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Solutions to The Nonhomogeneous Associated Laguerre's Equation by Means of N-Fractional Calculus Operator

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Abstract

In this article, solutions to the nonhomogeneous associated Laguerre's equations

\[
\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = f \quad (z \neq 0)
\]

\[
( \varphi_v = d^v \varphi / dz^v \text{ for } v > 0, \varphi_0 = \varphi = \varphi(z), f = f(z) )
\]

are discussed by means of N-fractional calculus operator (NFCO-Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

(i)

\[
\varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} \equiv \varphi_{[1](\alpha,\beta)}, \quad \text{(denote)}
\]

and

(ii)

\[
\varphi = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \equiv \varphi_{[2](\alpha,\beta)},
\]

where

\[
X_{[1]} = (f_\beta z^{\alpha+\beta} e^{-})_{-1}, \quad Y_{[1]} = e^z z^{-(\alpha+\beta+1)}.
\]

Group II.

(i)

\[
\varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \varphi_{[3](\alpha,\beta)},
\]

and

(ii)

\[
\varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \varphi_{[4](\alpha,\beta)},
\]

where

\[
X_{[2]} = ((fe^{-z})_{-(\alpha+\beta+1)} e^z z^{-(1+\beta)})_{-1}, \quad Y_{[2]} = e^z z^\beta.
\]
§0. Introduction (Definition of Fractional Calculus) and

§1. Preliminary

are omitted, then refer to the previous paper [35].

§ 2. Solutions to The Nonhomogeneous Associated Laguerre's

Equation by NFCO-Method

Theorem 1. Let $\varphi \in \mathcal{F}$ and $f \in \mathcal{F}$, then nonhomogeneous associated Laguerre's equation

\[\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = f \quad (z \neq 0)\]  

$(\varphi_\nu = \frac{\partial^n \varphi}{\partial z^n}$ for $\nu > 0$, $\varphi_0 = \varphi = \varphi(z)$, $f = f(z)$)

has particular solutions of the forms (fractional differintegrated form);

Group I.

(i) \[\varphi = (X_{(1)} \cdot Y_{(1)})_{-1}, \quad \varphi^* = \varphi_{[1]}(\alpha, \beta), \quad \text{denote}\]  

(ii) \[\varphi = (Y_{(1)} \cdot X_{(1)})_{-1}, \quad \varphi^* = \varphi_{[2]}(\alpha, \beta), \quad \text{denote}\]  

where

\[X_{(1)} = (f \cdot z^{\alpha+\beta} e^{-z})_{-1}, \quad Y_{(1)} = e^{-z} z^{-(\alpha+\beta+1)}\]  

Group II.

(i) \[\varphi = e^z (X_{(2)} \cdot Y_{(2)})_{-1}, \quad \varphi^* = \varphi_{[3]}(\alpha, \beta), \quad \text{denote}\]  

(ii) \[\varphi = e^z (Y_{(2)} \cdot X_{(2)})_{-1}, \quad \varphi^* = \varphi_{[4]}(\alpha, \beta), \quad \text{denote}\]  

where

\[X_{(2)} = ((f \cdot z^\alpha e^{-z})_{-1}) e^z z^{-(1+\alpha+\beta)}, \quad Y_{(2)} = e^{-z} z^\beta\]  

Group III.

(i) \[\varphi = z^{-\alpha} (X_{(3)} \cdot Y_{(3)})_{-1}, \quad \varphi^* = \varphi_{[5]}(\alpha, \beta), \quad \text{denote}\]  

(ii) \[\varphi = z^{-\alpha} (Y_{(3)} \cdot X_{(3)})_{-1}, \quad \varphi^* = \varphi_{[6]}(\alpha, \beta), \quad \text{denote}\]  

where

\[X_{(3)} = ((f \cdot z^\alpha)_{-1} e^z z^{-(1+\alpha+\beta)}), \quad Y_{(2)} = e^{-z} z^\beta\]  

Group IV.

(i) \[\varphi = z^{-\alpha} e^z (X_{(4)} \cdot Y_{(4)})_{-1}, \quad \varphi^* = \varphi_{[7]}(\alpha, \beta), \quad \text{denote}\]  

(ii) \[\varphi = z^{-\alpha} e^z (Y_{(4)} \cdot X_{(4)})_{-1}, \quad \varphi^* = \varphi_{[8]}(\alpha, \beta), \quad \text{denote}\]  

where

\[X_{(4)} = ((f \cdot z^\alpha e^{-z})_{-1} e^z z^{-(1+\alpha+\beta)}), \quad Y_{(2)} = e^{-z} z^\beta\]
Proof of Group I.

Operate \( N^{\nu} \) fractional calculus (NFC) operator to the both sides of equation (1), we have then

\[
(\varphi_{2} \cdot z)_{\nu} + (\varphi_{1} \cdot (-z + \alpha + 1))_{\nu} + (\varphi \cdot \beta)_{\nu} = f_{\nu}, \quad (f \neq 0) .
\]

Now we have

\[
(\varphi_{2} \cdot z)_{\nu} = \sum_{k=0}^{\nu} \frac{\Gamma(\nu + 1)}{k! \Gamma(\nu + 1 - k)} (\varphi_{2})_{\nu-k}(z)_{k},
\]

\[
= \varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot \nu,
\]

\[
(\varphi_{1} \cdot (-z + \alpha + 1))_{\nu} = \varphi_{1+\nu} \cdot (-z + \alpha + 1) - \varphi_{\nu} \cdot \nu
\]

and

\[
(\varphi \cdot \beta)_{\nu} = \varphi_{\nu} \cdot \beta,
\]

respectively, by Lemmas (i) and (iv).

Therefore, we have

\[
\varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot (-z + \alpha + 1 + \nu) + \varphi_{\nu} \cdot (\beta - \nu) = f_{\nu},
\]

from (14), applying (16), (17) and (18).

Choosing \( \nu \) such that

\[
\nu = \beta
\]

we obtain

\[
\varphi_{2+\beta} \cdot z + \varphi_{1+\beta} \cdot (-z + \alpha + \beta + 1) = f_{\beta}.
\]

Set

\[
\varphi_{1+\beta} = \phi = \phi(z) \quad (\varphi = \phi_{-(1+\beta)})
\]

we have then

\[
\phi_{1} + \phi \left( \frac{\alpha + \beta + 1}{z} - 1 \right) = f_{\beta} \cdot z^{-1}
\]

from (21). A particular solution to this linear first order equation is given by

\[
\phi = X_{[1]}Y_{[1]},
\]

where \( X_{[1]} \) and \( Y_{[1]} \) are the ones shown by (4), respectively.

Therefore, we obtain

\[
\varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} = \varphi^{*}_{[1] (\alpha, \beta)}
\]

from (24) and (22).
Inversely (24) satisfies equation (23). then (2) satisfies equation (1).

Next, changing the order

\[ X_{[1]} \text{ and } Y_{[1]} \text{ in parenthesis (}_0^{-(1+\beta)} \]

we obtain other solution \( \varphi_{[2](a,\beta)}^{*} \) which is different from (2) for \(-(1+\beta) \notin \mathbb{Z}_0^+ \), that is,

\[ \varphi = (Y_{[1]} \cdot X_{[1]}^{-(1+\beta)} = \varphi_{[2](a,\beta)}^{*} \quad (3) \]

(Refer to Theorem D in the previous paper [35].)

Proof of Group II.

Set

\[ \varphi = e^{\gamma z} \psi \quad (\psi = \psi(z)) \quad (25) \]

we have then

\[ \psi_2 \cdot z + \psi_1 \cdot \{z(2\gamma - 1) + \alpha + 1\} + \psi \cdot \{z\gamma(\gamma - 1) + \gamma(\alpha + 1) + \mathscr{F} = f e^{-\gamma z} \quad (26) \]

from (1).

Here we choose \( \gamma \) such that

\[ \gamma(\gamma - 1) = 0 \]

that is,

\[ \gamma = 0, 1 \quad (27) \]

When \( \gamma = 0 \), (26) is reduced to (1), therefore, we have the same solutions as Group I.

When \( \gamma = 1 \) we have

\[ \psi_2 \cdot z + \psi_1 \cdot \{z + \alpha + 1\} + \psi \cdot (\alpha + \beta + 1) = f e^{-z} \quad (28) \]

from (26)

Operate \( N^\nu \) to the both sides of equation (28), we have then

\[ (\psi_2 \cdot z)_\nu + (\psi_1 \cdot (z + \alpha + 1))_\nu + (\psi \cdot (\alpha + \beta + 1))_\nu = (f e^{-z})_\nu \quad (29) \]

Hence, using Lemma (i v), we obtain

\[ \psi_{2+v} \cdot z + \psi_{1+v} \cdot (z + \alpha + 1 + v) + \psi_v \cdot (\nu + \alpha + \beta + 1) = (f e^{-z})_\nu \quad (30) \]

Choosing \( \nu \) such that

\[ \nu = -(\alpha + \beta + 1) \quad (31) \]

we obtain

\[ \psi_{1-(a+\beta)} \cdot z + \psi_{-(a+\beta)} \cdot (z - \beta) = (f e^{-z})_{-(a+\beta+1)} \quad (32) \]
from (30).

Set

$$\psi_{-(\alpha+\beta)} = \phi = \phi(z) \quad (\psi = \phi_{\alpha+\beta}) \tag{33}$$

we have then

$$\phi_1 + \phi \cdot \left(1 - \frac{\beta}{z}\right) = (f e^{-z})_{-(\alpha+\beta+1)} \cdot z^{-1} \tag{34}$$

from (32). A particular solution to this equation is given by

$$\phi = X_{[2]} Y_{[2]} \tag{35}$$

Hence we obtain

$$\psi = (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \tag{36}$$

from (35) and (33).

Therefore, we obtain

$$\varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \varphi_{[3]}^{(*)} \tag{5}$$

from (25) and (36), having \( \gamma = 1 \).

Inversely, (35) satisfies (34), then (36) satisfies equation (28).

Hence (5) satisfies equation (1).

Next, changing the order

\( X_{[2]} \) and \( Y_{[2]} \) in parenthesis \( (\quad)_{\alpha+\beta} \) in (5)

we obtain other solution

$$\varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \varphi_{[4]}^{(*)} \tag{6}$$

which is different from (5) for \( (\alpha + \beta) \notin Z^+ \).

Proof of Group III.

Set

$$\varphi = z^\lambda \psi \quad (\psi = \psi(z)) \tag{37}$$

we have then

Hence we obtain

$$\psi_2 \cdot z^{\lambda+1} + \psi_1 \cdot \{- z^{\lambda+1} + z^\lambda (2 \lambda + \alpha + 1)\}$$

$$+ \psi \cdot \{z^\lambda (\beta - \lambda) + z^{\lambda-1} \lambda (\lambda + \alpha)\} = f \tag{38}$$

from (1).

Here we choose \( \lambda \) such that

$$\lambda (\lambda + \alpha) = 0 \ ,$$
that is,

$$\lambda = 0, -\alpha.$$  

When $\lambda = 0$, (38) is reduced to (1), therefore, we have the same solutions as Group I.

When $\lambda = -\alpha$ we have

$$\psi_2 \cdot z + \psi_1 \cdot \{-z + 1 - \alpha\} + \psi \cdot (\alpha + \beta) = f z^\alpha \tag{40}$$

from (38).

Operate $N^v$ to both sides of equation (40), we have then

$$\psi_{2+v} \cdot z + \psi_{1+v} \cdot \{-z + 1 - \alpha + v\} + \psi_v \cdot (\alpha + \beta - v) = (f z^\alpha)_v. \tag{41}$$

Choosing $v$ such that

$$v = \alpha + \beta \tag{42}$$

we obtain

$$\psi_{2+\alpha+\beta} \cdot z + \psi_{1+\alpha+\beta} \cdot \{-z + 1 + \beta\} = (f z^\alpha)_{\alpha+\beta}. \tag{43}$$

from (41), applying (42).

Set

$$\psi_{1+\alpha+\beta} = \phi = \phi(z) \quad (\psi = \phi_{-(1+\alpha+\beta)}) \tag{44}$$

we have then

$$\phi + \phi \cdot \left(\frac{\beta + 1}{z} - 1\right) = (f z^\alpha)_{\alpha+\beta} z^{-1} \tag{45}$$

from (43).

A particular solution to this equation is given by

$$\phi = X_{[3]} Y_{[3]} \tag{46}$$

Where $X_{[3]}$ and $Y_{[3]}$ are the ones given by (10).

Hence we obtain

$$\psi = (X_{[3]} Y_{[3]})_{-(1+\alpha+\beta)} \tag{47}$$

from (44) and (46).

Therefore, we obtain

$$\varphi = z^{-\alpha}(X_{[3]} Y_{[3]})_{-(1+\alpha+\beta)} \equiv \varphi^* \tag{8}$$

from (37) and (47), having $\lambda = -\alpha$. 


Inversely, (46) satisfies (equation (45)), then (47) satisfies equation (43).

Therefore, (8) satisfies equation (1).

Next, changing the order

\[ X_{[3]} \text{ and } Y_{[3]} \text{ in parenthesis } \] 

we obtain other solution

\[ \varphi = z^{-\alpha}(Y_{[3]}X_{[3]})-(1+\alpha+\beta) \equiv \varphi_{[6]}(\alpha, \beta) \]  

(9)

which is different from (8) for \(-(1+\alpha+\beta) \not\in Z_0^+\).

**Proof of Group IV.**

First set

\[ \varphi = z^{\lambda}\psi \]  

(37)

and substitute (37) into equation (1), we have then (38).

We have then (40) from (38), having

\[ \lambda = -\alpha \]  

Next set

\[ \psi = e^{\delta z}\phi \]  

(48)

We have then

\[ \phi_2 \cdot z + \phi_1 \cdot \{z(2\delta - 1) + 1 - \alpha\} + \phi \cdot \{z(\delta^2 - \delta) + \delta(1 - \alpha) + \alpha + \beta\} = f z^{\alpha}e^{-\delta z} \]  

(49)

from (40), applying (48).

Choose \(\delta\) such that

\[ \delta^2 - \delta = 0 \]  

that is,

\[ \delta = 0, 1 \]  

(50)

When \(\delta = 0\), we obtain (40) from (49). Then we have the same solutions as Group III.

When \(\delta = 1\) we have

\[ \phi_2 \cdot z + \phi_1 \cdot (z + 1 - \alpha) + \phi \cdot (1 + \beta) = f z^{\alpha}e^{-z} \]  

(51)

from (49).

Operate \(N^\nu\) to the both sides of equation (51), we have then

\[ \phi_{2+\nu} \cdot z + \phi_{1+\nu} \cdot (z + 1 - \alpha + \nu) + \phi_{\nu} \cdot (\nu + 1 + \beta) = (f z^{\alpha}e^{-z})_{\nu} \]  

(52)
Choose $\nu$ such that
\[ \nu = -(1 + \beta) \] (53)
we obtain
\[ \phi_{1-\beta} \cdot z + \phi_{-\beta} \cdot (z - \alpha - \beta) = (f z^a e^{-z})_{-(1+\beta)} \] (54)
from (52).

Therefore, setting
\[ \phi_{-\beta} = u = u(z) \quad (\phi = u_\beta) \] (55)
we have
\[ u_1 + u \cdot \left(1 - \frac{\alpha + \beta}{z}\right) = (f z^a e^{-z})_{-(1+\beta)} z^{-1} \] (56)
from (54). A particular solution to this equation is given by
\[ u = X_{[4]} \cdot Y_{[4]} \] (57)
where $X_{[4]}$ and $Y_{[4]}$ are the ones shown by (13).

Hence we obtain
\[ \phi = (X_{[4]} \cdot Y_{[4]})_{\beta} \] (58)
from (55) and (57).

Therefore, we have
\[ \psi = e^{z} (X_{[4]} \cdot Y_{[4]})_{\beta} \] (59)
from (58) and (48), having $\delta = 1$.

We have then
\[ \varphi = z^{-\alpha} e^{z} (X_{[4]} \cdot Y_{[4]})_{\beta} = \varphi^{*}_{(7\alpha,\beta)} \] (11)
from (59) and (37), having $\lambda = -\alpha$.

Inversely, the function shown by (57) satisfies equation (56), then (58) satisfies equation (54), and hence (59) satisfies (40).

Therefore, the function given by (11) satisfies equation (1), by (37) where $\lambda = -\alpha$.

Next, changing the order
$X_{[4]}$ and $Y_{[4]}$ in parenthesis ( ) in (11)
we obtain other solution
\[ \varphi = z^{-\alpha} e^{z} (Y_{[4]} \cdot X_{[4]})_{\beta} = \varphi^{*}_{[8](\alpha,\beta)} \] (12)
which is different from (11) for $\beta \notin \mathbb{Z}_0^*$. 
§3. Some Illustrative Example

(I) Let

\[ f(z) = e^z \]

we have then the nonhomogeneous Laguerre's equation

\[ \varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = e^z \quad (z \neq 0) \tag{1} \]

from §2. (1)

A particular solution to this equation is given by

\[ \varphi = \varphi^* \mid_{[1]}(\alpha; \beta) = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \quad \tag{2} \]

\[ = \left( \left( (e^z)_\beta z^{\alpha+\beta} e^{-z} \right)_{-1} (e^z z^{-\left(\alpha+\beta+1\right)}) \right)_{-(1+\beta)} \quad \tag{3} \]

\[ = \frac{1}{\alpha + \beta + 1} e^z \quad ; \quad \tag{4} \]

since

\[ (e^z)_\beta = e^z \quad \tag{5} \]

and

\[ (z^{\alpha+\beta})_{-1} = \frac{1}{\alpha + \beta + 1} z^{\alpha+\beta+1} \quad \tag{6} \]

Indeed we have

\[ \varphi_1 = \frac{1}{\alpha + \beta + 1} e^z \quad \text{and} \quad \varphi_2 = \frac{1}{\alpha + \beta + 1} e^z \quad \tag{7} \]

from (4). Hence applying (4) and (7) we obtain

\[ \text{LHS of (1)} = \frac{1}{\alpha + \beta + 1} e^z (z - z + \alpha + 1 + \beta) = e^z \quad . \tag{8} \]

(II) Let

\[ \alpha = 0, \quad \beta = -1 \quad \text{and} \quad f(z) = e^z \]

we have then the nonhomogeneous Laguerre's equation

\[ \varphi_2 \cdot z + \varphi_1 \cdot (-z + 1) - \varphi = e^z \quad (z \neq 0) \quad \tag{9} \]

from §2. (1)

A particular solution to this equation is given by
\[ \varphi = \varphi^*_{[3](0,-1)} = (X_{[3]} \cdot Y_{[3]})_{(\alpha+\beta)} = (f_{-1} z^{-\alpha} e^{-z})_{-1} e^z \]  

(10)

\[ = e^z \log z . \]  

(11)

Hence we obtain

\[ \varphi_1 = e^z \log z + e^z \frac{1}{z} \]  

(12)

and

\[ \varphi_2 = e^z \log z + 2 e^z \frac{1}{z} - e^z \frac{1}{z^2} . \]  

(13)

from (11), respectively.

Therefore, we have

LHS of (9) = \[ ze^z \log z + 2e^z - e^z \frac{1}{z} \]

\[ - ze^z \log z - e^z + e^z \log z + e^z \frac{1}{z} - e^z \log z \]

\[ = e^z \]  

(14)

applying (11), (12) and (13).

(III) Let

\[ f(z) = z^{-\alpha} e^z \]

we have then the nonhomogeneous Laguerre's equation

\[ \varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = z^{-\alpha} e^z \]  

(\( z \neq 0 \))  

(15)

from §2. (1)

A particular solution to this equation is given by

\[ \varphi = \varphi^*_{[3](\alpha, \beta)} = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \]  

(16)

\[ = e^z \left( (f e^{-z})_{-(\alpha+\beta+1)} e^z z^{-(1+\beta)} \right)_{-1} e^{-z} z^\beta \]  

(17)

\[ = \frac{1}{\beta + 1} e^z z^{-\alpha} \]  

(18)

since we have

\[ (f e^{-z})_{-(\alpha+\beta+1)} = (z^{-\alpha})_{-(\alpha+\beta+1)} \]  

(19)
\[ = e^{i\pi(\alpha+\beta+1)} \frac{\Gamma(-\beta-1)}{\Gamma(\alpha)} z^{\beta+1} \left( \left| \frac{\Gamma(-\beta-1)}{\Gamma(\alpha)} \right| < \infty \right) \quad (20) \]

and

\[ (z^\beta)^{\alpha+\beta} = e^{-i\pi(\alpha+\beta)} \frac{\Gamma(\alpha)}{\Gamma(-\beta)} z^{-\alpha} \left( \left| \frac{\Gamma(\alpha)}{\Gamma(-\beta)} \right| < \infty \right) \quad (21) \]

by Lemma (i), respectively.

Indeed we have

\[ \varphi_1 = \frac{1}{\beta+1} e^z (z^{-\alpha} - \alpha z^{-\alpha-1}) \quad (22) \]

and

\[ \varphi_2 = \frac{1}{\beta+1} e^z \left[ z^{-\alpha} - 2\alpha z^{-\alpha-1} + \alpha(\alpha+1)z^{-\alpha-2} \right] \quad (23) \]

from (18), respectively.

Therefore, we obtain

\[ \text{LHS of (15)} = \frac{1}{\beta+1} e^z \left[ z^{1-\alpha} - 2\alpha z^{-\alpha} + \alpha(\alpha+1)z^{-\alpha-1} \right. \]
\[ \left. - z^{1-\alpha} + \alpha z^{-\alpha} + \alpha z^{-\alpha} - \alpha^2 z^{-\alpha-1} + z^{-\alpha} - \alpha z^{-\alpha-1} + \beta z^{-\alpha} \right] \quad (24) \]
\[ = \frac{1}{\beta+1} e^z \left[ z^{-\alpha} + \beta z^{-\alpha} \right] \quad (25) \]
\[ = e^z z^{-\alpha} \quad , \quad (26) \]

applying (18), (22) and (23).

**References**


