbb{R}^N;\mathbb{C}),L^2(\mathbb{R}^N;\mathbb{C})).$$

dS_t is finitely additive and may be called an operator-valued "Riemann measure".

§ 7.2. Iterated Integral and Multiple Integral

From the definition of Riemann integral, we obtain that

$$\prod_{j=1}^m S_\theta e^{\theta V(\tau_j,x)}\varphi = R\text{-}\int_{\mathbb{R}^N} dS_\theta(x)e^{\theta V(\tau_m,x)}\cdots R\text{-}\int_{\mathbb{R}^N} dS_\theta(x)e^{\theta V(\tau_1,x)}\varphi.$$

This is the iterated integral. We shall express this by the multiple integral.

We denote by $C([0,t],C_\infty(\mathbb{R}^N;\mathbb{C}))$ the space of continuous functions on $[0,t]$ with values in $C_\infty(\mathbb{R}^N;\mathbb{C})$.

Lemma 7.3. *Let V in $C([0,t],C_\infty(\mathbb{R}^N;\mathbb{C}))$. Then we have*

$$(7.3)\qquad \prod_{j=1}^m S_\theta e^{\theta V(\tau_j,\cdot)}\varphi = \lim_{h\to 0}\prod_{j=1}^m S_\theta \sum_k \chi(I_k^h)(\cdot)e^{\theta V(\tau_j,x_h^k)}\varphi \quad for \quad x_h^k \in I_k^h.$$

Denote $\kappa = (k(1),\cdots,k(m)) \in \mathbb{Z}^{N\times m}$ where $k(j) = (k_1(j),\cdots,k_N(j)) \in \mathbb{Z}^N$.
Note that $\triangle_{k(j)}^h S_\theta$ and $e^{\theta V(\tau_j,x_h^{k(j)})}$ commute since each $e^{V(\tau_j,x_h^{k(j)})\theta}$ is a constant function. Thus we have

$$\prod_{j=1}^m S_\theta \sum_{k\in\mathbb{Z}^N}\chi(I_k^h)e^{\theta V(\tau_j,x_h^k)} = \sum_{\kappa\in\mathbb{Z}^{N\times m}}\prod_{j=1}^m\left(S_\theta\chi(I_{k(j)}^h)e^{\theta V(\tau_j,x_h^{k(j)})}\right) = \sum_{\kappa\in\mathbb{Z}^{N\times m}}\prod_{j=1}^m\left(\triangle_{k(j)}^h S_\theta e^{\theta V(\tau_j,x_h^{k(j)})}\right)$$

$$(7.4)\qquad = \sum_{\kappa\in\mathbb{Z}^{N\times m}}\prod_{j=1}^m\left(\triangle_{k(j)}^h S_\theta\right)e^{\sum_{l=1}^m \theta V(\tau_l,x_h^{k(l)})},$$

since the sum $\sum_{k\in\mathbb{Z}^N}\chi(I_k^h)e^{\theta V(\tau_j,x_h^k)}$ is unconditionally convergent. The multiple integral is defined by as follows:

Definition 7.4. The multiple integral of $\exp\left(\sum_{l=1}^m \theta V(\tau,\gamma(\tau_l))\right)$ is defined by

$$(7.5)\quad R\text{-}\int\cdots\int dS_\theta(\gamma(\tau_1))\cdots dS_\theta(\gamma(\tau_m))e^{\sum_{l=1}^m \theta V(\tau,\gamma(\tau_l))}$$

$$= \lim_{h\to 0}\sum_{\kappa\in\mathbb{Z}^{Nm}}\prod_{j=1}^m\left(\triangle_{k(j)}^h S_\theta\right)e^{\sum_{l=1}^m \theta V(\tau_l,x_h^{k(l)})}.$$

Let $h \to 0$ in (7.4) and we get the following lemma by (7.2) and (7.3),

Lemma 7.5. *Let $V \in C([0,t], C_\infty(\mathbb{R}^N;\mathbb{C}))$. Then we have*

$$\prod_{j=1}^{m} S_\theta e^{\theta V(\tau_j,x)} = R\text{-}\int \cdots \int dS_\theta(\gamma(\tau_1)) \cdots dS_\theta(\gamma(\tau_m)) e^{\sum_{j=1}^{m} \theta V(\tau,\gamma(\tau_j))},$$

where $\gamma(\tau_j)$ runs over $\mathbb{R}^N$ for each j and

$$\int dS_\theta(\gamma(\tau_j)) e^{\theta V(\tau,\gamma(\tau_j))} \quad \textit{means} \quad \int_{\mathbb{R}^N} dS_\theta(x) e^{\theta V(\tau_j,x)}.$$

Roughly speaking,

$$\prod_{j=1}^{m} \triangle_{k(j)}^h S_\theta \sim \prod_{j=1}^{m} dS_\theta(\gamma(\tau_j)), \ \textit{for}\ \gamma \in \Omega_{[0,t]},\ \gamma(\tau_j) \in I_{k(j)}^h \quad \textit{as}\ h \to 0.$$

§ 7.3. Path Integral of Riemann Type

Now we define the path integral of Riemann type.

Definition 7.6. The Riemann type path integral of $F(V;t,\gamma) = e^{\int_0^t V(\tau,\gamma(\tau))d\tau}$ is defined by

$$R\text{-}\int_{\Omega_{[0,t]}} e^{\int_0^t V(\tau,\gamma(\tau))d\tau} \varphi d\mu^{Q}(\gamma) = \lim_{m\to\infty} \lim_{h\to 0} \sum_{\kappa \in \mathbb{Z}^{Nm}} \prod_{j=1}^{m} \triangle_{k(j)}^h S_\theta e^{\sum \theta V(\tau_j, x_h^{k(j)})} \varphi,$$

$$= \lim_{m\to\infty} \lim_{h\to 0} \sum_{\kappa \in \mathbb{Z}^{Nm}} \triangle_\kappa^h S_\theta e^{\sum \theta V(\tau_j, x_h^{k(j)})} \varphi, \tag{7.6}$$

where $\triangle_\kappa^h S_\theta = \prod_{j=1}^{m} \triangle_{k(j)}^h S_\theta$.

Thus from Definition 7.4, Lemma 7.5 and Definition 7.6 we obtain that

$$R\text{-}\int_{\Omega_{[0,t]}} e^{\int_0^t V(\tau,\gamma(\tau))d\tau} \varphi d\mu^{Q}(\gamma) = \lim_{m\to\infty} R\text{-}\int \cdots \int dS_\theta(\gamma(\tau_1)) \cdots dS_\theta(\gamma(\tau_m)) e^{\sum_{j=1}^{m} \theta V(\tau_j,\gamma(\tau_j))} \varphi$$

$$= \lim_{m\to\infty} \left(\prod_{j=1}^{m} S_\theta e^{\theta V(\tau_j,x)} \varphi\right).$$

Remark. In general we have not defined the function $F(V;t,\gamma) = e^{\int_0^t V(\tau,\gamma(\tau))d\tau}$, nor the (generalized) measure μ^Q. Since $\int_0^t V(\tau,\gamma(\tau))d\tau$ might not exist for a path γ. Nevertheless the path integral (7.6) is defined for some V.

A sufficient (but not necessary) condition for a function $F(V;t,\gamma)$ to be μ^Q-integrable is given in our next theorem.

Theorem 7.7. *Let $V \in C([0,t], C_\infty(\mathbb{R}^N;\mathbb{C}))$. Then the function $e^{\int_0^t V(\tau,\gamma(\tau))d\tau}$ is μ^Q-integrable. That is,*

$$R\text{-}\int_{\Omega_{[0,t]}} e^{\int_0^t V(\tau,\gamma(\tau))d\tau} \varphi(\gamma(0)) d\mu^{Q}(\gamma) = \lim_{m\to\infty} \left(\prod_{j=1}^{m} S_\theta e^{\theta V(\tau_j,x)}\right) \varphi(x) \tag{7.7}$$

exists.

A direct consequence of Theorem 7.7 is the following theorem.

Theorem 7.8. *Let a real function U in $C([0,t],C_\infty(\mathbb{R}^N;\mathbb{C}))$. Then the mild solution to the Schrödinger equation* (5.1) *is expressed as the Riemann type integral*

$$u(t,x) = R\text{-}\int_{\Omega_{[0,t]}} e^{-i\int_0^t U(\tau,\gamma(\tau))d\tau}\varphi(\gamma(0))d\mu^{\mathcal{Q}}(\gamma). \tag{7.8}$$

§ 8. Integrable Functions

In this section we study Schrödinger equations with singular potentials.

§ 8.1. Integration on a Bounded Domain

Let subset D of $\mathbb{R}^N$ be a bounded open domain with smooth boundary and V be a continuous function on $\overline{D}$. Denote $\Omega_{[0,t]}(D) = \prod_{\alpha\in[0,t]} D_\alpha$ where $D_\alpha =$ a copy of $D = \{\gamma|\gamma(s)\in\overline{D}, \forall s\in[0,t]\}$. We consider the integration on $\Omega_{[0,t]}(D)$.

The family of solutions to the Schrödinger equation in D with Dirichlet boundary condition

$$\frac{\partial}{\partial t}u(t,x) = i\Delta u(t,x), \qquad u(t,x)|_{x\in\partial D} = 0, \quad u(0,x) = \varphi(x)|_{x\in D} \tag{8.1}$$

is written as $u(t) = S_t\varphi$ by a group $\{S_t| -\infty < t < \infty\}$ of unitary operators.

Let $\bigcup_{k\in\mathbb{Z}^N} I_k^h(D) = D$, $I_k^h(D) = D\bigcap\Big([hk_1,hk_1+h)\times\cdots\times[hk_N,hk_N+h)\Big)$, $k=(k_1,\cdots,k_N)$, $k_j\in\mathbb{Z}$.

Definition 8.1. If the Riemann sum $\sum_{k\in\mathbb{Z}^N}\mu(I_k^h(D))(\cdot)e^{V(x_h^k)}$ converges as $h\to 0$ independently of the choice of $\{I_k^h(D)\}$ and $\{x_h^k\}$, the function $e^{V(x)}$ is said to be *Riemann integrable*, where $\mu(I_k^h(D))$ is the volume of $I_k^h(D)$.

If the function $G(x) = e^{-iU(x)}$, $U(x)\in\mathbb{R}$, is Riemann integrable in a bounded domain D, the operator-valued integral $R\text{-}\int_D dS_t(x)e^{V(x)} = \lim_{h\to 0}\sum_{k\in\mathbb{Z}^N}\Delta_k^h S_t e^{V(x_h^k)}$ also exists. Moreover the multiple integral exists.

As is well known, a bounded function on a bounded domain is Riemann integrable if and only if the set of discontinuous points is of measure zero. In our case,

Lemma 8.2. *A function $G(x) = e^{-iU(x)}$, $U(x)\in\mathbb{R}$ for a bounded function U, is Riemann integrable in a bounded domain D if and only if the set of discontinuous points of U is of measure zero.*

Let $\mathcal{N}_{V(t)}(D) = \{x\in\overline{D}\mid V(t)$ is not continuous at $x\}$ and $\mathcal{N}_V(D) = \bigcup_{t\in[0,T]}\mathcal{N}_{V(t)}(D)$ Our next theorem is analogous to Theorem 7.7.

Theorem 8.3. *If a function V in $C([0.T];L^\infty(\mathbb{R}^n;\mathbb{C}))$ and for any t in $[0,T]$ $V(t)$ is Riemann integrable on $\overline{D}$ and $\mathcal{N}_V(D)$ is a closed set of measure zero, then the function $e^{\int_0^t V(\tau,\gamma(\tau))d\tau}$ is μ^Q-integrable on $\Omega_{[0,t]}(D)$. That is,*

$$R\text{-}\int_{\Omega_{[0,t]}(D)} e^{\int_0^t V(\tau,\gamma(\tau))d\tau}\varphi(\gamma(0))d\mu^Q(\gamma) = \lim_{m\to\infty}\prod_{j=1}^{m} S_\theta e^{\theta V(\tau_j,x)}\varphi(x), \quad x\in a.e.D$$

exists.

§ 8.2. Strong Integrability for Non-negative Potentials with Singularity

For simplicity we shall discuss the time-independent case. We use the following notations

$$(8.2) \qquad \mathcal{N} = \text{a fixed closed subset of } \mathbb{R}^N \text{ of measure } 0,$$

$$(8.3) \qquad C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+) = \{U\in C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}) \mid U(x)\geq 0, \text{ for all } x\in\mathbb{R}^N\}.$$

In this section we consider the integrability of the function $e^{-i\int_0^t U(\gamma(s))ds}$ for a function $U\in C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+)$. Let $D_n = \{x\in\mathbb{R}^N| \quad n > U(x)\}$ for $n\in\mathbb{N}$. $\{D_n\}_{n=1}^\infty$ is an increasing sequence such that $\overline{D}_n\subset D_{n+1}$ and $\bigcup_{n=1}^{\infty} D_n = \mathbb{R}^N\backslash\mathcal{N}$. Here D_n is a finite sum of E_k^n for $k\in\mathbb{N}$ and each E_k^n is a bounded open connected set with smooth boundary. For $U\in C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+)$ we define a sequence of functions U_n such that

$$U_n(x) = \min\{n,U(x)\} \quad for \quad n\in\mathbb{N}.$$

Lemma 8.4. *Let U in $C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+)$. Then $e^{-i\int_0^t U_n(\gamma(s))ds}$ is Riemann integrable.*

We denote that

$$T_n(t)\phi = \int_{\Omega_{[0,t]}(D_n)} e^{-i\int_0^t U_n(\gamma(s))ds}\phi d\mu^Q \quad for \quad \phi\in L^2(\mathbb{R}^N;\mathbb{C}).$$

When a function U is not bounded the Riemann integral of $e^{-i\int_0^t U(\gamma(s))ds}$ is not exist for U in $C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+)$. Therefore we introduce the definition of improper Riemann integration with respect to μ^Q.

Definition 8.5. For a function $U\in C(\mathbb{R}^n\backslash\mathcal{N},\mathbb{R}^+)$, the function $e^{-i\int_0^t U(\gamma(s))ds}$ is said to be improper Riemann integrable by μ^Q if

$$\lim_{n\to\infty} R\text{-}\int_{\Omega_{[0,t]}(D_n)} e^{-i\int_0^t U_n(\gamma(s))ds}\phi d\mu^Q$$

exists for any $\phi\in L^2(\mathbb{R}^N;\mathbb{C})$ independently of the choice of $\{D_n\}$.

The main results of this section is the following theorem:

Theorem 8.6. *Let U in $C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+)$. Then the function $F(-iU;t,\gamma) = e^{-i\int_0^t U(\gamma(s))ds}$ is improper Riemann integrable by μ^Q.*

For the proof of this theorem we shall use the subdifferential of convex functionals.

Denote $H^1_R = H^1(\mathbb{R}^N;\mathbb{R})$ and $H^2_R = H^2(\mathbb{R}^N;\mathbb{R})$, where $H^1(\mathbb{R}^N;\mathbb{R})$ is the first Sovolev space on the $\mathbb{R}^N$ and $H^2_R = H^2(\mathbb{R}^N;\mathbb{R})$ is the second Sovolev space on the $\mathbb{R}^N$ The subdifferential of a lower semicontinuous convex functional $\Psi : L^2_R \to (-\infty,\infty]$ is defined as

$$\partial\Psi\colon \psi \mapsto \{\phi \in L^2(\mathbb{R}^N;\mathbb{R}) \mid \Psi(\varphi) \geq \Psi(\psi) + \langle\phi,\varphi-\psi\rangle \text{ for all } \varphi \in L^2(\mathbb{R}^N;\mathbb{R})\}.$$

For the basic property of lower semicontinuous convex functionals and their subdifferentials, we refer to the book [2] by Brézis.

For $U \in C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+)$, the functional $\|\sqrt{U}\phi\|^2$ is lower semicontinuous and convex.

$H^0_1(\Omega)$ denote the Sobolev space defined as the closure of the space of test functions on the open set $\Omega \subset \mathbb{R}^N$ with respect to the Hilbert norm $(\|\cdot\|_2^2 + \|\nabla\cdot\|_2^2)^{1/2}$.

Lemma 8.7. *Let φ_1 and φ_2 be properly lower semicontinuous and convex such that $D(\varphi_1)\cap D(\varphi_2) \neq \emptyset$ where $D(\varphi_1)$ and $D(\varphi_2)$ are effective domain. Then $\varphi_1+\varphi_2$ is properly lower semicontinuous and convex and $\partial\varphi_1 + \partial\varphi_2 \subset \partial(\varphi_1+\varphi_2)$. Moreover $\partial\varphi_1+\partial\varphi_2$ is maximal monotone i f and only if $\partial\varphi_1+\partial\varphi_2 = \partial(\varphi_1+\varphi_2)$.*

Lemma 8.8. *Each functional*

$$\Psi_n(\phi) \equiv \frac{1}{2}(\|(-\triangle)^{\frac{1}{2}}\phi\|^2 + \|\sqrt{U_n}\phi\|^2) \quad or \quad \Psi(\phi) \equiv \frac{1}{2}(\|(-\triangle)^{\frac{1}{2}}\phi\|^2 + \|\sqrt{U}\phi\|^2),$$

is lower semicontinuous and convex. Its effective domain is $D(\Psi_n) \equiv \{f \in L^2(\mathbb{R}^N;\mathbb{R}) \mid \Psi_n(f) < \infty\} = H^1_R$ or $D(\Psi) = D((-\triangle)^{\frac{1}{2}})\bigcap D(\sqrt{U})$.

Roughly speaking, $\Psi(\phi)$ is a closed extension of $\langle -\triangle\phi + U\phi,\phi\rangle$.

Lemma 8.9. *The resolvent $\phi_n = (I+\partial\Psi_n)^{-1}\varphi_0$ of the subdifferential $\partial\Psi_n$ is given by the projection of φ_0 to B_n where $B_n = \{\phi \in L^2(\mathbb{R}^N;\mathbb{R}) \mid \Psi_n(\phi) \leq \Psi_n(\phi_n)\}$: $\mathrm{proj}_{B_n}\varphi_0 = (I+\partial\Psi_n)^{-1}\varphi_0$.*

Lemma 8.10. *The resolvent $(I+\partial\Psi_n)^{-1}$ strongly converges to the resolvent $(I+\partial\Psi)^{-1}$: $(I+\partial\Psi)^{-1}\varphi_0 = \lim_{n\to\infty}(I+\partial\Psi_n)^{-1}\varphi_0$, for any $\varphi_0 \in L^2(\mathbb{R}^N;\mathbb{R})$.*

Proposition 8.11. *$-\partial\Psi$ generates a C_0-semigroup, hence it is a linear operator and $R(I+\partial\Psi) = L^2(\mathbb{R}^N;\mathbb{R})$.*

Proposition 8.12. *Let $S(t)$ and $S_n(t)$ be the semigroups generated by infinitesimal generator $-\partial\Psi$ and $-\partial\Psi_n$ respectively. Then we obtain the following equation*

$$\lim_{n\to\infty} S_n(t)\varphi = S(t)\varphi \quad for\ all \quad \varphi \in L^2(\mathbb{R}^N;\mathbb{R}). \tag{8.4}$$

Proof. We obtain (8.4) by using (8.10) and Trotter-Kato Theorem.

Let $\partial\tilde{\Psi}$, $\partial\tilde{\Psi}_n$: $L^2(\mathbb{R}^N;\mathbb{C}) \to L^2(\mathbb{R}^N;\mathbb{C})$ be the complex extension of $\partial\Psi$, $\partial\Psi_n$: $L^2(\mathbb{R}^N;\mathbb{R}) \to L^2(\mathbb{R}^N;\mathbb{R})$ respectively. From Proposition 1, $R(I+\partial\Psi) = L^2(\mathbb{R}^N;\mathbb{R})$. Hence we obtain $R(I+\partial\tilde{\Psi}) = L^2(\mathbb{R}^N;\mathbb{C})$.

Lemma 8.13. *The operator $\partial\tilde{\Psi}$ is a self-adjoint positive operator.*

Proof. If a symmetric operator T satisfies $R(I+T)=L^2(\mathbb{R}^N;\mathbb{C})$, then it is self-adjoint. The positivity of $\partial\tilde{\Psi}$ is evident, since $\langle\partial\Psi(\phi),\phi\rangle \geq 0$ for all $\phi\in L^2(\mathbb{R}^N;\mathbb{R})$.

Theorem 8.14 (Stone). *A is the infinitesimal generator of a C_0 group of unitary operator on a Hilbert space H if and only if iA is self-adjoint.*

Theorem 8.15. *If a function U in $C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}^+)$ then the Schrödinger equation*

$$\frac{d}{dt}u(t) = -i\partial\tilde{\Psi}(u(t)) \equiv i(\triangle - U)u(t)$$

has a unique solution. Moreover the semigroup $\{T(t)\}$ of solution family is unitary.

Proof. Since $\partial\tilde{\Psi}\equiv -(\triangle - U)$ is self-adjoint, $-i\partial\tilde{\Psi}\equiv i(\triangle - U)$ generates a semigroup of unitary operators by virtue of Stone's Theorem. □

Definition 8.16. Let A be the linear operator in complex Hilbert space $H=(H,\|\cdot\|)$.

(a) The operator A is called monotone if and only if $\mathrm{Re}(x,Ax)\geq 0$ for all $x\in D(A)$.

(b) The operator A is called maximal monotone if and only if any monotone extension of A coincides with A.

Lemma 8.17. *Let $-A$ be a maximal monotone operator. Then $\|A(I-A)^{-1}\|\leq 1$.*

Lemma 8.18. *Let $-A$ and $-A_n$ be maximal monotone operators. Then $(I-(1+\alpha)A)^{-1}$ and $(I-(1+\alpha)A_n)^{-1}$ are bounded operators for $|\alpha|<1$. Moreover if $\lim_{n\to\infty}(I-A_n)^{-1}\varphi=(I-A)^{-1}\varphi$ for all $\varphi\in H$, then we have $\lim_{n\to\infty}(I-(1+\alpha)A_n)^{-1}\varphi=(I-(1+\alpha)A)^{-1}\varphi$ for all $\varphi\in H$.*

Lemma 8.19. *Let $-A$ and $-A_n$ be self-adjoint positive operators. If $(I-e^{i\theta}A)^{-1}$ and $(I-e^{i\theta}A_n)^{-1}$ are bounded operators for $0\leq\theta\leq\pi/2$ and*

$$\lim_{n\to\infty}(I-A_n)^{-1}\varphi=(I-A)^{-1}\varphi,\quad \text{for all}\quad \varphi\in H,$$

then we obtain tnat $\lim_{n\to\infty}(I-iA_n)^{-1}\varphi=(I-iA)^{-1}\varphi$ for all $\varphi\in H$.

Remark. $(-i+c)A$ and $(-i+c)iA_n$ are not maximal monotone operators for any $c>0$.

Proposition 8.20. *Let $T(t)$ and $T_n(t)$ be the semigroups generated by infinitesimal generator $-i\partial\tilde{\Psi}$ and $-i\partial\tilde{\Psi}_n$ respectively. Then it follows that*

$$\lim_{n\to\infty}T_n(t)\phi=T(t)\phi\quad \text{for all}\quad \phi\in L^2(\mathbb{R}^N;\mathbb{C}).$$

Proof of Theorem 8.6. From Proposition 8.20, it follows that

$$\lim_{n\to\infty}R\text{-}\int_{\Omega_{[0,t]}} e^{-i\int_0^t U_n(\gamma(s))ds}\phi d\mu^Q = \lim_{n\to\infty}T_n(t)\phi$$

uniquely exists. Therefore we obtain that $e^{-i\int_0^t U(\gamma(s))ds}$ is Riemann integrable by μ^Q. □

Corollary 8.21. *Let a function $U\in C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R})$ and there exist $m\in\mathbb{R}$ such that $U(x)\geq m$ for any $x\in\mathbb{R}^N\backslash\mathcal{N}$. Then the function $F(-iU;t,\gamma)=e^{-i\int_0^t U(\gamma(s))ds}$ is improper Riemann integrable by μ^Q.*

For a time dependent case we give the following theorem

Theorem 8.22. *Let $U(t,\cdot)$ be a $C(\mathbb{R}^N;\mathbb{R})\cap L^\infty(\mathbb{R}^N;\mathbb{R})$ -valued function and be continuous in t on every compact set $\in \mathbb{R}$. Then the function $F(-iU;t,\gamma) = e^{-i\int_0^t U(s,\gamma(s))ds}$ is Riemann integrable by μ^Q.*

§ 8.3. Weak Integrability for Real Potentials with Singularity

In this section we study about more general potentials. We consider the following equation :

$$\frac{\partial}{\partial t}u(t,x) = i\triangle u(t,x) - iU(x)u(t,x), \quad u(0,x) = \varphi(x),\ \varphi \in H^{(2)}(\mathbb{R}^N;\mathbb{C}), \tag{8.5}$$

where $H^{(2)}(\mathbb{R}^N;\mathbb{C}) \equiv \{\varphi \in L^2(\mathbb{R}^N;\mathbb{C}) \mid \partial^2\varphi \in L^2(\mathbb{R}^N;\mathbb{C})\}$. Recall that we set $\mathcal{N}$ = a fixed closed subset of $\mathbb{R}^N$ of measure 0. Let $\mathcal{D} = \{D\}$ be the maximum family such that each element $D \subset \overline{D} \subset \mathbb{R}^N\backslash\mathcal{N}$ is a finite union of connected bounded open sets. The family $\mathcal{D} = \{D\}$ satisfies $\bigcup_{D\in\mathcal{D}} D = \mathbb{R}^N\backslash\mathcal{N}$. We denote the restriction of f to D by $f|_D$, or simply, by f_D. We use the following notation

$$L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R}) = \{f \mid f(x) \in \mathbb{R},\ f|_D \in L^\infty(D;\mathbb{R})), \forall D \in \mathcal{D}\}. \tag{8.6}$$

Let U in $L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R})$. We assume for any neighbourhood of any point of $\mathcal{N}$, U is not essentially bounded.

Definition 8.23. For a function U in $C(\mathbb{R}^n\backslash\mathcal{N};\mathbb{R})$, the function $e^{-i\int_0^t U(\gamma(s))ds}$ is said to be weakly Riemann integrable by μ^Q if

$$w-\lim_{m,n\to\infty} R\text{-}\int_{\Omega_{[0,t]}(D_{m,n})} e^{-i\int_0^t U_{m,n}(\gamma(s))ds}\phi d\mu^Q$$

exists for any ϕ in $L^2(\mathbb{R}^N;\mathbb{C})$ independently of the choice of $\{D_{m,n}\}$.

Now we return to (8.5). Let U in $C(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R})$. In order to use the previous theorem we define a sequence of functions $U_{m,n}$ and $D_{m,n}$ such that $U_{m,n}(x) = min\{m,\ max\{-n,U(x)\}\}$, $D_{m,n} = \{x \in \mathbb{R}^N | m > U(x) > -n\}, m,n = 1,2,3,\cdots$. By virtue of Corollary 8.21 the solution $u_{m,n}$ to the Schrödinger equation in $\mathbb{R}^N$

$$\begin{cases} \frac{\partial}{\partial t}u_{m,n}(t,x) = i\triangle u_{m,n}(t,x) - iU_{m,n}(x)u_{m,n}(t,x), \\ u_{m,n}(0,x) = \varphi(x),\ \varphi \in L^2(\mathbb{R}^N;\mathbb{C}) \end{cases} \tag{8.7}$$

exists.

Theorem 8.24. *For any U in $L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R})$, there exists a closed extension of the operator $i(\Delta - U)|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N})}$ in $L^2(\mathbb{R}^N;\mathbb{C}) \to L^2(\mathbb{R}^N;\mathbb{C})$ which generates a contraction C_0-semigroup $\{T(t) \mid t \geq 0\}$ such that*

$$T(t)\varphi = w\text{-}\lim_{n\to\infty} T_{m,n}(t)\varphi, \quad \forall\varphi \in L^2(\mathbb{R}^N;\mathbb{C}), \tag{8.8}$$

where $T_{m,n}(t)\varphi$ is the solution to

$$\frac{d}{dt}u_{m,n}(t) = A_{m,n}u_{m,n}(t), \qquad where \quad A_{m,n} = i(\Delta - U_{m,n}) \tag{8.9}$$

and w-lim *means the weak convergence.*

Proof. Let $U^+_{m,n}(x) = \max\{0, U_{m,n}(x)\}$ and $U^-_{m,n}(x) = \max\{0, -U_{m,n}(x)\}$. Then

$$U_{m,n}(x) = U^+_{m,n}(x) - U^-_{m,n}(x).$$

Note that: let $m, n \in \mathbb{N}$,
(A) In the case that there exists $M \geq 0$ such that $U^+_{m,n}(x) \leq M$ for $x \in D_{m,n}$ and there exists $n_0 \in \mathbb{N}$ such that $D_{m,n} \subset B(n)$ for any $n \geq n_0 \geq M$.
(B) In the case that there exists $M \geq 0$ such that $U^-_{m,n}(x) \leq M$ for $x \in D_{m,n}$ and there exists $m_0 \in \mathbb{N}$ such that $D_{m,n} \subset B(m)$ for any $m \geq m_0 \geq M$.
(C) Other case we obtain that $\max\{B(n), B(m)\} \supset D_{m,n} \supset \min\{B(n), B(m)\}$.
Note that $D = \bigcup_{n,m=1}^{\infty} D_{m,n}$. Therefore from the result of Theorem 4.1 we obtain the consequence.

Note that $\{T_t\}$ is independent of the choice of $\{D_{m,n}\}$. □

We conclude this section with a condition for $F(-iU; t, \gamma)$ to be weakly Riemann integrable.

Theorem 8.25. *Let the associated scalar function $G(x) = e^{-iU(x)}$ is Riemann integrable on any bounded domain in $\mathbb{R}^N$. Then the function $F(-iU; t, \gamma) = e^{-i\int U(\gamma(s))ds}$ is weakly Riemann integrable.*

Corollary 8.26. *Let U be continuous and real valued function on the complement of $\mathcal{N}$. Then the function $F(-iU; t, \gamma) = e^{-i\int_0^t U(\gamma(s))ds}$ is weakly Riemann integrable.*

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