数理解析研究所講究録
第 1727 巻 2011 年 35-47

Solutions to The Nonhomogeneous Associated Laguerre's Equation by Means of N-Fractional Calculus Operator

Katsuyuki Nishimoto

Institute for Applied Mathematics, Descartes Press Co.
2 - 13 - 10 Kaguike, Koriyama 963 - 8833, JAPAN.

Abstract

In this article, solutions to the nonhomogeneous associated Laguerre's equations

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = f \qquad (z \neq 0)$$

$$(\varphi_\nu = d^\nu \varphi / dz^\nu \text{ for } \nu > 0,\ \varphi_0 = \varphi = \varphi(z),\ f = f(z))$$

are discussed by means of N-fractional calculus operator (NFCO- Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

(i) $$\varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} \equiv \varphi^{*}_{[1](\alpha,\beta)}, \quad \text{(denote)}$$

and

(ii) $$\varphi = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \equiv \varphi^{*}_{[2](\alpha,\beta)},$$

where

$$X_{[1]} = (f_\beta z^{\alpha+\beta} e^{-z})_{-1}, \qquad Y_{[1]} = e^z z^{-(\alpha+\beta+1)} .$$

Group II.

(i) $$\varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \varphi^{*}_{[3](\alpha,\beta)},$$

and

(ii) $$\varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \varphi^{*}_{[4](\alpha,\beta)},$$

where

$$X_{[2]} = ((f e^{-z})_{-(\alpha+\beta+1)} e^z z^{-(1+\beta)})_{-1}, \qquad Y_{[2]} = e^{-z} z^{\beta} .$$

§0. Introduction (Definition of Fractional Calculus) and §1.Preliminary

are omitted, then refer to the previous paper [35] .

§ 2. Solutions to The Nonhomogeneous Associated Laguerre's Equation by NFCO-Method

Theorem 1. *Let $\varphi \in F$ and $f \in F$,then nonhomogeneous associated Laguerre's equation*

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = f \qquad (z \neq 0) \tag{1}$$

$$(\varphi_\nu = d^\nu \varphi / dz^\nu \text{ for } \nu > 0 ,\ \varphi_0 = \varphi = \varphi(z) ,\ f = f(z))$$

has particular solutions of the forms (fractional differintegrated form);

Group I.

(i)
$$\varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} \equiv \overset{*}{\varphi}_{[1](\alpha,\beta)} , \quad \text{(denote)} \tag{2}$$

(ii)
$$\varphi = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \equiv \overset{*}{\varphi}_{[2](\alpha,\beta)} , \tag{3}$$

where

$$X_{[1]} = (f_\beta z^{\alpha+\beta} e^{-z})_{-1} , \qquad Y_{[1]} = e^z z^{-(\alpha+\beta+1)} . \tag{4}$$

Group II.

(i)
$$\varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \overset{*}{\varphi}_{[3](\alpha,\beta)} ; \tag{5}$$

(ii)
$$\varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \overset{*}{\varphi}_{[4](\alpha,\beta)} ; \tag{6}$$

where

$$X_{[2]} = ((f e^{-z})_{-(\alpha+\beta+1)} e^z z^{-(1+\beta)})_{-1} ; \qquad Y_{[2]} = e^{-z} z^\beta . \tag{7}$$

Group III.

(i)
$$\varphi = z^{-\alpha} (X_{[3]} \cdot Y_{[3]})_{-(1+\alpha+\beta)} \equiv \overset{*}{\varphi}_{[5](\alpha,\beta)} ; \tag{8}$$

(ii)
$$\varphi = z^{-\alpha} (Y_{[3]} \cdot X_{[3]})_{-(1+\alpha+\beta)} \equiv \overset{*}{\varphi}_{[6](\alpha,\beta)} ; \tag{9}$$

where

$$X_{[3]} = ((f z^\alpha)_{\alpha+\beta} z^\beta e^{-z})_{-1} ; \qquad Y_{[3]} = e^z z^{-(\beta+1)} . \tag{10}$$

Group IV.

(i)
$$\varphi = z^{-\alpha} e^z (X_{[4]} \cdot Y_{[4]})_\beta \equiv \overset{*}{\varphi}_{[7](\alpha,\beta)} ; \tag{11}$$

(ii)
$$\varphi = z^{-\alpha} e^z (Y_{[4]} \cdot X_{[4]})_\beta \equiv \overset{*}{\varphi}_{[8](\alpha,\beta)} ; \tag{12}$$

where

$$X_{[4]} = ((f z^\alpha e^{-z})_{-(1+\beta)} e^z z^{-(1+\alpha+\beta)})_{-1} ; \qquad Y_{[4]} = e^{-z} z^{\alpha+\beta} . \tag{13}$$

Proof of Group I.

Operate N-fractional calculus (NFC) operator N^{ν} to the both sides of equation (1), we have then

$$(\varphi_2 \cdot z)_{\nu} + (\varphi_1 \cdot (-z+\alpha+1))_{\nu} + (\varphi \cdot \beta)_{\nu} = f_{\nu} \ ; \quad (f \neq 0) . \tag{14}$$

Now we have

$$(\varphi_2 \cdot z)_{\nu} = \sum_{k=0}^{1} \frac{\Gamma(\nu+1)}{k!\,\Gamma(\nu+1-k)} (\varphi_2)_{\nu-k} (z)_k \tag{15}$$

$$= \varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot \nu \ , \tag{16}$$

$$(\varphi_1 \cdot (-z+\alpha+1))_{\nu} = \varphi_{1+\nu} \cdot (-z+\alpha+1)) \ - \varphi_{\nu} \cdot \nu \tag{17}$$

and

$$(\varphi \cdot \beta)_{\nu} = \varphi_{\nu} \cdot \beta \ , \tag{18}$$

respectively, by Lemmas (i) and (i v).

Therefore, we have

$$\varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot (-z+\alpha+1+\nu) + \varphi_{\nu} \cdot (\beta-\nu) = f_{\nu} \tag{19}$$

from (14), applyimg (16) , (17) and (18).

Choosing ν such that

$$\nu = \beta \tag{20}$$

we obtain

$$\varphi_{2+\beta} \cdot z + \varphi_{1+\beta} \cdot (-z+\alpha+\beta+1) = f_{\beta} \ . \tag{21}$$

Set

$$\varphi_{1+\beta} = \phi = \phi(z) \qquad (\varphi = \phi_{-(1+\beta)}) \ , \tag{22}$$

we have then

$$\phi_1 + \phi \cdot \left(\frac{\alpha+\beta+1}{z} - 1 \right) = f_{\beta} \cdot z^{-1} \tag{23}$$

from (21). A particular solution to this linear first order equation is given by

$$\phi = X_{[1]} Y_{[1]} \ , \tag{24}$$

Where $X_{[1]}$ and $Y_{[1]}$ are the ones shown by (4), respectively.

Therefore, we obtain

$$\varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} \equiv \overset{*}{\varphi}_{[1](\alpha,\beta)} \ , \tag{2}$$

from (24) and (22).

Inversely (24) satisfies equation (23). then (2) satisfies equation (1).

Next, changing the order

$$X_{[1]} \text{ and } Y_{[1]} \text{ in parenthesis } (\quad)_{-(1+\beta)}$$

we obtain other solution $\overset{*}{\varphi}_{[2](\alpha,\beta)}$ which is different from (2) for $-(1+\beta)\notin Z_0^+$, that is,

$$\varphi = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \equiv \overset{*}{\varphi}_{[2](\alpha,\beta)} \, . \qquad (3)$$

(Refer to Theorem D in the previous paper [35].)

Proof of Group II.

Set

$$\varphi = e^{\gamma z}\psi \qquad (\psi = \psi(z)) \, , \qquad (25)$$

we have then

$$\psi_2 \cdot z + \psi_1 \cdot \{z(2\gamma - 1) + \alpha + 1\} + \psi \cdot \{z\gamma(\gamma - 1) + \gamma(\alpha + 1) + \beta\} = f e^{-\gamma z} \qquad (26)$$

from (1).

Here we choose γ such that

$$\gamma(\gamma - 1) = 0 \, ,$$

that is,

$$\gamma = 0, \; 1 \, . \qquad (27)$$

When $\gamma = 0$, (26) is reduced to (1), therefore, we have the same solutions as Group I.

When $\gamma = 1$ we have

$$\psi_2 \cdot z + \psi_1 \cdot \{z + \alpha + 1\} + \psi \cdot (\alpha + \beta + 1) = f e^{-z} \qquad (28)$$

from (26)

Operate N^ν to the both sides of equation (28), we have then

$$(\psi_2 \cdot z)_\nu + (\psi_1 \cdot (z + \alpha + 1))_\nu + (\psi \cdot (\alpha + \beta + 1))_\nu = (f e^{-z})_\nu \, . \qquad (29)$$

Hence, using Lemma (i v), we obtain

$$\psi_{2+\nu} \cdot z + \psi_{1+\nu} \cdot (z + \alpha + 1 + \nu) + \psi_\nu \cdot (\nu + \alpha + \beta + 1) = (f e^{-z})_\nu \, . \qquad (30)$$

Choosing ν such that

$$\nu = -(\alpha + \beta + 1) \qquad (31)$$

we obtain

$$\psi_{1-(\alpha+\beta)} \cdot z + \psi_{-(\alpha+\beta)} \cdot (z - \beta) = (f e^{-z})_{-(\alpha+\beta+1)} \, . \qquad (32)$$

from (30).

Set

$$\psi_{-(\alpha+\beta)} = \phi = \phi(z) \qquad (\psi = \phi_{\alpha+\beta}) , \tag{33}$$

we have then

$$\phi_1 + \phi \cdot \left(1 - \frac{\beta}{z}\right) = (f e^{-z})_{-(\alpha+\beta+1)} \cdot z^{-1} \tag{34}$$

from (32). A particular solution to this equation is given by

$$\phi = X_{[2]} Y_{[2]} \ . \tag{35}$$

Hence we obtain

$$\psi = (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \tag{36}$$

from (35) and (33).

Therefore, we obtain

$$\varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \overset{*}{\varphi}_{[3](\alpha,\beta)} \tag{5}$$

from (25) and (36), having $\gamma = 1$.

Inversely , (35) satisfies (34), then (36) satisfies equation (28).

Hence (5) satisfies equation (1).

Next, changing the order

$X_{[2]}$ and $Y_{[2]}$ in parenthesis $(\qquad)_{\alpha+\beta}$ in (5)

we obtain other solution

$$\varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \overset{*}{\varphi}_{[4](\alpha,\beta)} \ ; \tag{6}$$

which is different from (5) for $(\alpha + \beta) \notin Z_0^+$,

Proof of Group III.

Set

$$\varphi = z^\lambda \psi \qquad (\psi = \psi(z)) , \tag{37}$$

we have then

Hence we obtain

$$\psi_2 \cdot z^{\lambda+1} + \psi_1 \cdot \{-z^{\lambda+1} + z^\lambda (2\lambda + \alpha + 1)\}$$
$$+ \psi \cdot \{z^\lambda (\beta - \lambda) + z^{\lambda-1} \lambda (\lambda + \alpha)\} = f \tag{38}$$

from (1).

Here we choose λ such that

$$\lambda(\lambda + \alpha) = 0 ,$$

that is,

$$\lambda = 0\,,\ -\alpha\ . \tag{39}$$

When $\lambda = 0$, (38) is reduced to (1), therefore, we have the same solutions as Group I.

When $\lambda = -\alpha$ we have

$$\psi_2 \cdot z + \psi_1 \cdot \{-z + 1 - \alpha\} + \psi \cdot (\alpha + \beta) = f z^{\alpha} \tag{40}$$

from (38)

Operate N^{ν} to the both sides of equation (40), we have then

$$\psi_{2+\nu} \cdot z + \psi_{1+\nu} \cdot (-z + 1 - \alpha + \nu) + \psi_{\nu} \cdot (\alpha + \beta - \nu) = (f z^{\alpha})_{\nu}\ . \tag{41}$$

Choosing ν such that

$$\nu = \alpha + \beta \tag{42}$$

we obtain

$$\psi_{2+\alpha+\beta} \cdot z + \psi_{1+\alpha+\beta} \cdot (-z + \beta + 1) = (f z^{\alpha})_{\alpha+\beta}\ . \tag{43}$$

from (41), applying (42).

Set

$$\psi_{1+\alpha+\beta} = \phi = \phi(z) \qquad (\psi = \phi_{-(1+\alpha+\beta)})\,, \tag{44}$$

we have then

$$\phi_1 + \phi \cdot \left(\frac{\beta + 1}{z} - 1 \right) = (f z^{\alpha})_{\alpha+\beta} z^{-1} \tag{45}$$

from (43).

A particular solution to this equation is given by

$$\phi = X_{[3]} Y_{[3]}\ . \tag{46}$$

Where $X_{[3]}$ and $Y_{[3]}$ are the ones given by (10).

Hence we obtain

$$\psi = (X_{[3]} Y_{[3]})_{-(1+\alpha+\beta)} \tag{47}$$

from (44) and (46).

Therefore, we obtain

$$\varphi = z^{-\alpha} (X_{[3]} Y_{[3]})_{-(1+\alpha+\beta)} \equiv \overset{*}{\varphi}_{[5](\alpha,\beta)} \tag{8}$$

from (37) and (47), having $\lambda = -\alpha$.

Inversely , (46) satisfies (equation (45), then (47) satisfies equation (43). Therefore, (8) satisfies equation (1)

Next, changing the order

$$X_{[3]} \text{ and } Y_{[3]} \text{ in parenthesis } (\quad)_{-(1+\alpha+\beta)} \text{ in } (8)$$

we obtain other solution

$$\varphi = z^{-\alpha}(Y_{[3]}X_{[3]})_{-(1+\alpha+\beta)} \equiv \varphi^{*}_{[6](\alpha,\beta)} \tag{9}$$

which is different from (8) for $-(1+\alpha+\beta) \notin Z_0^+$,

Proof of Group IV.

First set

$$\varphi = z^{\lambda}\psi \qquad (\psi = \psi(z)) , \tag{37}$$

and substitute (37) into equation (1), we have then (38).

We have then (40) from (38), having

$$\lambda = -\alpha .$$

Next set

$$\psi = e^{\delta z}\phi \qquad (\phi = \phi(z)) , \tag{48}$$

We have then

$$\phi_2 \cdot z + \phi_1 \cdot \{z(2\delta - 1) + 1 - \alpha\}$$
$$+ \phi \cdot \{z(\delta^2 - \delta) + \delta(1-\alpha) + \alpha + \beta\} = f z^{\alpha} e^{-\delta z} \tag{49}$$

from (40), applying (48).

Choose δ such that

$$\delta^2 - \delta = 0 ,$$

that is,

$$\delta = 0,\ 1 . \tag{50}$$

When $\delta = 0$, we obtain (40) from (49). Then we have the same solutions as Group III .

When $\delta = 1$ we have

$$\phi_2 \cdot z + \phi_1 \cdot (z + 1 - \alpha) + \phi \cdot (1 + \beta) = f z^{\alpha} e^{-z} \tag{51}$$

from (49).

Operate N^{ν} to the both sides of equation (51), we have then

$$\phi_{2+\nu} \cdot z + \phi_{1+\nu} \cdot (z + 1 - \alpha + \nu) + \phi_{\nu} \cdot (\nu + 1 + \beta) = (f z^{\alpha} e^{-z})_{\nu} . \tag{52}$$

Choose ν such that

$$\nu = -(1+\beta) \tag{53}$$

we obtain

$$\phi_{1-\beta} \cdot z + \phi_{-\beta} \cdot (z - \alpha - \beta) = (f z^{\alpha} e^{-z})_{-(1+\beta)} \tag{54}$$

from (52).

Therefore, setting

$$\phi_{-\beta} = u = u(z) \qquad (\phi = u_{\beta}) \, , \tag{55}$$

we have

$$u_1 + u \cdot \left(1 - \frac{\alpha + \beta}{z}\right) = (f z^{\alpha} e^{-z})_{-(1+\beta)} z^{-1} \tag{56}$$

from (54). A particular solution to this equation is given by

$$u = X_{[4]} Y_{[4]} \, , \tag{57}$$

where $X_{[4]}$ and $Y_{[4]}$ are the ones shown by (13).

Hence we obtain

$$\phi = (X_{[4]} \cdot Y_{[4]})_{\beta} \tag{58}$$

from (55) and (57).

Therefore, we have

$$\psi = e^{z} (X_{[4]} \cdot Y_{[4]})_{\beta} \tag{59}$$

from (58) and (48), having $\delta = 1$.

We have then

$$\varphi = z^{-\alpha} e^{z} (X_{[4]} \cdot Y_{[4]})_{\beta} \equiv \overset{*}{\varphi}_{[7](\alpha,\beta)} \tag{11}$$

from (59) and (37), having $\lambda = -\alpha$.

Inversely , the function shown by (57) satisfies equation (56), then (58) satisfies equation (54), and hence (59) satisfies (40).

Therefore, the function given by (11) satisfies equation (1), by (37) where $\lambda = -\alpha$.

Next, changing the order

$X_{[4]}$ and $Y_{[4]}$ in parenthesis $(\quad)_{\beta}$ in (11)

we obtain other solution

$$\varphi = z^{-\alpha} e^{z} (Y_{[4]} X_{[4]})_{\beta} \equiv \overset{*}{\varphi}_{[8](\alpha,\beta)} \tag{12}$$

which is different from (11) for $\beta \notin Z_0^+$,

§3. Some Illustrative Example

(I) Let

$$f(z)=e^z$$

we have then the nonhomogeneous Laguerre's equation

$$\varphi_2 \cdot z+\varphi_1 \cdot (-z+\alpha+1)+\varphi \cdot \beta = e^z \quad (z \neq 0) \tag{1}$$

from §2. (1)

A particular solution to this equation is given by

$$\varphi = \varphi^{*}_{[1](\alpha,\beta)} = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \tag{2}$$

$$= \left(((e^z)_\beta z^{\alpha+\beta} e^{-z})_{-1} (e^z z^{-(\alpha+\beta+1)})\right)_{-(1+\beta)} \tag{3}$$

$$= \frac{1}{\alpha+\beta+1} e^z ; \tag{4}$$

since

$$(e^z)_\beta = e^z \tag{5}$$

and

$$(z^{\alpha+\beta})_{-1} = \frac{1}{\alpha+\beta+1} z^{\alpha+\beta+1} . \tag{6}$$

Indeed we have

$$\varphi_1 = \frac{1}{\alpha+\beta+1} e^z \quad \text{and} \quad \varphi_2 = \frac{1}{\alpha+\beta+1} e^z \tag{7}$$

from (4). thence applying (4) and (7) we obtain

$$\text{LHS of (1)} = \frac{1}{\alpha+\beta+1} e^z (z-z+\alpha+1+\beta) = e^z . \tag{8}$$

(II) Let

$$\alpha = 0, \quad \beta = -1 \quad \text{and} \quad f(z) = e^z$$

we have then the nonhomogeneous Laguerre's equation

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z+1) - \varphi = e^z \quad (z \neq 0) \tag{9}$$

from § 2. (1)

A particular solution to this equation is given by

$$\varphi = \varphi^{*}_{[5](0,-1)} = (X_{[3]} \cdot Y_{[3]})_{-(1+\beta)} = (f_{-1} z^{-1} e^{-z})_{-1} e^{z} \tag{10}$$

$$= e^{z} \log z \ . \tag{11}$$

Hence we obtain

$$\varphi_1 = e^{z} \log z + e^{z} \frac{1}{z} \tag{12}$$

and

$$\varphi_2 = e^{z} \log z + 2 e^{z} \frac{1}{z} - e^{z} \frac{1}{z^2} \ . \tag{13}$$

from (11), respectively.

Therefore, we have

$$\text{LHS of (9)} = z e^{z} \log z + 2 e^{z} - e^{z} \frac{1}{z}$$

$$- z e^{z} \log z - e^{z} + e^{z} \log z + e^{z} \frac{1}{z} - e^{z} \log z$$

$$= e^{z} \tag{14}$$

applying (11). (12) and (13).

(**III**) Let

$$f(z) = z^{-\alpha} e^{z}$$

we have then the nonhomogeneous Laguerre's equation

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = z^{-\alpha} e^{z} \quad (z \neq 0) \tag{15}$$

from §2. (1)

A particular solution to this equation is given by

$$\varphi = \varphi^{*}_{[3](\alpha,\beta)} = e^{z} (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \tag{16}$$

$$= e^{z} \left(((f e^{-z})_{-(\alpha+\beta+1)} e^{z} z^{-(1+\beta)})_{-1} e^{-z} z^{\beta} \right)_{\alpha+\beta} \tag{17}$$

$$= \frac{1}{\beta + 1} e^{z} z^{-\alpha} \tag{18}$$

since we have

$$(f e^{-z})_{-(\alpha+\beta+1)} = (z^{-\alpha})_{-(\alpha+\beta+1)} \tag{19}$$

$$= e^{i\pi(\alpha+\beta+1)} \frac{\Gamma(-\beta-1)}{\Gamma(\alpha)} z^{\beta+1} \qquad \left(\left| \frac{\Gamma(-\beta-1)}{\Gamma(\alpha)} \right| < \infty \right) \tag{20}$$

and

$$(z^{\beta})_{\alpha+\beta} = e^{-i\pi(\alpha+\beta)} \frac{\Gamma(\alpha)}{\Gamma(-\beta)} z^{-\alpha} \qquad \left(\left| \frac{\Gamma(\alpha)}{\Gamma(-\beta)} \right| < \infty \right) . \tag{21}$$

by Lemma (i), respectively.

Indeed we have

$$\varphi_1 = \frac{1}{\beta+1} e^z (z^{-\alpha} - \alpha z^{-\alpha-1}) \tag{22}$$

and

$$\varphi_2 = \frac{1}{\beta+1} e^z [z^{-\alpha} - 2\alpha z^{-\alpha-1} + \alpha(\alpha+1) z^{-\alpha-2}] \tag{23}$$

from (18), respectively.

Therefore, we obtain

$$\text{LHS of (15)} = \frac{1}{\beta+1} e^z [z^{1-\alpha} - 2\alpha z^{-\alpha} + \alpha(\alpha+1) z^{-\alpha-1}$$

$$- z^{1-\alpha} + \alpha z^{-\alpha} + \alpha z^{-\alpha} - \alpha^2 z^{-\alpha-1} + z^{-\alpha} - \alpha z^{-\alpha-1} + \beta z^{-\alpha}] \tag{24}$$

$$= \frac{1}{\beta+1} e^z [z^{-\alpha} + \beta z^{-\alpha}] \tag{25}$$

$$= e^z z^{-\alpha} \quad , \tag{26}$$

applying (18). (22) and (23).

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5 (1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century) ; Integrals and Differentiations of Arbitrary Order (1991), Descartes Press , Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^{ν} (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol.6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; Ring and Field produced from The Set of N-Fractional Calculus Operator, J. Frac. Calc. Vol. 24, Nov. (2003), 29 - 36.
[6] K. Nishimoto ; An application of fractional calculus to the nonhomogeneous Gauss equations, J. Coll. Engng. Nihon Univ., B - 28 (1987), 1 - 8.
[7] K. Nishimoto and S. L. Kalla ; Application of Fractional Calculus to Ordinary Differential Equation of Fuchs Type, Rev. Tec. Ing. Univ. Zulia, Vol. 12, No. 1, (1989).
[8] K. Nishimoto ; Application of Fractional Calculus to Gauss Type Partial Differential Equations, J. Coll. Engng. Nihon UNiv., B - 30 (1989), 81 - 87.
[9] Shih - Tong Tu, S. - J. Jaw and Shy - Der Lin ; An application of fractional calculus to Chebychev's equation, Chung Yuan J. Vol. XIX (1990), 1 - 4.
[10] K. Nishimoto, H. M. Srivastava and Shih - Tong Tu ; Application of fractional Calculus in Solving Certain Classes of Fuchsian Differential Equations, J. Coll. Engng. Nihon Univ., B - 32 (1991), 119 - 126.
[11] K. Nishimoto ; A Generalization of Gauss Equation by Fractional Calculus Method, J. Coll. Engng. Nihon Univ., B - 32 (1991), 79 - 87.
[12] Shy - Der Lin, Shih - Tong Tu and K. Nishimoto ; A generalization of Legendre's equation by fractional calculus method, J. Frac. Calc. Vol.1, May (1992),35 - 43.
[13] N. S> Sohi, L. P. Singh and K. Nishimoto ; A generalization of Jacobi's equation by fractional calculus method, J. Frac. Calc. Vol. 1, May (1992), 45 - 51.
[14] K. Nishimoto ; Solutions of Gauss equation in fractional calculus, J. Frac. Calc. Vol. 3, May (1993), 29 - 37.
[15] K. Nishimoto ; Solutions of homogeneous Gauss equations, which have a logarithmic function, in fractional calculus, J. Frac. Calc. Vol. 5, May (1994), 11 - 25.
[16] K. Nishimoto ; Application of N- transformation and N- ftactional calculus method to nonhomogeneous Bessel equations (I), J. Frac. Calc. Vol. 8, Nov. (1995),25 - 30.
[17] K. Nishimoto ; Operator N^{ν} method to nonhomogeneous Gauss and Bessel equations, J. Frac. Calc. Vol. 9, May. (1996), 1 - 15.
[18] K. Nishimoto and Susana S. de Romero ; N- fractional calculus operator N^{ν} method to nonhomogeneous and homogeneous Whitttaker equations (I), J. Frac. Calc. Vol. 9, May. (1996) 17 - 22.
[19] K. Nishimoto and Judith A. de Duran ; N- fractional calculus operator N^{ν} method to nonhomogeneous Fukuhara equations (I), J. Frac. Calc. Vol. 9, May. (1996) ,23 - 31.
[20] K. Nishimoto ; N- fractional calculus operator N^{ν} method to nonhomogeneous Gauss equations , J. Frac. Calc. Vol. 10, Nov. (1996), 33 - 39.
[21] K. Nishimoto ; Kummer's twenty - four functions and N- fractional calculus, Nonlinear Analysis, Theory, Method & Applications, Vol.30, No.2, (1997),1271 - 1282.
[22] Shih-Tong Tu, Ding-Kuo Chyan and Wen-Chieh Luo ; Some solutions to the nonhomogeneous Jacobi equations via fractional calculus operator N^{ν} method, , J. Frac. Calc. Vol. 12, Nov. (1997), 51 - 60.
[23] Shih-Tong Tu, Ding-Kuo Chyan and Erh-Tsung Chin ; Solutions of Gegenbauer and Chebysheff equations via operator N^{ν} method, , J. Frac. Calc. Vol. 12, Nov. (1997), 61 - 69
[24] K. Nishimoto ; N- method to Hermite equations, J. Frac. Calc. Vol. 13, May (1998),21 - 27.
[25] K. Nishimoto ; N- method to Weber equations, J. Frac. Calc. Vol. 14, Nov. (1998), 1 - 8.
[26] K. Nishimoto ; N- method to generalized Laguerre equations, J. Frac. Calc. Vol. 14, Nov. (1998), 9 - 21
[27] Shy-Der Lin, Jaw-Chian Shyu, Katsuyuki Nishimoto and H.M. Srivastava ; Explicit Solutions of Some General Families of Ordinary and Partial Differential Equations Associated with the Bessel Equation by Means of Fractional Calculus, J. Frac. Calc. Vol. 25, May (2004), 33 - 45..
[28] K. Nishimoto ; Solutions to Some Extended Hermite's equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 29, May (2006), 45 - 56.

[29] Tsuyako Miyakoda ; Solutions to An Extended Hermite's Equation by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 30, Nov. (2006), 23 - 32.
[30] K. Nishimoto ; Solutions to Some Extended Weber's Equations by Means of N-Fractional Calculus, J. Frac. Calc. Vol. 30, Nov. (2006), 1 - 11.
[31] K. Nishimoto ; N- fractional Calculus of Products of Some Power Functions, J. Frac. Calc. Vol. 27, May (2005), 83 - 88.
[32] K. Nishimoto ; N-Fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May (2006),35 -44.
[33] Shy-Der Lin, Yen-Shung Tsai, H.M. Srivastava and Pin-Yu Wang ; Fractional Calculus Derivations of the Explicit Solutions of Confluent Hypergeometric and Associated Laguerre Equations, J. Frac. Calc. , May (2006), 75 - 86.
[34] K. Nishimoto ; N-Fractional Calculus Operator Method to Chebyshev's Equations, J. Frac. Calc. Vol. 33, May (2008), 71 - 90.
[35] K. Nishimoto ; N-Fractional Calculus Operator Method to Laguerre's Equations (I), J. Frac. Calc. Vol. 35, May (2009), 119 - 141.
[36] David Dummit and Richard M. Foote ; Abstract Algebra, Prentice Hall (1991).
[37] K. B. Oldham and J. Spanier ; The Fractional Calculus, Academic Press (1974).
[38] A. C. Mcbride ; Fractional Calculus and Integral Transforms of Generalized Functions, Research Notes, Vol. 31, (1979), Pitman.
[39] S. G. Samko, A. A. Kilbas and O.I. Marichev ; Fractional Integrals and Derivatives, and Some Their Applications (1987), Nauka, USSR.
[40] K. S. Miller and B. Ross ; An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, (1993).
[41] V. Kiryakova ; Generalized fractional calculus and applications, Piman Research Notes, No.301, (1994), Longman.
[42] Igor Podlubny ; Fractional Differential Equations (1999), Academic Press.
[43] R. Hilfer (ed.) ; Applications of Fractional Calculus in Physics, (2000), World Scientific, Singapore, New Jersey, London, Hong Kong.
[44] Anatoly A. Kilbas, Hari M. Srivastava and Juan J. Trujillo ; Theory and Applications of Fractional Differential Equations (2006), Elsevier, North-Holland, Mathematics Studies 204.