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NEW FRACTALS WOVEN BY OLD ONES

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ABSTRACT. We propose a new combinatoric study of Fractals.

1. INTRODUCTION.

First, please look at the following figure.
This Figure 1 is made from two Apollonian gaskets. The Apollonian gasket is one of well-known fractals as in Fig.2. made from a circle packing. Make a copy of Fig.2 of smaller size and attach it along the circle of some hole of the original Fig.2; then we get Figure 1. The purpose of this lecture is to investigate such Fractals as in Figure 1. We will focus on "ideas" rather than detailed proofs, which will be published later sometime somewhere.

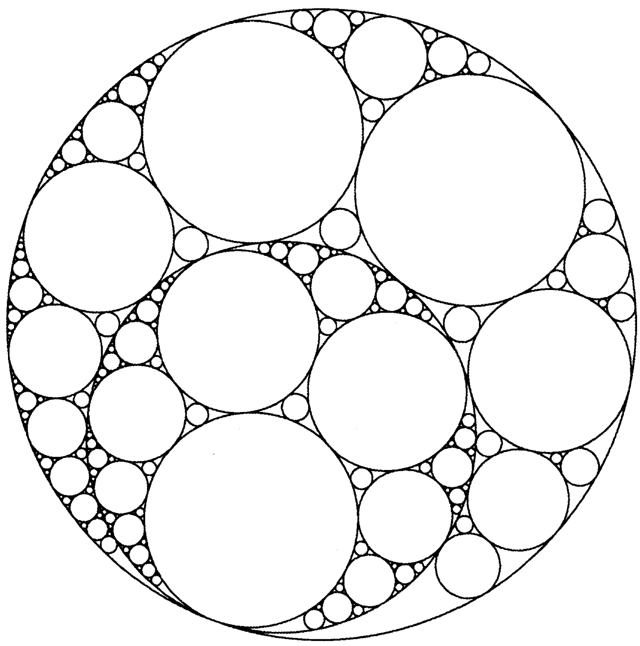

FIG. 1. Fusion of two Apollonian gaskets.

First, you may wonder if this Fig.1 is really a new fractal ?, or it may happen that it is homeomorphic with Fig.2. We can show later that this is really a new one.

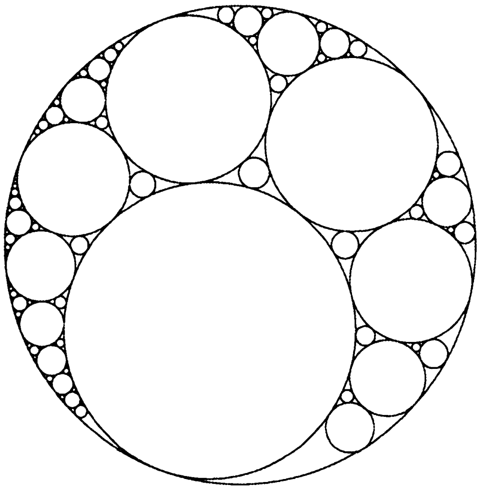

FIG. 2. The Apollonian Gasket.

Before we go into details, let us see how we can produce new fractals just by combining already well known fractals.

Example 1.
Consider in the plane the Sierpinski gasket S (Figure 3) made from the equilateral triangle of size (=side length) 1 with its base [0,1] on the x-axis.

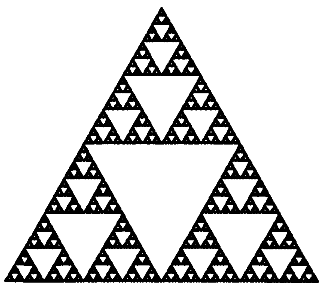

FIG. 3. The Sierpinski Gasket.

Let $S \cup \bar{S}$ be the join of S with its reflection $\bar{S}$ in the x-axis.
This is Figure 4.

Rotate S about the origin by degrees $k\pi/3\,(k < 6)$, then we get Figure 5 and this is called "the Hexagonal gasket" :

$$H = \bigcup_{k<6} S_k \;\; where \;\; S_k = \rho^k \cdot S \;\; and \;\; \rho = \exp(\mathrm{i}\frac{\pi}{3}).$$

This hexagonal figure seems to be well known, historically as old as the Sierpinski gasket itself.

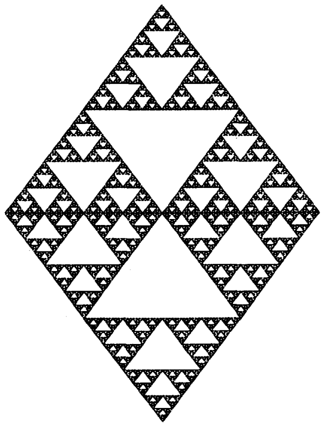

FIG. 4. Fusion of two Sierpinski gaskets.

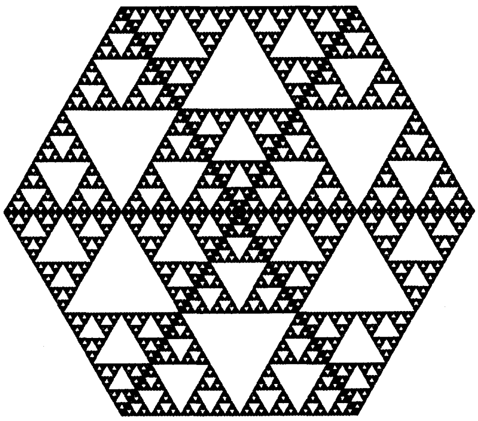

FIG. 5. The Hexagonal gasket.

Next, consider the middle hole of S, the upside-down triangle of size $1/2$. Attach here a copy of Sierpinski gasket of size $1/2$; this is Figure 6.

We can repeat this procedures of filling up holes by copies of the Sierpinski gasket. If we do this infinitely many times, we will finally get a fractal set without any holes of finite sizes, which we called the Sierpinski " *Sheet*" in the paper [2]. Figure 7 illustrates an example on the way to the Sheet.

Further, applying this technique to the triangular grid of the plane, we can get the "spread" version of the sheet; the Sierpinski " *Spread Sheet*." Figure 8 shows how it looks like on some intermediate step of its construction.

Observe that Figure 4 is very fundamental in the sense that its homeomorphic copy is contained in all of Figures 5,6,7,8.

Example 2. The Sierpinski carpet is a fractal made from the square as in Fig.9. Figure 10 illustrates the fusion of two Sierpinski carpets along

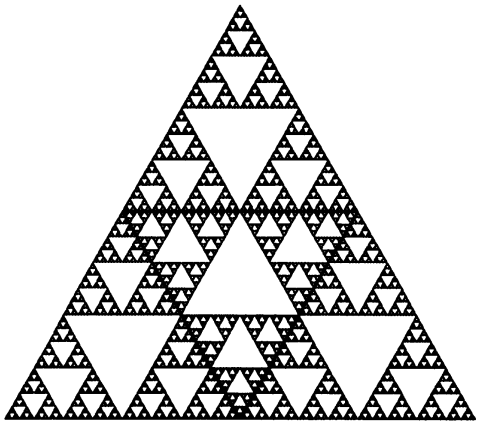

FIG. 6. Filling up the middle hole.

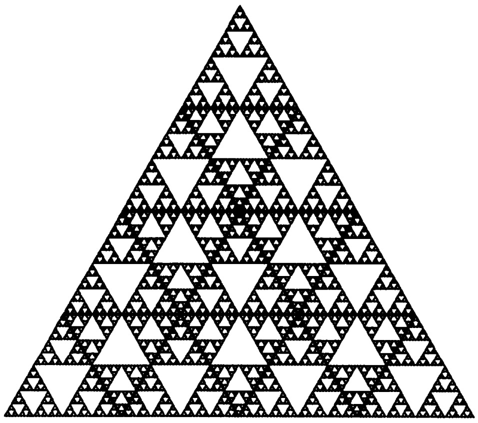

FIG. 7. On the way to the Sierpinski Sheet.

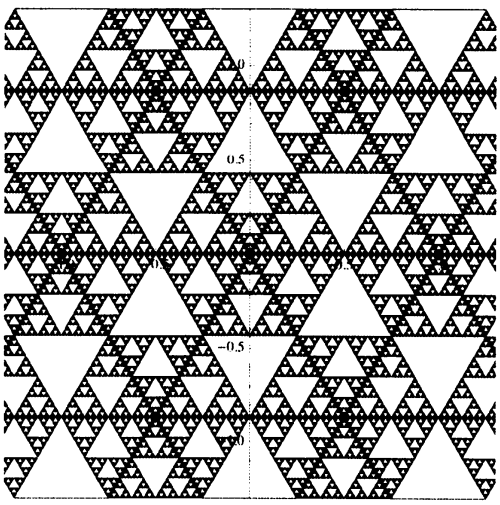

FIG. 8. On the way to the Sierpinski Spread Sheet.

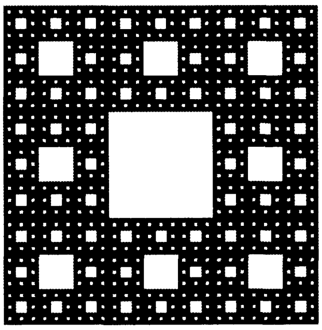

FIG. 9. The Sierpinski Carpet.

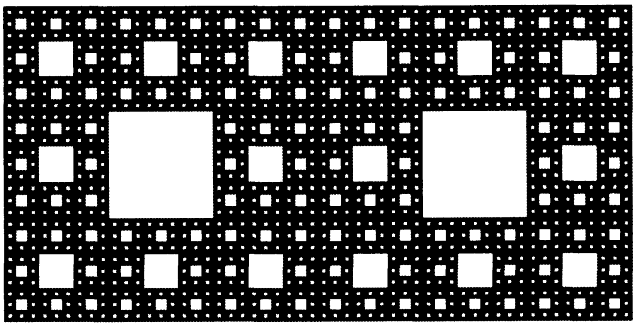

FIG. 10. Fusion of two Sierpinski carpets.

their boundary segments. In contrast with the above cases this space is topologically the same with one Sierpinski carpet, because of the celebrated theorem due to Whyburn that any two Sierpinski curves are homeomorphic. (For the definition of the Sierpinski curve see Section 4.)

2. PACKING GASKETS AND FUSION.

For a unified approach to the hitherto examples and constructions, it will be very natural to introduce the following notions we call "Packing Gaskets" and "Fusion."

(I) Packing Gaskets

Let D_0 be a homeomorph of the closed unit disc in the plane. In this D_0 we consider a packing $\{D_i \,|\, i \in \omega\}$ such that each D_i is a homeomorph of the disc. Let $C_i = \partial D_i$ denotes the boundary of D_i so that C_i is a simple closed curve homeomorphic with the circle. We assume the following conditions:

(1) $\bigcup_{i>0} D_i$ is a dense subset of D_0;
(2) open discs $\mathrm{O}(D_i) = D_i \backslash C_i$ $(i > 0)$ are disjoint, and moreover, $D_i \cap D_j = C_i \cap C_j$ is a finite set for each $0 \leqslant i < j < \omega$;

(3) the diameter $|D_i|$ of D_i tends to 0 as $i \to \infty$.

This packing $\{D_i \mid i \in \omega\}$ naturally determines a compact nowhere dense subset $A = D_0 \setminus \bigcup_{i>0} \mathrm{O}(D_i)$, which we call the Generalized Circle Packing Gasket, or briefly, "*Packing Gasket*." Each C_i will be called the "boundary Jordan circle", or briefly, "*boundary circle*". Note that a sphere minus one point is homeomorphic with the plane, so that the above definition can be done on a sphere rather than the plane. Also note that discs in the definition need not be the standard ones. Hence almost all examples we presented above are our Packing Gaskets. Exceptional are the non-compact Sheet or Spread Sheet (see Fig.s 7 and 8), but even those are expressed as the increasing union of (compact) packing gaskets.

Whyburn [6] [7] defined a term "E-continuum": A plane continuum M is called an *E-continuum* provided that for any $\epsilon > 0$ there are at most a finite number of complementary domains of M of diameter greater than ϵ. The condition (3) above implies that our packing gasket is an E-continuum, and so, all of Whyburn's results can be applied to the packing gaskets. Especially we get from Theorem 4.4 in [7]

Fact 1. *Every packing gasket A is a Peano continuum, that is, compact, connected and locally connected. Hence, every two points of A can be joined by an arc in A.*

(II) **Fusion**

Given two packing gaskets A_1 and A_2, identifying some parts of their boundary circles, we can get a new packing gasket $A_1 \sharp A_2$, which we call *fusion* of A_1 and A_2. We always assume the identified parts are arcs or boundary circles themselves. The idea of "fusion" will be clear if we see the hitherto examples. Figures 1 and 6 are the fusions by boundary circles, and Figures 4 and 10 are the fusions by arcs. Repeating fusions we get Figures 5, 7 and 8.

Inverting process of fusion will be called "*fission.*" A question that naturally arises is:

What kind of packing gasket A can be split into the form $A_1 \sharp A_2$?

We will come back to this problem after introducing the notion of "degree" by which we can distinguish given fractals topologically.

Note that "fusion" or "fission" will also be considered on one gasket.

3. Degree.

Given a point x in a topological space X. How many disjoint arcs can we draw from the point x ? The maximal number of such arcs is the "*degree*"

$\deg(x, X)$ and this is obviously the natural generalization of the notion of "degree of a vertex" in the graph theory. At first glance it seems for example that every point of the Sierpinski gasket would have an infinite degree. On the contrary, its degree does not exceed 4, which we want to show now. Another basic notion in the continuum theory called "order" is helpful : the order of a point x in X, denoted $\mathrm{ord}(x, X)$, is the minimum of a cardinal m such that for every neighborhood U of x in X we can find an open set V with $x \in V \subset U$ and $|\partial V| \leqslant m$, where ∂V means the topological boundary $cl_X(V)\backslash V$ of V. Noting that an arc is a connected subset, it is easy to see that the inequality $\deg(x, X) \leqslant \mathrm{ord}(x, X)$ holds.

Let S be the Sierpinski gasket made from the triangle Δ of vertices α_1, α_2, α_3. Since each vertex α_i has an arbitrary small neighborhood V such that $\partial_S V$ consists of two points, we have $\deg(\alpha_i, S) \leqslant \mathrm{ord}(\alpha_i, S) \leqslant 2$. Obviously, the two edges of Δ starting at α_i show $\deg(\alpha_i, S) \geqslant 2$. Hence we get $\deg(\alpha_i, S) = 2$ $(i = 1, 2, 3)$. Any vertex other than α_1, α_2, α_3 is a local cut point of S, and it is easy to see that $\deg(x, S) = 4$ for such a vertex x. Now let x be any non-vertex point. Such a point is either a point on some edge or a point on no edge; we call the former point *irrational* and the latter *inner*. We will show that $\deg(x, S) = 3$. Since x is not a vertex, we can choose a decreasing sequence of triangle-shaped neighborhoods of x such that

$$\Delta = \Delta_0 \supset \Delta_1 \supset \Delta_1^\circ \supset \Delta_2 \supset \Delta_2^\circ \supset \Delta_3 \supset \cdots$$

where Δ_n° is the triangle Δ_n without its three vertices α_1^n, α_2^n, α_3^n. Therefore, $\mathrm{ord}(x, S) \leqslant 3$. For each $i = 1, 2, 3$ let $L_i(x)$ be a path that traces points α_i^0, α_i^1, $\alpha_i^2 \cdots$ consecutively by edges.

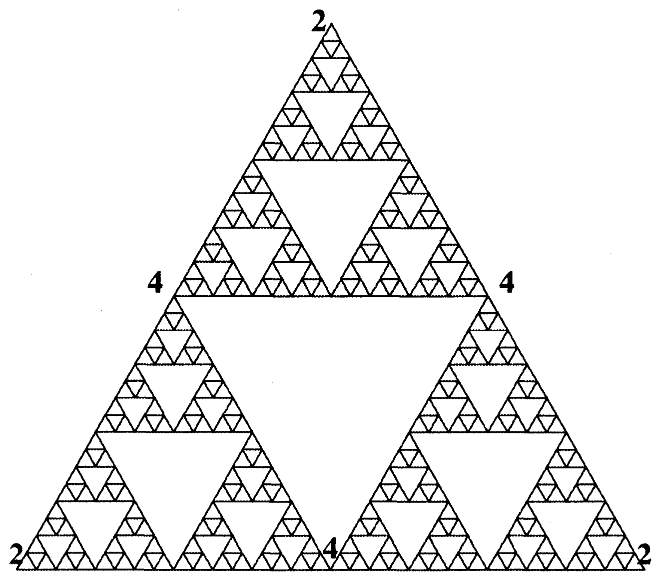

FIG. 11. Degrees of vertices.

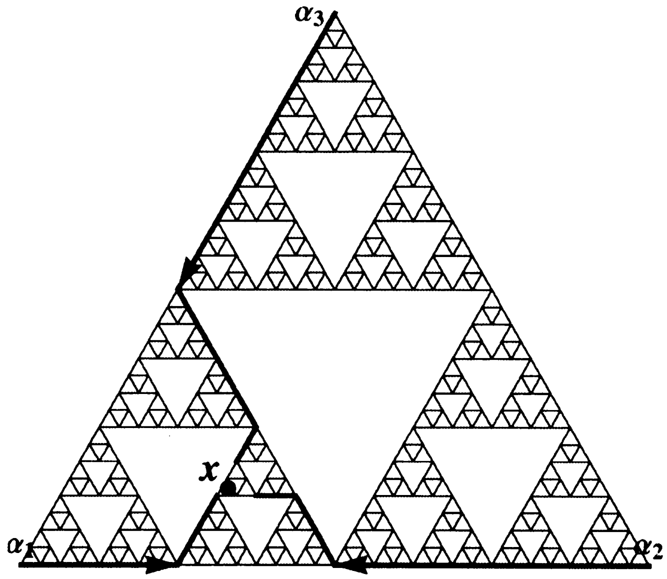

FIG. 12. x "inner".

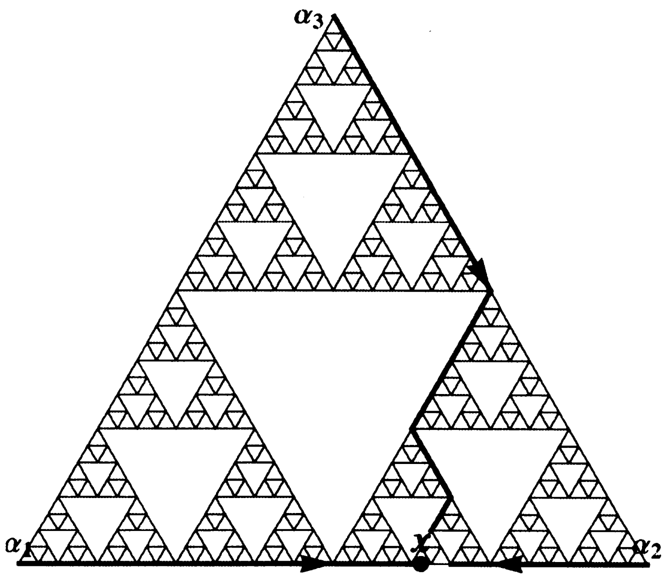

FIG. 13. x "irrational".

See Figures 12 and 13. Since these three paths $L_1(x)$, $L_2(x)$, $L_3(x)$ are disjoint and reaching to x, we get the result that $\deg(x, S) = 3$. Thus we can conclude that the degrees of points of the Sierpinski gasket does not exceed 4. Since the Apollonian gasket is made from two homeomorphic copies of Sierpinski gaskets with the corresponding three vertices identified, the points of degree 2 disappear, and so, the degrees of points of the Apollonian gasket are 3 or 4. Summarizing the above results, we get

Property 1. *The degrees of points of the Sierpinski gasket or the Apollonian gasket do not exceed 4.*

Now let us consider the example $S \cup \bar{S}$ of Figure 4 which is a fusion of two Sierpinski gaskets by the edge [0,1]. By directly counting the number of arcs starting a point, it will be easy to see that in this fusion the points 0 and 1 have degree 3, the vertices on (0,1) (that is, binary rationals between 0 and 1) have degree 6, and that the non-vertex points on [0,1] have degree 4. Hence the points of big degree 6 appear on the identified edge densely. Note that in general the degree does not decrease by a homeomorphic embedding. So, combining with Property 1 we get

Property 2. *In the fusion $S \cup \bar{S}$ of two Sierpinski gaskets the identified edge $[0,1]$ has a dense subset of points of degree 6. Consequently, $S \cup \bar{S}$ is not embeddable into either the Sierpinski gasket or the Apollonian gasket, and any homeomorphic embedding of $S \cup \bar{S}$ into itself maps the edge $[0,1]$ into itself.*

We can now examine Fig.1, the fusion of two Apollonian gaskets $A_1 \cup A_2$ along some boundary circle. By the same reason as above, any embedding of this fusion into itself maps the identified boundary circle into itself. Since any homeomorphic embedding of the circle into itself is an onto homeomorphism, we can conclude that

Property 3. *In the fusion (Figure 1) of two Apollonian gaskets along some boundary circle any homeomorphic embedding of this fusion into itself maps the identified boundary circle onto itself.*

4. Sierpinski Curve or Carpet.

Now we consider the Sierpinski curve or carpet, and this is topologically quite different from the Sierpinski gasket; for example, we will see that in a Sierpinski curve any point has the degree $\mathbf{c}$, the cardinal of the continuum.

A *Sierpinski curve* is, by definition, a compact, connected, locally connected, nowhere-dense subset of the plane that has the property that any two boundaries of complementary domains are pairwise-disjoint simple closed curves. The most well known example of a Sierpinski curve is the Sierpinski carpet made from the square in the plane. Whyburn showed that

Fact 2 (Whyburn [4]). *Any two Sierpinski curves are homeomorphic.*

Let $A = D_0 \setminus \bigcup_{i>0} \mathrm{O}(D_i)$ be a Sierpinski curve so that the boundary circles $C_i = \partial D_i$ $(i \in \omega)$ are disjoint.

Collapse each C_i $(i > 0)$ to a point c_i, then we get a quotient space homeomorphic with the standard disc where points c_i $(i > 0)$ form a countable dense subset. That this quotient space is Hausdorff is ensured by our condition (3) of Packing Gaskets in Section 2 that the diameter $|D_i|$ decreases to 0 as $i \to \infty$. Since in the standard disc it is easy to find $\mathbf{c}$-many disjoint arcs avoiding any given countable set, we get the following result. (For the detailed proof see [3].)

Property 4. *Every point of the Sierpinski curve has the degree* $\mathbf{c}$, *the cardinal of the continuum.*

From this result we know that the Sierpinski carpet can not be embedded into any packing gaskets of finite degrees such as the Sierpinski gasket or the Apollonian gasket. But the converse is true; namely,

Fact 3. *Any packing gasket is embeddable into the Sierpinski carpet.*

This is due to the well-known universal property of the Sierpinski curve or carpet that any compact nowhere dense subset in the plane is embeddable into it. Without recourse to this general property, we can give an alternative, very geometric proof by using fusions:

Proof. Let $A = D_0 \setminus \bigcup_{i>0} \mathrm{O}(D_i)$ be an arbitrary packing gasket with the boundary circles $C_i = \partial D_i$ $(i \in \omega)$, and place it on the sphere. Attach along every boundary circle C_i $(i \in \omega)$ a Sierpinski curve B_i. Then the resulting fusion $\widehat{A} = A \sharp B_0 \sharp B_1 \sharp B_2 \cdots$ is another Sierpinski curve, because all the boundary circles of A are melted away in $\widehat{A}$. □

As for "fusion" of Sierpinski curves, due to Fact 2, nothing new is produced by fusion, and this is what we pointed out at the end of Section 1. Figure 9 is homeomorphic with Figure 10 ! In spite of this fact, if we repeat fusions infinitely many times, we can get a new topological object, called "the universal 1-dimensional pseudo-boundary of the Euclidean plane", which is introduced by Geoghagen and Summerhill and well studied in the field of infinite-dimensional topology (see [1],[2]).

5. Splitting or Fission.

Now let us examine the inverting process of fusion which we may call *fission*. Given a packing gasket A, can we split it into two packing gaskets $A = A_1 \sharp A_2$?
For the Sierpinski curve or carpet the answer is definitely Yes, as illustrated in Example 2, thanks to Fact 2. We will show that the answer is No for the Sierpinski gasket. Before going into the proof, we need to clarify the notion of our splitting or fission.

Let $A = D_0 \setminus \bigcup_{i>0} \mathrm{O}(D_i)$ be a packing gasket with the boundary circles $C_i = \partial D_i$ $(i \in \omega)$. Let l be a curve in A that is a simple closed curve or a finite union of disjoint arcs. If A is split into two packing gaskets $A = A_1 \sharp A_2$ in such a way as $A_1 \cap A_2 = l$, we call this splitting "*fission*" and the curve l as its "*fission curve*." In case l is a finite union of disjoint arcs, we further assume that the end points of the arcs belong to some boundary circles. Note that, because of the condition (2) of packing gasket (see Section 2), this fission curve l must satisfy the following condition:

(*) $l \cap C_i$ is a finite set for each $i \in \omega$.

Property 5. *The Sierpinski gasket does not contain any arc l as in the above (*). Consequently, neither the Sierpinski gasket nor the Apollonian gasket can be decomposed as the fusion of two packing gaskets.*

Proof. Let S be the Sierpinski gasket made from the triangle Δ of vertices α_1, α_2, α_3, and let V denote the set of all vertices other than α_1, α_2, α_3, that is, the set of all points of degree 4. Suppose there exists an arc l satisfying the above condition (*). Let a, b be the end points of l. Since in the Sierpinski gasket any two points are separated by some finite subset of V, we can find a vertex $v \in V \cap l$ which separates a and b. Let e_i $(i = 1, 2, 3, 4)$ be the four edges starting from v. Then our condition (*) implies that $l \cap (e_1 \cup e_2 \cup e_3 \cup e_4)$ is a finite set. By shortening the length of the edges e_i we can make that $l \cap (e_1 \cup e_2 \cup e_3 \cup e_4) \setminus \{v\}$ is an empty set. Then l and e_i $(i = 1, 2, 3, 4)$ form six disjoint arcs starting from v. Hence $\deg(v, S) \geqslant 6$ and this contradicts Property 1. $\square$

The notion of "fission" will be closely related with that of "percolation."

6. Appendix.

With respect to fusion or fission the Sierpinski gasket and the Sierpinski carpet are the extremes. The former can not be split, but the latter can be very easily. We may consider various packing gaskets which are in-between these two extremes. Let us call a sequence $S_1, S_2, \cdots S_n$ of distinct boundary circles as a *loop* of *size* n if each $S_i \cap S_{i+1}$ is non-empty for each $1 \leqslant i \leqslant n$, where $S_{n+1} = S_1$. Then we can define the *loop size* of a packing gasket as a minimal number of sizes of such loops in the gasket. The loop size of the Apollonian gasket is 3. The construction of the Apollonian gasket was already generalized to make packing gaskets of loop size $\geqslant 4$, by geophysicists; they call "space-filling bearings." See for example, G.Oron, H.J.Herrmann "Generalization of space-filling bearings to arbitrary loop size" J.Phys.A: Math. Gen. 33(2000), 1417-1434.

We can also consider an example which is almost a Sierpinski curve except that some boundary circles are tangent on the outmost circle. See Fig.14. Precisely speaking, consider a packing gasket $A = D_0 \setminus \bigcup_{i>0} \mathrm{O}(D_i)$ with two kinds of boundary circles $C_i = \partial D_i$ $(i \in \omega \setminus \{0\} = \Lambda_0 \oplus \Lambda_1)$ such that

(1) each C_i $(i \in \Lambda_0)$ is tangent at one point to the outmost circle C_0, and those tangent points form a dense subset of C_0;

(2) each C_i $(i \in \Lambda_1)$ is disjoint from others C_j $(j \in \omega \setminus \{i\})$.

Researches on this kind of packing gaskets will be our future task.

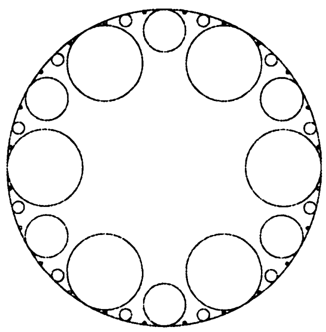

FIG. 14. C_i ($i \in \Lambda_0$).

Finally, please enjoy the following picture, Fig.15.

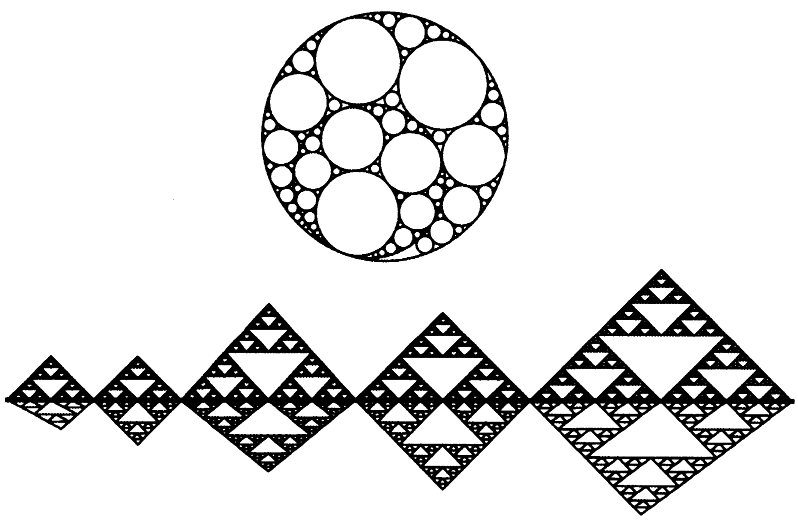

FIG. 15. The Apollo, Pyramids and the Nile.

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