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距離の構成と色々な次元

Constructions of metrics and dimensions

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The key word of this study is normal sequence of finite open covers. In general topology, the notion of normal sequence of open covers is one of the most useful tools for the study. In fact, the notion is the essence of metrizability of spaces. We obtain directly the numerical properties of normal sequences of finite open covers on a given separable metric space X and we give another proof of Pontrjagin-Schnirelmann theorem. Furtheremore, by use of normal sequences we can construct desired metrics d which control the values of $\log N(\epsilon, d)/|\log \epsilon|$. We investigate strong relations between topological dimension $\dim X$, metrics d and lower and upper box-counting dimensions $\underline{\dim}_B(X,d)$, $\overline{\dim}_B(X,d)$ of separable metric spaces X from a point of view of general topology. In particular, we construct *chaotic metrics* with respect to the determination of the upper and lower box-counting dimensions.

For a totally bounded metric d on X and $\epsilon > 0$, let

$$N(\epsilon, d) = \min\{|\mathcal{U}| \mid \mathcal{U} \text{ is a finite open cover of } X \text{ with } \mathrm{mesh}_d(\mathcal{U}) \leq \epsilon\},$$

where $|A|$ denotes the cardinality of a set A. Then the lower and upper box-counting dimensions of (X,d) are given by

$$\underline{\dim}_B(X,d) = \liminf_{\epsilon \to 0} \frac{\log N(\epsilon,d)}{|\log \epsilon|}, \quad \overline{\dim}_B(X,d) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon,d)}{|\log \epsilon|}.$$

If $\underline{\dim}_B(X,d) = \overline{\dim}_B(X,d)$, we define $\dim_B(X,d) = \underline{\dim}_B(X,d)$.
Suppose that $\mathcal{U}$ is an open cover of X and $A \subset X$. Let

$$St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \phi\}$$

$$\mathcal{U}^\star = \{St(U,\mathcal{U}) \mid U \in \mathcal{U}\} \text{ and } \mathcal{U}^\Delta = \{St(x,\mathcal{U}) \mid x \in X\}$$

$$\mathcal{U}^{\star^0} = \mathcal{U}, \quad \mathcal{U}^{\Delta^0} = \mathcal{U}$$

$$\mathcal{U}^{\star^{p+1}} = (\mathcal{U}^{\star^p})^\star = \{St(W,\mathcal{U}^{\star^p}) \mid W \in \mathcal{U}^{\star^p}\}$$

$$\mathcal{U}^{\Delta^{p+1}} = (\mathcal{U}^{\Delta^p})^\Delta = \{St(x,\mathcal{U}^{\Delta^p}) \mid x \in X\}.$$

An open cover $\mathcal{V}$ of X is a *star-refinement* of an open cover $\mathcal{U}$ of X if $\mathcal{V}^\star$ is a refinement of $\mathcal{U}$. An open cover $\mathcal{V}$ of X is a *delta-refinement* of an open cover $\mathcal{U}$ of X if $\mathcal{V}^\Delta$ is a refinement of $\mathcal{U}$. Let $\mathcal{U}_i$ $(i = 1,2,...)$ be open covers of X. Then the sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *normal star-sequence* if $\mathcal{U}_{i+1}$ is a star-refinement of $\mathcal{U}_i$ $(i = 1,2,...)$. Also, the sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *normal delta-sequence* if $\mathcal{U}_{i+1}$ is a delta-refinement of $\mathcal{U}_i$ $(i = 1,2,...)$. The sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *development* black of X if $\{St(x,\mathcal{U}_i) \mid i = 1,2,...\}$ is a neighborhood base for each point x of X. The following theorem is well known as Alexandroff-Urysohn metrization theorem.

Theorem 0.1. (Alexandroff-Urysohn metrization theorem) *A T_1-space X is metrizable if and only if there exists a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of open covers of X such that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a normal sequence and a development of X.*

For any normal space X ($\neq \phi$) and natural numbers k and p, we define the following indices:
(1) The index $\star_k^p(X)$ is defined as the least natural number m such that for every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = k$, there is an open cover $\mathcal{V}$ of X such that $|\mathcal{V}| \leq m$ and $\mathcal{V}^{\star^p} \leq \mathcal{U}$.
(2) The index $\Delta_k^p(X)$ is defined as the least natural number m such that for every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = k$, there is an open cover $\mathcal{V}$ of X such that $|\mathcal{V}| \leq m$ and $\mathcal{V}^{\Delta^p} \leq \mathcal{U}$.
By C_m^k, we denote the set of all m-element subsets of the set $\{1, 2, ..., k\}$ and by $\begin{pmatrix} k \\ m \end{pmatrix}$ its cardinality, i.e., $\begin{pmatrix} k \\ m \end{pmatrix} = \frac{k!}{m!(k-m)!}$.
For natural numbers k, m and p with $k \geq m$, we define the following indices;

$$\tilde{\Delta}(k; m; p) = \Sigma_{m \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \begin{pmatrix} k \\ j_1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \cdots \begin{pmatrix} j_{p-1} \\ j_p \end{pmatrix},$$

$$\tilde{\star}(k; m; p) = \Sigma_{m \geq j_1 \geq j_2 \geq \cdots \geq j_p \geq 1} \begin{pmatrix} k \\ j_1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \cdots \begin{pmatrix} j_{p-1} \\ j_p \end{pmatrix} j_p.$$

The following result follows from Bruijning-Nagata [4], Hashimoto-Hattori [8], Bogatyi-Karpov [2] and Koto-Matsumoto [10].

Theorem 0.2. *Let X be an infinite normal space with $\dim X = m < \infty$ and let k and p be natural numbers. Then*

$$\star_k^p(X) = \begin{cases} \tilde{\star}(k; k; (1/2)(3^p - 1)) = k[(1/2)(3^p - 1) + 1)]^{k-1} \ (k \leq m+1) \\ \tilde{\star}(k; m+1; (1/2)(3^p - 1)) \ (k \geq m+1), \end{cases}$$

$$\Delta_k^p(X) = \begin{cases} \tilde{\Delta}(k; k; 2^{p-1}) = (2^{p-1} + 1)^k - (2^{p-1})^k \ (k \leq m+1) \\ \tilde{\Delta}(k; m+1; 2^{p-1}) \ (k \geq m+1). \end{cases}$$

Lemma 0.3. (Koto-Matsumoto [10]) *Let X be an infinite separable metric space with $\dim X = m \geq 0$. Then the followings hold.*

1. *If $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a normal star-sequence of finite open covers of X and a development of X, then there is some i_0 such that*

$$|\mathcal{U}_i| \geq \tilde{\star}(m+1; m+1; (1/2)(3^{(i-i_0)} - 1))$$

for $i \geq i_0$. In particular, $\liminf_{i \to \infty} \frac{\log_3 |\mathcal{U}_i|}{i} \geq m$.

2. *If $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a normal delta-sequence of finite open covers of X and a development of X, then there is some i_0 such that*

$$|\mathcal{U}_i| \geq \tilde{\Delta}(m+1; m+1; 2^{(i-i_0-1)})$$

for $i \geq i_0$. In particular, $\liminf_{i \to \infty} \frac{\log_2 |\mathcal{U}_i|}{i} \geq m$.

Theorem 0.4. (Kato-Matsumoto [10], Kato [11]) *Let X be a nonempty separable metric space. Then*

$$\dim X = \min\{\liminf_{i\to\infty} \frac{\log_3 |\mathcal{U}_i|}{i} \mid \{\mathcal{U}_i\}_{i=1}^{\infty} \text{ is a normal star-sequence of finite open covers of } X \text{ and a development of } X\}$$

$$= \min\{\liminf_{i\to\infty} \frac{\log_2 |\mathcal{U}_i|}{i} \mid \{\mathcal{U}_i\}_{i=1}^{\infty} \text{ is a normal delta-sequence of finite open covers of } X \text{ and a development of } X\}.$$

Moreover, there exists a normal star (resp. delta)-sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of finite open covers of X which is a development of X such that

$$\dim X = \lim_{i\to\infty} \frac{\log_3 |\mathcal{U}_i|}{i} \ (\textit{resp.} \dim X = \lim_{i\to\infty} \frac{\log_2 |\mathcal{U}_i|}{i}).$$

Consider the following indices:

$$\star^p(X,\mathcal{U}) = \min\{|\mathcal{V}| \mid \mathcal{V} \text{ is a finite open covering of } X \text{ such that } \mathcal{V}^{\star^p} \leq \mathcal{U}\},$$

$$\Delta^p(X,\mathcal{U}) = \min\{|\mathcal{V}| \mid \mathcal{V} \text{ is a finite open covering of } X \text{ such that } \mathcal{V}^{\Delta^p} \leq \mathcal{U}\}.$$

Theorem 0.5. (Kato-Matsumoto [10]) *Let X be a normal space. Then*

$$\dim X = \sup\{\limsup_{p\to\infty} \frac{\log_3 \star^p(X,\mathcal{U})}{p} \mid \mathcal{U} \text{ is a finite open covering of } X\}$$

and

$$\dim X = \sup\{\limsup_{p\to\infty} \frac{\log_2 \Delta^p(X,\mathcal{U})}{p} \mid \mathcal{U} \text{ is a finite open covering of } X\}.$$

The next proposition implies that for any separable metric space X there is a natural bijection from the set of all totally bounded metrics on X to the set of Alexandroff-Urysohn metrics on X induced by normal sequences of finite open covers which are developments of X, up to Lipschitz equivalence.

Proposition 0.6. (Koto-Matsumoto [10]) *Let X be a separable metric space and let ρ be a totally bounded metric on X. Then there is a normal star (resp. delta)-sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of finite open covers of X such that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a development of X and ρ is Lipschitz equivalent to d, where d is the Alexandroff-Urysohn metric induced by $\{\mathcal{U}_i\}_{i=1}^{\infty}$.*

For separable metric spaces, we need the Alexandroff-Urysohn metrics induced by normal sequences of finite open covers. Define the functions $D_\star : X \times X \to [0,9]$ and $D_\Delta : X \times X \to [0,4]$ as follows:
($\star$) Let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a normal star-sequence of finite open covers of X and a development of X. For any pair of points x, y of X, we define the function $D_\star : X \times X \to [0,9]$ by

1. $D_\star(x,y) = 9$ if $\{x,y\}$ is not contained in any element of $\mathcal{U}_1$,

2. $D_\star(x,y) = 1/3^{(i-2)}$ if $\{x,y\}$ is contained in an element of $\mathcal{U}_i$ and $\{x,y\}$ is not contained in any element of $\mathcal{U}_j$ for $j > i$,

3. $D_\star(x,y) = 0$ if $\{x,y\}$ is contained in an element of $\mathcal{U}_i$ for each i.

(Δ) Let $\{\mathcal{U}_i\}_{i=1}^\infty$ be a normal delta-sequence of finite open covers of X and a development of X. For any pair of points x, y of X, we define the function $D_\Delta : X \times X \to [0,4]$ by

1. $D_\Delta(x,y) = 4$ if $\{x,y\}$ is not contained in any element of $\mathcal{U}_1$,

2. $D_\Delta(x,y) = 1/2^{(i-2)}$ if $\{x,y\}$ is contained in an element of $\mathcal{U}_i$ and $\{x,y\}$ is not contained in any element of $\mathcal{U}_j$ for $j > i$,

3. $D_\Delta(x,y) = 0$ if $\{x,y\}$ is contained in an element of $\mathcal{U}_i$ for each i.

Proposition 0.7. (see Nagami's book [14] for (2)) *Let X be a T_1-space.*

1. *If $\{\mathcal{U}_i\}_{i=1}^\infty$ is a normal star-sequence of finite open covers of X and a development of X, then $\{\mathcal{U}_i\}_{i=1}^\infty$ induces a totally bounded metric $d_\star$ on X such that for any $x, y \in X$,*
$$d_\star(x,y) \le D_\star(x,y) \le 6d_\star(x,y).$$

2. *If $\{\mathcal{U}_i\}_{i=1}^\infty$ is a normal delta-sequence of finite open covers of X and a development of X, then $\{\mathcal{U}_i\}_{i=1}^\infty$ induces a totally bounded metric d_Δ on X such that for any $x, y \in X$,*
$$d_\Delta(x,y) \le D_\Delta(x,y) \le 4d_\Delta(x,y).$$

By use of the above results, we obtain an another proof of the following well-known theorem.

Corollary 0.8. (Pontrjagin-Schnirelmann [19], Bruijning theorem [3] and Kato [11]) *Let X be a separable metric space. Then*
$$\dim X = \min\{\underline{\dim}_B(X,\rho) |\ \rho \text{ is a totally bounded metric for } X\}.$$
Moreover,
$$\dim X = \min\{\overline{\dim}_B(X,\rho) |\ \rho \text{ is a totally bounded metric for } X\}.$$

Lemma 0.9. (Kato [11]) *Let X be an infinite separable metric space and let $\{\mathcal{U}_i\}_{i=1}^\infty$ be a normal star-sequence of finite open covers and a development of X such that*
$$\lim_{i\to\infty} \frac{\log_3 |\mathcal{U}_i|}{i} = \dim X.$$
Then for any α, β with $\dim X \le \alpha \le \beta \le \infty$, there is a subsequence $\{\mathcal{U}_{i_j}\}_{j=1}^\infty$ of $\{\mathcal{U}_i\}_{i=1}^\infty$ such that
$$[\alpha,\beta] = \{\liminf_{k\to\infty} \frac{\log_3 |\mathcal{U}_{i_{n_k}}|}{n_k} \mid \{n_k\}_{k=1}^\infty \text{ is an increasing subsequence of natural numbers}\}.$$

Theorem 0.10. (Kato-Matsumoto [10], Kato [11]) *Let X be an infinite separable metric space. For any $\alpha, \beta \in [\dim X, \infty]$ with $\alpha \leq \beta$, there is a totally bounded metric $d = d_{\alpha\beta}$ on X such that*

$$[\alpha, \beta] = \{\liminf_{k\to\infty} \frac{\log N(\epsilon_k, d)}{|\log \epsilon_k|} \mid \{\epsilon_k\}_{k=1}^{\infty} \text{ is a decreasing sequence}$$

$$\text{of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0\}.$$

In particular, $\underline{\dim}_B(X, d) = \alpha$ and $\overline{\dim}_B(X, d) = \beta$.

Corollary 0.11. (Keesling [12], Kato-Matsumoto [10], Kato [11]) *Let X be a separable metric space with $\dim X \geq 1$. For any $\alpha, \beta \in [\dim X, \infty]$ with $\alpha \leq \beta$, there is a totally bounded metric $d = d_{\alpha\beta}$ on X such that*

$$[\alpha, \beta] = \{\liminf_{k\to\infty} \frac{\log N(\epsilon_k, d)}{|\log \epsilon_k|} \mid \{\epsilon_k\}_{k=1}^{\infty} \text{ is a decreasing sequence}$$

$$\text{of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0\}.$$

In particular, $\dim_H(X, d) = \underline{\dim}_B(X, d) = \alpha$ and $\overline{\dim}_B(X, d) = \beta$, where $\dim_H(X, d)$ is the Hausdorff dimension of (X, d).

Finally, we have the following problems.

Problem 0.12. *(1) Give an another proof of the following theorem of E. Marczewski (=Szpilrajn) by use of normal sequence of open covers: For a separable metric space X, $\dim X = \min\{\dim_H(X, d) \mid d \text{ is a metric on } X\}$.*
(2) Let X be a separable metric space with $\dim X \geq 1$. For any $\alpha, \beta, \gamma \in [\dim X, \infty]$ with $\alpha \leq \beta \leq \gamma$, does there esist a totally bounded metric d on X such that $\dim_H(X, d) = \alpha \leq \underline{\dim}_B(X, d) = \beta \leq \overline{\dim}_B(X, d) = \gamma$?
(3) What kinds of metrics can be embedded into Euclidean spaces, up to Lipschitz equivalence? If $\overline{\dim}_B(X, d) \leq n \in \mathbb{N}$, is it true that d can be embedded into $(2n+1)$-dimensional Euclidean space, up to Lipschitz equivalence? Note that if $\dim X = n$, there is a metric d on X such that d can be embedded into $(2n+1)$-dimensional Euclidean space with $\overline{\dim}_B(X, d) = n$.

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