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Chain recurrent points have been introduce by C. Conley [7]. They play an important role in the theory of attractors and in several other aspects of topological dynamics of a continuous map $f$ on a compact metric space $X$. The key theorem here is Conley's Decomposition Theorem which says that the space $X$ decomposes into the chain recurrent set $\text{CR}(f)$ (see §2 for definition) and the rest, where the action is gradient-like (see [7] for definition). Note that the chain recurrent set contains all nonwandering points in that including the “genuine” recurrent points $x$ (i.e., such that $x$ belongs to the closure of its forward orbit), minimal subsets and periodic orbits.

Another motivation for studying chain recurrent sets in this particular context (of $n$-dimensional locally $(n-1)$-connected spaces) is provided by two other results: The first one is Pugh’s Closing Lemma, which allows to replace chain recurrent points by periodic ones (by slightly perturbing the map):

**Theorem** ([13] for manifolds). Let $(X, d)$ be an $n$-dimensional locally $(n-1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$, skip the local connectedness assumption), and $f : X \rightarrow X$ be a map. If $x \in \text{CR}(f)$, then for every $\varepsilon > 0$, there exists a map $g : X \rightarrow X$ such that the uniform distance $d(f, g) < \varepsilon$ and $x$ is a periodic point of $g$.

**Sketch of proof.** We give here an outline in the case when $X$ is $n$-dimensional locally $(n-1)$-connected, $n \in \mathbb{N}$. Let $x \in \text{CR}(f)$, and any $\varepsilon > 0$ is given. We may assume $x \not\in \text{Per}(f)$.

Since $X$ is locally $(n-1)$-connected, we have a $\xi$ such that $0 < \xi < \varepsilon/2$ and

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(1) for every map \( \varphi : A \to X \) from a closed set \( A \) of a compact metric space \( Z \) with \( \dim Z \leq n \) and \( \text{diam} [\text{Im} \varphi] < \xi \), there exists an extension \( \tilde{\varphi} : Z \to X \) of \( \varphi \) satisfying \( \text{diam} [\text{Im} \tilde{\varphi}] < \varepsilon/2 \).

Using uniform continuity of \( f \), we also take a \( \delta > 0 \) such that

(2) if \( A \subseteq X \) with \( \text{diam} [A] < \delta \), then \( \text{diam} [f(A)] < \xi/2 \).

Then take a \( \xi/2 \)-chain \( \{x_0 = x, x_1, \ldots, x_k = x\} \) of least possible length \( k \); hence, \( k \geq 1 \) and \( x_i \neq x_j \) for \( 0 \leq i < j \leq k - 1 \). We have an open neighborhood \( U_i \) of \( x_i \) in \( X \), \( 0 \leq i \leq k - 1 \), such that \( \text{diam} [\text{Cl} U_i] < \delta \) for \( 0 \leq i \leq k - 1 \), and \( \text{Cl} U_i \cap \text{Cl} U_j = \emptyset \) for \( 0 \leq i < j \leq k - 1 \). For each \( i \in \{0, \ldots, k - 1\} \), we define the map \( \varphi_i : \text{Bd} U_i \cup \{x_i\} \to X \) by \( \varphi_i = f \) on \( \text{Bd} U_i \) and \( \varphi_i(x_i) = x_{i+1} \). Since \( \text{diam} [\text{Im} \varphi_i] < \xi \) by (2), we have an extension \( \tilde{\varphi}_i : \text{Cl} U_i \to X \) of \( \varphi_i \) with \( \text{diam} [\text{Im} \tilde{\varphi}_i] < \varepsilon/2 \) by (1).

Now we define the map \( g : X \to X \) by \( g = f \) on \( X \setminus \cup_{i=0}^{k-1} U_i \) and \( g = \tilde{\varphi}_i \) on \( \text{Cl} U_i \) for \( 0 \leq i < j \leq k - 1 \). Then it is easy to see that \( d(f, g) < \varepsilon \) and \( x \in \text{Per}(g) \).

The second is the result by Block and Franke [4, Theorem H], which characterizes the case where all chain recurrent points are nonwandering, in terms of stability of the nonwandering set under perturbations:

**Theorem** ([4] for manifolds). Let \( (X,d) \) be an \( n \)-dimensional locally \((n-1)\)-connected compact metric space, where \( n \geq 0 \) (for \( n = 0 \), skip the local connectedness assumption), and \( f : X \to X \) be a map. Then \( \Omega(f) = \text{CR}(f) \) if and only if \( f \) does not permit \( \Omega \)-explosions; that is, for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( g : X \to X \) with \( d(f, g) < \delta \), then each point of \( \Omega(g) \) belongs to the \( \varepsilon \)-neighborhood of \( \Omega(f) \), where \( \Omega(h) \) means the nonwandering set of a map \( h \).

It is hence quite important to know how large the set \( \text{CR}(f) \) is. In many systems the chain recurrent set indeed turns out to be small, for example, Franzová [9] proved that if \( X \) denotes the interval then for a generic (in the uniform metric) continuous maps the chain recurrent set has Lebesgue measure zero.

2. 鎖回帰集合の測度零性

We now give the terminology and notation needed in what follows. A map on \( X \) is a continuous function \( f : X \to X \) from a space \( X \) to itself; \( f^0 \) is the identity map, and for every \( n \geq 0 \), \( f^{n+1} = f^n \circ f \). The dimension \( \text{dim} X \) of a space \( X \) means the covering dimension (see [8] and [12]). By a graph, we mean a connected one-dimensional compact polyhedron. We let \( f : X \to X \) be a map from a compact metric space \((X,d)\) to itself. Let \( x, y \in X \). An \( \varepsilon \)-chain from \( x \) to \( y \) is a finite sequence of points \( \{x_0, x_1, \ldots, x_n\} \) of \( X \) such that \( x_0 = x, x_n = y \) and
$d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \ldots, n$. We say $x$ can be chained to $y$ if for every $\varepsilon > 0$ there exists an $\varepsilon$-chain from $x$ to $y$, and we say $x$ is chain recurrent if it can be chained to itself. The set of all chain recurrent points is called the chain recurrent set of $f$ and denoted by $\text{CR}(f)$. The chain recurrent set is non-empty, closed in $X$ and $f$-strongly invariant, and the set depends only on the topology. A point $x \in X$ is said to be wandering if for some neighborhood $V$ of $x$, $f^n(V) \cap V = \emptyset$ for all $n > 0$. The set of points which are not wandering is called the nonwandering set and denoted by $\Omega(f)$.

We state fundamental facts from geometric topology. A space $X$ is said to be locally $(n-1)$-connected if for every $x \in X$ and every neighborhood $U$ of $x$ in $X$, there exists a neighborhood $V \subseteq U$ of $x$ in $X$ such that every map $f : S^k \to V$ extends to a map $\tilde{f} : B^{k+1} \to U$ for every $0 \leq k \leq n-1$, where $S^k$ and $B^{k+1}$ stand for the unit $k$-dimensional sphere and the unit $(k+1)$-dimensional ball of the $(k+1)$-dimensional Euclidean space, respectively.

Here is our main result.

**Theorem 2.1** ([15]). Let $(X, d)$ be an $n$-dimensional locally $(n-1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$ we simply skip the local connectedness assumption), and $\mu$ be a finite Borel measure on $X$ without atoms at the isolated points of $X$. Then the set of maps on $X$ with the chain recurrent set of $\mu$-measure zero is residual in the space of all maps on $X$.

**Remark 1.**

1. The interval case modulo Lebesgue measure of the theorem above was proved by Franzová [9].
2. Analogous results to Theorem 2.1, Corollary 2.2 and Theorem 3.1 (below) hold for the nonwandering set of a map.
3. The main theorem is false if $\mu$ has an atom at the isolated points of $X$.
4. It is well known that any $f$-invariant finite measure $\mu$ is supported by the set of recurrent points ([14]). In particular $\mu(\text{CR}(f)) > 0$. This implies that with all the assumptions of Theorem 2.1, a generic map $f$ does not preserve a given finite measure $\mu$.

We note that a manifold and a polyhedron are locally contractible. The $n$-dimensional universal Menger compactum $M^{2n+1}_n$ is obtained by a process of successively deleting cubes from the $(2n+1)$-cube (see [8, p. 96], [2], [11]). When $n = 0$, we obtain the Cantor set, and when $n = 1$, the Menger curve (which is referred to as the Menger sponge in the fractal literature). A compact $n$-dimensional Menger manifold
is a compact metric space locally homeomorphic to the $n$-dimensional universal Menger compactum $M_n^{2n+1}$. A topological characterization of a compact $n$-dimensional Menger manifold obtained by Bestvina [2] (cf. Anderson [1] for $n = 1$) is: a compact metric space $X$ is an $n$-dimensional Menger manifold if and only if it is $n$-dimensional, locally $(n - 1)$-connected, and satisfies the disjoint $n$-cells property. Kato, Kawamura, Tuncali and Tymchatyn [11] studied measure theoretic properties of the dynamics of Menger manifolds.

**Corollary 2.2** ([15]). Let $X$ be a compact and $n$-dimensional either manifold, Menger manifold or polyhedron with no isolated points, where $n \in \mathbb{N}$, and $\mu$ be a finite Borel measure on $X$. Then the set of maps on $X$ with the chain recurrent set of $\mu$-measure zero is residual in the space of all maps on $X$.

3. 鎖回帰集合の連結性

We give an application of the main theorem to dynamical systems of graph maps.

**Theorem 3.1** ([15]). Let $G$ be a graph. Then the set of maps on $G$ with the chain recurrent set being totally disconnected is residual in the space of all maps on $G$.

Motivated by the result above, we discuss the relation between the chain recurrent set and its connectivity. We need some definitions. A map $f : X \to X$ is said to be **chain transitive** if for every $x, y \in X$, $x$ can be chained to $y$.

The next is a slight extension of Theorem 2.8 in [6] to the case of the chain recurrent sets of arbitrary surjective maps.

**Proposition 3.2** ([15]). Let $f : X \to X$ be a surjective map on a compact metric space $(X, d)$. If the restriction $f|_{\text{CR}(f)} : \text{CR}(f) \to \text{CR}(f)$ is chain transitive, then $\text{CR}(f) = X$.

**Proposition 3.3** ([15]). Let $f : X \to X$ be a surjective map on a compact metric space $(X, d)$. If the chain recurrent set $\text{CR}(f)$ of $f$ is connected, then $\text{CR}(f) = X$.

**Remark 2.** If $f : X \to X$ is surjective and $\text{CR}(f) \neq X$, then $\text{CR}(f)$ must be disconnected by Proposition 3.3. Using a similar argument to that in the proof (without measurable argument) of Theorem 2.1, the property $\text{CR}(f) \neq X$ is generic if $X$ is an $n$-dimensional locally $(n - 1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$, skip the local connected condition, but on further condition “with an accumulation point”).
以上のことにより、連結性に関する次の問いは自然であるが、この話題についてはまた別の機会としたい。

**Question.** Is a totally disconnected property of the chain recurrent set generic?

**References**


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