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# On Laplace transforms of certain probability densities

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## 1 Introduction

A probability distribution function $F(x)$ is called an infinitely divisible probability distribution if for each integer $n>1$ there is a probability distribution $F_n(x)$ such that the following relation holds,

$$F(x)=(F_n*\cdots*F_n)(x),$$

where * denotes the convolution. If a probability distribution function $F(x)$ is concentrated on the interval $[0, \infty)$ and an infinitely divisible probability distribution, and if we set

$$\eta(s)=\int_0^{\infty}e^{-sx}dF(x), \quad \eta_n(s)=\int_0^{\infty}e^{-sx}dF_n(x),$$

the following relation

$$\eta(s)=(\eta_n(s))^n$$

holds. It is known that the Laplace-Stieltjes transform of an infinitely divisible probability distribution $F(x)$ which is concentrated on the interval $[0, \infty)$ can be written as follows:

$$\eta(s)=\exp\{-ds+\int_{+0}^{\infty}(e^{-sx}-1)\frac{1}{x}dK(x)\}$$

where

(c1) $K(x)$ is nondecreasing,

(c2) $K(-0) = 0$,

(c3) $\int_1^\infty 1/x \, dK(x) < \infty$.

Here, let us assume $d = 0$ in what follows. If an infinitely divisible probability distribution $F(x)$ which is concentrated on the interval $[0, \infty)$ and if the probability distribution function $F(x)$ has a density function $f(x)$, the density funcion $f(x)$ satisfies the following integral equation:

$$xf(x) = \int_0^x f(x-t)dK(t), \; x > 0.$$

If $dK(t) = k(t)dt$ we have

$$xf(x) = \int_{(0,x)} f(x-t)k(t)dt, \quad x > 0.$$

We will discuss about the Student $t$ distribution. The density function of the Student $t$ distribution with degrees of freedom $r$ is as follows:

$$T(t) = \frac{\Gamma((r+1)/2))}{\sqrt{\pi r}\,\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}}$$

If $r$ is an odd integer, $r = 2n+1$ for a nonnegative integer $n$ and if we make a change of variable, $t/\sqrt{r} = x$, we have the density function

$$\frac{\Gamma(n+1)}{\sqrt{\pi}\,\Gamma(n+1/2)} \frac{1}{(1+x^2)^{n+1}}.$$

The purpose of this note is to show that we can prove the infinite divisibility of the $t$ distribution with the odd degrees of freedom $2n+1$ without making use of the Bessel functons (cf. [3]). We will make use of the fact that if $h$ tends to $+0$ the density function of the Student $t$ distribution can be obtained by the following relation

$$\begin{aligned} f(x;1,h) &= \frac{c}{(1+x^2)((1+h)^2+x^2)\cdots((1+nh)^2+x^2)} \\ &\to \frac{c_0}{(1+x^2)^{n+1}}, \end{aligned}$$

where $c$ and $c_0$ are normalised constants.

# 2 The hypergeometric function

Let $a$ be a positive constant. In what follows, suppose that $a_1 = a$, $a_2 = a + h$, ..., $a_{n+1} = a + nh$. Let us consider the following density function

$$f(x; a, h) = \frac{c}{\Pi_{j=1}^{n+1}(a_j^2 + x^2)} \tag{1}$$

where $c$ is a normalized constant. It holds that

$$f(x; a, h) = c \sum_{j=1}^{n+1} \frac{1}{\Pi_{l=1, l \neq j}^{n+1}(-a_j^2 + a_l^2)(a_j^2 + x^2)}. \tag{2}$$

From the relation

$$\begin{aligned} \frac{1}{a_j^2 + x^2} &= \int_0^\infty e^{-t(a_j^2 + x^2)} dt \\ &= \int_0^\infty \frac{1}{\sqrt{\pi v}} e^{-x^2/v} \sqrt{\pi}\, e^{-a_j^2/v} v^{-3/2} dv \end{aligned}$$

we obtain the following equality

$$\begin{aligned} f(x; a, h) = \int_0^\infty \frac{1}{\sqrt{\pi v}} e^{-x^2/v} \\ \sum_{j=1}^{n+1} \frac{c\sqrt{\pi}}{\Pi_{l=1, l \neq j}^{n+1}(-a_j^2 + a_l^2)}\, e^{-a_j^2/v} v^{-3/2}\, dv. \end{aligned} \tag{3}$$

Let us denote the mixing density function in the integrand of (3) by $g(v)$. The mixing density $g(v)$ is positive on $[0, \infty)$ and a probability density function. We take the Laplace transform of $g(v)$. Since it holds that

$$\int_0^\infty e^{-sv}\, e^{-a_j^2/v} v^{-3/2} dv = \frac{\sqrt{\pi}}{a_j} e^{-2a_j\sqrt{s}}$$

we obtain

$$\begin{aligned} \eta(s) &= c\sqrt{\pi} \\ &\sum_{j=1}^{n+1} \frac{1}{\Pi_{l=1, l \neq j}^{n+1}(-a_j^2 + a_l^2)} \int_0^\infty e^{-sv}\, e^{-a_j^2/v} v^{-3/2} dv \\ &= c\pi \sum_{j=1}^{n+1} \frac{1}{a_j \Pi_{l=1, l \neq j}^{n+1}(-a_j^2 + a_l^2)} e^{-2a_j\sqrt{s}}. \end{aligned} \tag{4}$$

For $n = 3$ we obtain

$$\eta(s) = \frac{c\pi}{a3!h^3(2a+h)(2a+2h)(2a+3h)} e^{-2a\sqrt{s}}$$
$$\cdot\Big(1 + \frac{(-3)(2m)z}{(2m+4)} + \frac{(-3)(-2)(2m)(2m+1)z^2}{(2m+4)(2m+5)2!}$$
$$+ \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)z^3}{(2m+4)(2m+5)(2m+5)3!}\Big). \tag{5}$$

Making use of hypergeometric function we obtain the simple expression

$$\eta(s) = \frac{2c\pi}{n!h^{2n+1}(2m)_{n+1}} z^m F(-n, 2m; 2m+n+1; z) \tag{6}$$

where we let $z = e^{-2h\sqrt{s}}$ and $m = a/h$. Concerning the roots of the hypergeometric function $F(-n, 2m; 2m+n+1; z)$ the author obtained the following result (cf.[11]).

**Theorem 1.** *If $m$ is a positive constant and $n$ is a natural number the hypergeometric function $F(-n, 2m; 2m+n+1; z)$ has roots outside the unit disk.*

# 3 The Student $t$ distributions

We show that the probability distribution with density function (2) is infinitely divisible and obtain the Lévy measure of the Student $t$ distribution from the Lévy mesure of the distribution with the density functon (2).

**Theorem 2.** *The probability distribution with density function (2) is infinitely divisible for each positive numbers $a$, $h$ and every positive integer $n$.*

*Proof.* Let us show that the density function $g(v)$ is an infinitely divisible density for every positive integer $n$. To show the infinite divisibility of the distribution with $g(v)$, it suffices to show that if $dK(x) = k(x)dx$ the following relation

$$-\eta'(s) = \eta(s) \int_0^\infty e^{-sx} k(x) dx$$

holds and $k(x)$ is a nonnegative function and satisfies the conditions (c1), (c2), (c3) imposed on an infinitely divisible probability distribution. From (6) we have

$$\eta(s) = \frac{2c\pi}{n!h^{2n+1}(2m)_{n+1}} z^m \sum_{j=0}^{n} \frac{(-n)_j (2m)_j}{(2m+n+1)_j j!} z^j, \tag{7}$$

where we set $z = e^{-2h\sqrt{s}}$ and $m = a/h$. From this we obtain

$$\eta'(s) = -\frac{2c\pi h}{n!h^{2n+1}(2m)_{n+1}\sqrt{s}} \sum_{j=0}^{n} \frac{(-n)_j(2m)_j(m+j)}{(2m+n+1)_j j!} z^{m+j} \tag{8}$$

and hence

$$-\frac{\eta'(s)}{\eta(s)} = \frac{h}{\sqrt{s}} \Big( \sum_{j=0}^{n} \frac{(-n)_j(2m)_j(m+j)}{(2m+n+1)_j j!} z^{m+j} \Big) / \Big( \sum_{j=0}^{n} \frac{(-n)_j(2m)_j(m+j)}{(2m+n+1)_j j!} z^{m+j} \Big). \tag{9}$$

If we set $z = e^{-2h\sqrt{s}}$ and $\Re\{\sqrt{s}\,\} \geq 0$, then $|z| \leq 1$. We note that

$$F(-n, 2m; 2m+n+1; z) \neq 0.$$

The denominator of (9) does not vanish in the whole complex plane except at the origin. By the contour integration of the figure after the reference we can calculate the inverse Laplace transform of the following formula

$$k(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\xi - iR_1}^{\xi + iR_1} e^{ts}(-1)\frac{\eta'(s)}{\eta(s)} ds,$$

$$(\xi > 0,\ t > 0,\ R_1 = R\cos\epsilon).$$

Let

$$D = \sqrt{s} \sum_{j=0}^{3} \frac{(-3)_j(2m)_j z^j}{(2m+4)_j j!},$$

$$N = h \sum_{j=0}^{3} \frac{(-3)_j(2m)_j(m+j) z^j}{(2m+4)_j j!}.$$

(A) The integral along a small circle with the center at O.

From $s = re^{i\theta}$, $\sqrt{s} = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$ for $-\pi < \theta < \pi$, we see that

$$\oint e^{st}\frac{N}{D}ds = -\int_{-\pi}^{\pi} e^{re^{i\theta}t}$$

$$\left[\left\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{r}e^{i\theta/2}}}{(2m+4)_j j!}\right\}\right.$$

$$\left./\left\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{r}e^{i\theta/2}}}{(2m+4)_j j!}\right\}\right]\sqrt{r}e^{i\theta/2}id\theta.$$

Since it holds that for every $0 < r \leq 1$ and $0 \leq \theta \leq \pi$

$$F(-1, 2m; 2m+2; e^{-2\sqrt{r}e^{i\theta/2}}) \neq 0$$

we have

$$\oint e^{st}\frac{N}{D}ds \to 0 \text{ as } r \to +0.$$

(B) The integral along $B \frown D$.
From $s = Re^{i\theta}$we have $\sqrt{s} = \sqrt{R}(\cos\theta/2 + i\sin\theta/2)$ and we see that

$$\int_{B\frown D} e^{st}\frac{N}{D}ds = \int_{\frac{\pi}{2}-\epsilon}^{\pi} e^{Re^{i\theta}t}$$

$$\left[\left\{h\sum_{j=0}^{3}(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{R}e^{i\theta/2}}\right\}\right.$$

$$\left./\left\{\sqrt{R}e^{i\theta/2}\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{R}e^{i\theta/2}}}{(2m+4)_j j!}\right\}\right]Re^{i\theta}id\theta$$

$$= i\int_{\frac{\pi}{2}}^{\pi} e^{Re^{i\theta}t}\left[\left\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{R}e^{i\theta/2}}}{(2m+4)_j j!}\right\}\right.$$

$$\left./\left\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{R}e^{i\theta/2}}}{(2m+4)_j j!}\right\}\right]\sqrt{R}e^{i\theta/2}d\theta$$

$$+i\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} e^{Re^{i\theta}t}\left[\left\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{R}e^{i\theta/2}}}{(2m+4)_j j!}\right\}\right.$$

$$\left./\left\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{R}e^{i\theta/2}}}{(2m+4)_j j!}\right\}\right]\sqrt{R}e^{i\theta/2}d\theta. \qquad (10)$$

We see that

$$\int_{\frac{\pi}{2}}^{\pi} \sqrt{R}|e^{i\theta/2}e^{Re^{i\theta}t}|d\theta = \int_{\frac{\pi}{2}}^{\pi} \sqrt{R}e^{tR\cos\theta}d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{R}e^{-tR\sin\phi}d\phi$$

$$\leq \int_0^{\frac{\pi}{2}} \sqrt{R}e^{-2tR\phi/\pi}d\phi = \sqrt{R}[-\frac{\pi}{2tR}e^{-2tR\phi/\pi}]_0^{\frac{\pi}{2}}$$

$$= \sqrt{R}\{\frac{\pi}{2tR}(-e^{-tR}+1)\} \to 0 \tag{11}$$

as $R \to +\infty$. We show that

$$\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} |\sqrt{R}e^{i\theta/2}e^{Re^{i\theta}t}|d\theta \to 0 \tag{12}$$

as $R \to \infty$. From the fact that

$$\cos\theta = \cos(\phi + \frac{\pi}{2} - \epsilon) = \sin\phi, \quad 0 \leq \phi \leq \epsilon,$$

$$\sin\epsilon = \frac{\xi}{R} \geq \sin\phi \geq 0,$$

we see that

$$\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}|e^{i\theta/2}e^{Re^{i\theta}t}|d\theta = \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}e^{tR\cos\theta}d\theta$$

$$= \int_0^{\epsilon} \sqrt{R}e^{tR\sin\phi}d\phi \leq \int_0^{\epsilon} \sqrt{R}e^{tR\xi/R}d\phi = \sqrt{R}e^{t\xi}\epsilon$$

$$= e^{t\xi}(\sqrt{R}\sin\epsilon)\frac{\epsilon}{\sin\epsilon} = e^{t\xi}(\sqrt{R}\frac{\xi}{R})\frac{\epsilon}{\sin\epsilon} \to 0 \tag{13}$$

as $R \to \infty$.

(C) The integrals along $D \to G$ and $H \to E$.

From $s = \rho e^{i\pi}$, $r \leq \rho \leq R$ on $D \to G$ and from $\sqrt{s} = \sqrt{\rho}e^{i\pi/2} = i\sqrt{\rho}$, we see that

$$\int_{D\to G} e^{st}\frac{N}{D}ds$$

$$= \int_{D\to G} e^{\rho e^{i\pi}t}\Big[\Big\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{\rho}e^{i\pi/2}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sqrt{\rho}e^{i\pi/2}\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{\rho}e^{i\pi/2}}}{(2m+4)_j j!}\Big\}\Big]e^{i\pi}d\rho$$

$$= -\int_{R}^{r} e^{-\rho t}\Big[\Big\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}i}$$

$$= \int_{r}^{R} e^{-\rho t}\Big[\Big\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}i}.$$

From $s = \rho e^{-i\pi} = -\rho,\;\; r \le\; \rho \le R$ on $H \to E$ and from $\sqrt{s} = -i\sqrt{\rho}$ we see that

$$\int_{H\to E} e^{st}\frac{N}{D}ds$$

$$= \int_{H\to E} e^{\rho e^{-i\pi}t}\Big[\Big\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{\rho}e^{-i\pi/2}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sqrt{\rho}e^{-i\pi/2}\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{\rho}e^{-i\pi/2}}}{(2m+4)_j j!}\Big\}\Big]e^{-i\pi}d\rho$$

$$= -\int_R^r e^{-\rho t}\Big[\Big\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}\Big]\frac{d\rho}{\sqrt{\rho i}}$$

$$= \int_r^R e^{-\rho t}\Big[\Big\{h\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}\Big]\frac{d\rho}{\sqrt{\rho i}}. \tag{14}$$

Therefore we see that

$$\frac{1}{2\pi i}\int_r^R e^{-\rho t}\Big[\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$+\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}\Big]\frac{h d\rho}{\sqrt{\rho i}}$$

$$\rightarrow -\frac{1}{2\pi}\int_0^\infty e^{-\rho t}\Big[\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$+\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{+j2h\sqrt{\rho i}}}{(2m+4)_j j!}\Big\}\Big]\frac{h d\rho}{\sqrt{\rho}}$$

as $r \to 0$ and $R \to \infty$. From the Cauchy theorem we see that

$$\frac{1}{2\pi i}\int_{A\to B} e^{st}\frac{N}{D}ds = \frac{1}{2\pi i}\int_{\xi-iR_1}^{\xi+iR_1} e^{st}\frac{N}{D}ds$$

$$\to \frac{1}{2\pi}\int_0^\infty e^{-\rho t}\Big[\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}$$

$$+\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{+j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{+j2h\sqrt{\rho}i}}{(2m+4)_j j!}\Big\}\Big]\frac{hd\rho}{\sqrt{\rho}} \tag{15}$$

as $R \to \infty$. By change of variable, $\sqrt{\rho} = y$, we obtain

$$k(t) = \frac{1}{\pi}\int_0^\infty e^{-ty^2}\Big[\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{-j2hyi}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{-j2hyi}}{(2m+4)_j j!}\Big\}$$

$$+\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j(m+j)e^{+j2hyi}}{(2m+4)_j j!}\Big\}$$

$$/\Big\{\sum_{j=0}^{3}\frac{(-3)_j(2m)_j e^{+j2hyi}}{(2m+4)_j j!}\Big\}\Big]hdy.$$

For the general case $n$ we obtain

$$k(t) = \frac{1}{\pi}\int_0^\infty e^{-ty^2}\Big[\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_j(m+j)e^{-j2hyi}}{(2m+n+1)_j}\Big\}$$

$$/\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_j e^{-j2hyi}}{(2m+n+1)_j}\Big\}$$

$$+\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_j(m+j)e^{+j2hyi}}{(2m+n+1)_j}\Big\}$$

$$/\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_j e^{+j2hyi}}{(2m+n+1)_j}\Big\}\Big]hdy. \tag{16}$$

To show the infinite divisibility it is necessary to show that the following function

$$\Re\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_j(m+j)e^{-j2hyi}}{(2m+n+1)_j}\Big\}$$
$$\cdot\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_je^{+j2hyi}}{(2m+n+1)_j}\Big\}$$

is nonnegative for $y \geq 0$. Let $2hy = \theta$ in the above and let

$$\begin{aligned} A &= \Re\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_j(m+j)e^{-i(m+j)\theta}}{(2m+n+1)_j}\Big\} \\ &\quad\cdot\Big\{\sum_{j=0}^{n}\frac{(-n)_j(2m)_je^{+i(m+j)\theta}}{(2m+n+1)_j}\Big\}. \end{aligned}$$

If $n = 0$ we obtain

$$A = (1/m^2)(\cos\theta\cdot m\cos\theta + m\sin\theta\cdot\sin\theta) = 1/m.$$

If $n = 1$ we obtain

$$A = \frac{2m(2m+1)}{2m+2}(1-\cos\theta).$$

For the general $n$ we obtain

$$A = \frac{2^{n-1}(2m)_{n+1}}{(2m+n+1)_n}(1-\cos\theta)^n.$$

Let

$$\begin{aligned} B &= \Big|\sum_{j=0}^{n}\frac{(-n)_j(2m)_je^{i(m+j)\theta}}{(2m+n+1)_jj!}\Big|^2 \\ &= |F(-n,2m;2m+n+1;e^{i\theta})|^2. \end{aligned} \tag{17}$$

From the fact that $A$ is nonnegative for $\theta = 2hy \geq 0$ we see that the function

$$k(t) = \frac{1}{\pi}\int_0^\infty e^{-ty^2}\frac{2A}{B}hdy \tag{18}$$

is positive for $t>0$ and we obtain

$$\frac{2A}{B}=\frac{2^n(2m)_{n+1}}{(2m+n+1)_n}\frac{(1-\cos\theta)^n}{|F(-n,2m;2m+n+1;e^{i\theta})|^2}$$
$$=2^n(2m)_{n+1}(1-\cos 2hy)^n/$$
$$\{\sum_{j=0}^{n}\frac{(2m)_j}{(2m+n+1)_{n-j}}\binom{n}{j}\binom{2n-j}{n}$$
$$(2(n-j))!2^j(1-\cos 2hy)^j\}. \tag{19}$$

After all, by change of variable, $y=\sqrt{w}$, we obtain

$$k(t)=\int_0^\infty e^{-tw}\Big(2^{n-1}(2m)_{n+1}(1-\cos 2h\sqrt{w})^n h\Big)dw$$
$$\Big/\Big(\pi\sum_{j=0}^{n}\frac{(2m)_j\,2^j\sqrt{w}}{(2m+n+1)_{n-j}}\binom{n}{j}\binom{2n-j}{n}$$
$$(2(n-j))!(1-\cos 2h\sqrt{w})^j\Big)$$

and therefore

$$k(t)=\int_0^\infty e^{-tw}\Big(2^{2n-1}(2m)_{n+1}(\sin h\sqrt{w})^{2n}h\Big)dw$$
$$\Big/\Big(\pi\sum_{j=0}^{n}\frac{(2m)_j\,2^{2j}\sqrt{w}}{(2m+n+1)_{n-j}}\binom{n}{j}\binom{2n-j}{n}$$
$$(2(n-j))!(\sin h\sqrt{w})^{2j}\Big). \tag{20}$$

We can show that $k(t)$ satisfies the conditions (c1), (c2) and (c3). Therefore the density function $g(v)$ is an infinitely divisible density, and the probability distribution with the density function (2) is infinitely divisible since it is a mixture density of the normal distributions. □

Let us denote the characteristic fuction of the probability distribution with the density function (2) in the following form

$$\phi(t)=\exp\Big[\int_{R-\{0\}}\Big(e^{itx}-1-\frac{itx}{1+x^2}\Big)\frac{l(x)}{x}dx\Big].$$

In what follows we will obtain the measure $l(x)dx/x$. We have

$$\begin{aligned}\phi(t) &= \int_{-\infty}^{+\infty}e^{itx}\Big(\int_0^\infty\frac{1}{\sqrt{\pi v}}e^{-x^2/v}g(v)dv\Big)dx\\ &= \int_0^\infty e^{-vt^2/4}g(v)dv\end{aligned}$$

and

$$\log \phi(t) = \int_{+0}^{+\infty} (e^{-sx} - 1)\frac{k(x)}{x} dx$$
$$= \int_0^\infty (e^{-sx} - 1)\frac{1}{x}\Big(\int_{+0}^\infty e^{-xw} U(w)dw\Big)dx$$
$$= -\int_0^\infty \log(1 + \frac{t^2}{4w})U(w)dw \tag{21}$$

where we set $s = t^2/4$ and

$$U(w) = \Big(2^{2n-1}(2m)_{n+1}(\sin h\sqrt{w})^{2n}h\Big)$$
$$\Big/\Big(\pi \sum_{j=0}^{n} \frac{(2m)_j\, 2^{2j}\sqrt{w}}{(2m+n+1)_{n-j}} \binom{n}{j}\binom{2n-j}{n}$$
$$(2(n-j))!(\sin h\sqrt{w})^{2j}\Big).$$

By using the following equality

$$-\log\,(1 + \frac{t^2}{4w}) = \int_{R-\{0\}} e^{-2\sqrt{w}|u|}\Big(e^{itu} - 1 - \frac{itu}{1+u^2}\Big)\frac{du}{|u|}$$

we obtain

$$\phi(t)$$
$$= \exp\Big[\int_0^\infty \Big\{\int_{R-\{0\}} e^{-2\sqrt{w}|u|}\Big(e^{itu} - 1 - \frac{itu}{1+u^2}\Big)\frac{du}{|u|}\Big\}U(w)dw\Big]$$
$$= \exp\Big[\int_{R-\{0\}} \Big(e^{itu} - 1 - \frac{itu}{1+u^2}\Big)\frac{1}{|u|}\Big(\int_0^\infty e^{-2\sqrt{w}|u|}U(w)dw\Big)du\Big].$$

We see that the function $l(x)$ can be given in the following form

$$l(x) = (sgn\ x)\int_0^\infty e^{-2\sqrt{w}\,|x|}U(w)dw$$
$$= (sgn\ x)\int_0^\infty e^{-|x|\,v}\, 2^{2n-1}$$
$$\Big[(2m)_{n+1}h(\sin(hv/2))^{2n}\Big/$$
$$\Big(\pi \sum_{j=0}^{n} \frac{(2m)_j\, 2^{2j}}{(2m+n+1)_{n-j}} \binom{n}{j}\binom{2n-j}{n}$$
$$(2(n-j))!(\sin(hv/2))^{2j}\Big)\Big]dv. \tag{22}$$

Let us denote the characteristic function of the Student $t$ distribution with odd degrees of freedom in the following form

$$\phi(t) = \exp\Big[\int_{R-\{0\}}\Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big)\frac{l_{st}(x)}{x}dx\Big].$$

**Theorem 3.** *The function $l_{st}(x)$ can be given in the explicit form*

$$l_{st}(x) = (sgn\ x)\int_0^\infty e^{-|x|v}$$
$$(2a)^{2n+1}v^{2n}dv/\{2\pi\sum_{j=0}^{n}(2a)^{2j}\binom{n}{j}\binom{2n-j}{n}(2(n-j))!v^{2j}\}. \quad (23)$$

*We take $a = 1$ for the Student t distribution.*

*Proof.* By (22) and $hm = a$ we see that

$$l(x) = (sgn\ x)\int_0^\infty e^{-|x|\,v}$$
$$\Big[2^{2n-1}(2m)_{n+1}h\Big((\sin(hv/2))/(hv/2)\Big)^{2n}$$
$$(hv/2)^{2n}/\Big\{\pi\sum_{j=0}^{n}\frac{(2m)_j\ 2^{2j}}{(2m+n+1)_{n-j}}\binom{n}{j}\binom{2n-j}{n}$$
$$(2(n-j))!\Big((\sin(hv/2))/(hv/2)\Big)^{2j}(hv/2)^{2j}\Big\}\Big]dv \quad (24)$$

From the above we see that

$$l_{st}(x) = (sgn\ x)\int_0^\infty e^{-|x|\,v}\Big[(2a)^{2n+1}v^{2n}/$$
$$\Big(2\pi\sum_{j=0}^{n}(2a)^{2j}\binom{n}{j}\binom{2n-j}{n}(2(n-j))!v^{2j}\Big)\Big]dv$$

as $h$ tends $+0$ and we obtain (23). □

If $a = 1$ we have

$$l_{st}(x) = (sgn\ x)\int_0^\infty e^{-|x|y}\Big[2(2y)^{2n}/$$
$$\Big\{2\pi\sum_{j=0}^{n}\binom{n}{j}\binom{2n-j}{n}(2(n-j))!(2y)^{2j}\Big\}\Big]dy.$$

(25)

In order to show that the results here coincide with those formulae of which have been already obtained we write down the several cases (cf. [1]).
If $n = 1$

$$l_{st}(x) = (sgn\ x) \int_0^\infty e^{-|x|y} \frac{y^2}{\pi(1+y^2)} dy.$$

If $n = 2$

$$l_{st}(x) = (sgn\ x) \int_0^\infty e^{-|x|y} \frac{y^4}{\pi(3^2+3y^2+y^4)} dy.$$

If $n = 3$

$$l_{st}(x) = (sgn\ x) \int_0^\infty e^{-|x|y} \frac{y^6}{\pi(225+45y^2+6y^4+y^6)} dy.$$

If $n = 4$

$$l_{st}(x) = (sgn\ x) \int_0^\infty e^{-|x|y}$$
$$\frac{y^8}{\pi(11025+1575y^2+135y^4+10y^6+y^8)} dy. \tag{26}$$

From the above we see that the function $l_{st}(x)$ can be decomposed to the two terms and we can obtain the convolutional decomposition.
If $n = 1$

$$\begin{aligned} l_{st}(x) &= \frac{1}{\pi x} - (sgn\ x) \int_0^\infty e^{-|x|y} \frac{1}{\pi(1+y^2)} dy \\ &= \frac{1}{\pi x} - \frac{sgn\ x}{\pi} \left[ \cos|x| \int_{|x|}^{\to\infty} \frac{\sin y}{y} dy - \sin|x| \int_{|x|}^{\to\infty} \frac{\cos y}{y} dy \right], \\ &(x \neq 0). \end{aligned} \tag{27}$$

If $n = 2$

$$l_{st}(x) = \frac{1}{\pi x} - (sgn\ x) \int_0^\infty e^{-|x|y} \frac{3^2+3y^2}{\pi(3^2+3y^2+y^4)} dy.$$

If $n = 3$

$$\begin{aligned} l_{st}(x) &= \frac{1}{\pi x} \\ &-(sgn\ x) \int_0^\infty e^{-|x|y} \frac{225+45y^2+6y^4}{\pi(225+45y^2+6y^4+y^6)} dy. \end{aligned}$$

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