数理解析研究所講究録
第 1729 巻 2011 年 167-177

Determinacy of Infinite Games and Inductive Definition in Second Order Arithmetic

増子加奈 (Kana Mashiko)
東北大学・理学研究科 田中一之 (Kazuyuki Tanaka)
Mathematical Institute, Tohoku University
東北大学・理学研究科 吉居啓輔 (Keisuke Yoshii)
Mathematical Institute, Tohoku University

1 Introduction

In this paper, we review some recent works on proof-theoretic strength of determinacy of infinite games.

In 1953, Gale and Stewart introduced the following infinite games. Two players I and II alternatively choose an element from $\mathbb{N}$ and construct an infinite sequence $f \in \mathbb{N}^{\mathbb{N}}$. Player I is said to win the game if the resulting sequence f belongs to a given set A, and player II wins otherwise. We denote this game G_A, and call A the winning set. If A is an open set (a closed set, or a Borel set), G_A is called an open game (a closed game, or a Borel game, respectively). A strategy for player I (for player II) is a function σ from $\mathbb{N}^{<\mathbb{N}}_{\mathrm{even}}$ to $\mathbb{N}$ (a function τ from $\mathbb{N}^{<\mathbb{N}}_{\mathrm{odd}}$ to $\mathbb{N}$), where $\mathbb{N}^{<\mathbb{N}}_{\mathrm{even}}$ ($\mathbb{N}^{<\mathbb{N}}_{\mathrm{odd}}$) is a set of finite sequences of natural numbers with even (odd) length. A game G_A is said to be determinate if one of the players has a winning strategy in G_A, i.e. a strategy with which the player always wins.

Researches on the determinacy of infinite games have been conducted in descriptive set theory. It is provable in ZFC that a Borel game is determinate, but not provable in ZFC that any analytic game is determinate. These facts simply represent that the strength of the determinacy of G_A varies depending on the complexity of the winning set A.

Aside from descriptive set theory, researches classifing the strength of determinacy have been started by J. Steel and Tanaka, based on the Reverse Mathematics program. The goal of the program is to answer the next questions: *What set existance axioms are needed to prove the theorems of ordinary mathematics*? See Simpson [6] for the major results. It is now known that the determinacy of Δ^0_4 games is not provable over Z_2 ([3]), and the determinacy of Σ^0_3 games is provable over $\Pi^1_3\text{-}\mathsf{CA}_0$, yet the the highest class of the games whose exact determinacy strength has been pinned down is Δ^0_3 (see [2]). In [2], the difference hierarchy $(\Sigma^0_2)_\alpha, 0 < \alpha < \omega_1^{\mathrm{ck}}$, between Σ^0_2 and Δ^0_3 were studied and the result on Δ^0_3 games was obtained as the limit. For more refined classes, Wadge classes were studied by Nemoto in [4]. Medsalem, Nemoto, Tanaka also proved that below Δ^0_3 the strength of the determinacy of the games differs between Cantor space and Baire space.

The following diagram shows the results on determinacy of infinite games in second order arithmetic. The left most colum contains subsystems of second order arithmetic from weaker to stronger. The center colum and the right most colum contain classes of the games in Baire space and Cantor spaces, respectively. Each row represents that a certain axiom is equivalent to the determinacy of the corresponding games over appropriate systems (RCA_0, but over $\mathsf{ACA}_0 + \Pi^1_3\text{-}\mathsf{TI}$ for the last row).

Let $\mathcal{C}$ be a class of formulas. $\mathcal{C}$-ID(resp. $\mathcal{C}$-MI) is an axiom scheme which guarantees a pre-wellordering set defined by the inductive definitions (monotone inductive definitions). It is proved that Σ^1_1-ID and Σ^1_1-MI are equivalent over RCA_0 by Medsalem and Tanaka in [2]. By using the same technique, we proved that:

Theorem 1.1. $\mathsf{RCA}_0 \vdash \Sigma^1_1\text{-}\mathsf{IDTR} \leftrightarrow \Sigma^1_1\text{-}\mathsf{MITR}$.

$\mathcal{C}$-IDTR means a transfinite iteration of $\mathcal{C}$-ID along a given well-ordering. We give the formal definition in section 2. Also, $\mathcal{C}$-Det (resp. $\mathcal{C}$-Det^*) represents the determinacy of games in Baire space (in Cantor space).

In this paper we prove the following:

Theorem 1.2. The following equivalences hold in RCA_0.

1. $\Sigma^1_1\text{-}\mathsf{ID} \leftrightarrow \mathsf{Sep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det} \leftrightarrow \mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}$,

2. $\Sigma^1_1\text{-}\mathsf{IDTR} \leftrightarrow \Delta((\Sigma^0_2)_2)\text{-}\mathsf{Det}$,

3. $[\Sigma^1_1]^k\text{-}\mathsf{IDTR} \leftrightarrow \Delta((\Sigma^0_2)_{k+1})\text{-}\mathsf{Det} \leftrightarrow \Delta((\Sigma^0_2)_{k+2})\text{-}\mathsf{Det}^*$ $(k \geq 1)$.

In the diagram, the shaded parts are obtained new.

Subsystem of SOA	Determinacy in Baire space	Determinacy in Cantor space
ATR_0	Δ^0_1 Σ^0_1	Δ^0_2 Σ^0_2
$\mathsf{ATR}_0 + \Sigma^1_1$ induction		$\mathsf{Sep}(\Delta^0_1, \Sigma^0_2)$
$\Pi^1_1\text{-}\mathsf{CA}_0$	$\Delta((\Sigma^0_1)_2)$ $(\Sigma^0_1)_2$	$\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$
$\Pi^1_1\text{-}\mathsf{TR}_0$	Δ^0_2	$\Delta((\Sigma^0_2)_2)$
$\Sigma^1_1\text{-}\mathsf{ID}_0$	Σ^0_2 $\mathsf{Sep}(\Delta^0_1, \Sigma^0_2)$ $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$	$(\Sigma^0_2)_2$
$\Sigma^1_1\text{-}\mathsf{IDTR}_0$	$\Delta((\Sigma^0_2)_2)$	$\Delta((\Sigma^0_2)_3)$
$[\Sigma^1_1]^2\text{-}\mathsf{ID}_0$	$(\Sigma^0_2)_2$	$(\Sigma^0_2)_3$
$[\Sigma^1_1]^2\text{-}\mathsf{IDTR}$	$\Delta((\Sigma^0_2)_3)$	$\Delta((\Sigma^0_2))_4$
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^k\text{-}\mathsf{ID}_0$	$(\Sigma^0_2)_k$	$(\Sigma^0_2)_{k+1}$
$[\Sigma^1_1]^k\text{-}\mathsf{IDTR}$	$\Delta((\Sigma^0_2)_{k+1})$	$\Delta((\Sigma^0_2))_{k+2}$
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^{\mathrm{TR}}\text{-}\mathsf{ID}_0$	Δ^0_3	Δ^0_3

2 Determinacy of Games and Inductive Definitions

2.1 Inductive Definitions

First, we see the general definition of the inductive defition.

Definition 2.1. $X \subset \mathbb{N}$ is a pre-well ordering if the following conditions hold:

1. $\forall x(x \leq_X x)$
2. $\forall x \forall y \forall z((x \leq_X y \wedge y \leq_X z) \rightarrow x \leq_X z)$,
3. $\forall x \forall y((x, y \in \mathrm{field}(X)) \rightarrow (x \leq_X y \vee y \leq_X x))$,

A pre-well ordering is a well-ordering if the next aditional condition is satisfied.

4 $\forall x \forall y((x \leq_X y \wedge y \leq_X x) \rightarrow x = y)$.

A mapping from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})$ is called an operator.

If "$x \in \Gamma(X)$" is represented by a $\mathcal{C}$-formula, Γ is called a $\mathcal{C}$ operator. If, for any X and Y, $Y \subseteq X$ implies $\Gamma(X) \subset \Gamma(Y)$, then Γ is called a monotone operator. By mon-$\mathcal{C}$ we denote the set of all monotone operators.

Definition 2.2. The axiom $\mathcal{C}$-ID is the following scheme: for any $\mathcal{C}$ operator $\Gamma(X)$, there exists a pre-wellordering set $W \subset \mathbb{N} \times \mathbb{N}$ such that:

1. W is a pre-wellordering on its field $F = \mathrm{field}(W)$,
2. $\forall x \in F(W_x = \Gamma(W_{<x}) \cup W_{<x})$,
3. $\Gamma(F) \subset F$.

where $W_x = \{y \in F : y \leq_W x\}$, $W_{<x} = \{y \in F : y <_W x\}$.

Especially, we write [mon-$\mathcal{C}$]-ID as $\mathcal{C}$-MI. For $\mathcal{C}$-MI, the second condition above becomes

$$\forall x \in F(W_x = \Gamma(W_{<x}))$$

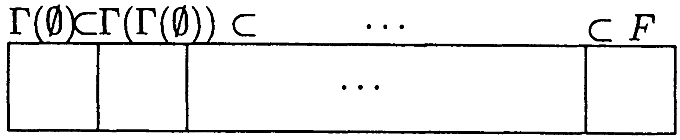

Figure 1: $\mathcal{C}$-MI

2.2 Σ^1_1-ID and Determinacy of Games

In this section we consider Σ^1_1 inductive definition. As the researches of Reverse Mathematics have been proceeded, it is shown that most of mathematics theorems are equivalent to one of Big 5 ($\mathsf{RCA}_0, \mathsf{WKL}_0, \mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1$-$\mathsf{CA}_0$ from weaker to stronger). However, in [8], Tanaka defined Σ^1_1-ID, much stronger than Π^1_1-CA_0 and proved it is equivalent to Σ^0_2-Det.

Theorem 2.3 (Tanaka [8]). Over RCA_0, the following are equivalent.

- Σ^0_2-Det,
- Σ^1_1-MI.
- Σ^1_1-ID

Then, it must be natural to consider more complex game than Σ^0_2. We consider the next classes of formulas.

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be classes of formulas. We denote the classes of formulas in the form $\phi \land \psi$ $(\phi \in \mathcal{C}, \psi \in \mathcal{C}')$ as $\mathcal{C} \land \mathcal{C}'$, and $\neg\psi$ $(\psi \in \mathcal{C})$ as $\neg\mathcal{C}$.

Definition 2.5. For $n \geq 1$ and $k < \mathbb{N}$, we define the classes of the formulas $(\Sigma^0_n)_k, (\Pi^0_n)_k$ as follows.

- $(\Pi^0_n)_k = \neg(\Sigma^0_n)_k$,
- $(\Sigma^0_n)_1 = \Sigma^0_n$,
- $(\Sigma^0_n)_k = \Sigma^0_n \land (\Pi^0_n)_{k-1}$ if $k > 1$.

Lemma 2.6. For any $n \geq 1$ and $k < \mathbb{N}$, the following hold.

- $(\Sigma^0_n)_k = \Pi^0_n \land (\Sigma^0_n)_{k-1}$ if k is even.
- $(\Sigma^0_n)_k = \Sigma^0_n \lor (\Sigma^0_n)_{k-1}$ if k is odd.

Definition 2.7. Let $\mathcal{C}$ be a class of formulas.

- If there exists a $\neg\mathcal{C}$-formula φ' such that
$$\forall f \in \mathbb{N}^{\mathbb{N}}(\varphi(f) \leftrightarrow \varphi'(f)),$$
we call a $\mathcal{C}$-formula φ a $\Delta(\mathcal{C})$-formula.
- If there exist $\mathcal{C}$-formula ψ, $\neg\mathcal{C}'$-formula η, $\mathcal{C}'$-formula η' such that
$$\forall f \in X^{\mathbb{N}}(\varphi(f) \leftrightarrow ((\psi(f) \land \eta(f)) \lor (\neg\psi(f) \land \eta'(f)))),$$
we call a formula φ a $\mathsf{Sep}(\mathcal{C}, \mathcal{C}')$-formula.

We show the determinacy of $\mathsf{Sep}(\Delta^0_1, \Sigma^0_2)$ and that of $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$ are equivalent, while the determinacy statements for those two classes are known to be separated in Cantor space.

Theorem 2.8. Over RCA_0, the following are equivalent.

- Σ^1_1-ID_0,
- $\mathsf{Sep}(\Delta^0_1, \Sigma^0_2)$-$\mathsf{Det}$,
- $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}$.

2.3 Iteration of Inductive Definition

We here consider sets constructed by transfinite iteration of the inductive definitions. The following is a formal definition of $\mathcal{C}$-IDTR.

Definition 2.9. The axiom $\mathcal{C}$-IDTR is the following scheme.

For any well-ordering $\preceq$ and $\mathcal{C}$ operator Γ, there exists a sequence of reflexive sets $\langle V^r : r \in \mathrm{field}(\preceq)\rangle$ such that: for any $r \in \mathrm{field}(\preceq)$,

1. V^r is pre-wellordering on its field $F^r = \mathrm{field}(V^r)$,
2. $\forall x \in F^r(V^r_x = \Gamma^{F^{\prec r}}(V^r_{<x}) \cup V^r_{<x})$,
3. $\Gamma^{F^{\prec r}}(F^r) \subset F^r$,

where $V^r_x = \{y \in F^r : y \leq_{V^r} x\}$, $V^r_{<x} = \{y \in F^r : y <_{V^r} x\}$, $F^{\prec r} = \bigcup\{F^{r'} : r' \prec r\}$.

Especially, [mon-$\mathcal{C}$]-IDTR is written as $\mathcal{C}$-MITR. For $\mathcal{C}$-MITR, the second condition of the definition may be replaced by:

$$\forall x \in F^r(V^r_x = \Gamma^{F^{\prec r}}(V^r_{<x})).$$

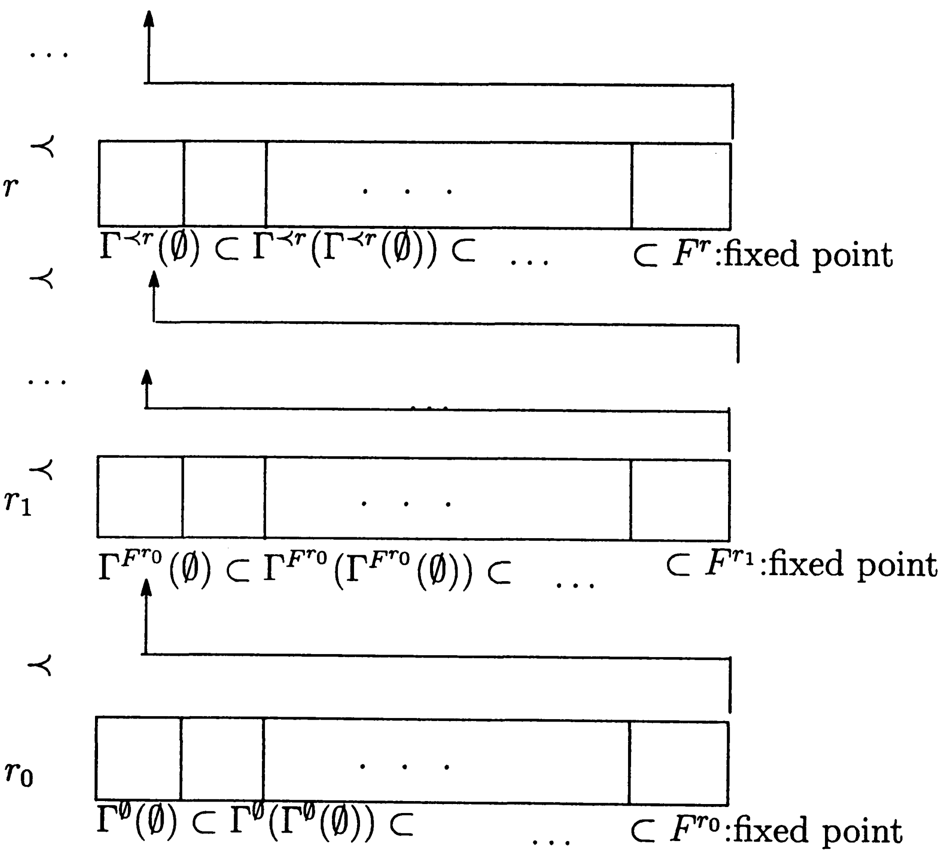

Figure 2: $\mathcal{C}$-IDTR

As mentioned above, in $\mathcal{C}$-IDTR the inductive definitions with $\mathcal{C}$ operator Γ^Y are iterated transfinitely many times. More precisely, we apply the inductive definition and obtain a pre-wellordering V^{r_0} on its field F^{r_0}. Then, by taking F^{r_0} as an oracle, we again apply the inductive definition with operator $\Gamma^{F^{r_0}}$. Then, we obtain a pre-wellordering V^{r_1} on its field $F^{r_1} = \text{field}(V^{r_1})$. We iterate this procedure transfinitely many times along the given well-ordering $\preceq$, and then we obtain the sequence of pre-wellorderings $\langle V^r : r \in \text{field}(\preceq)\rangle$.

2.4 Σ^1_1-IDTR and Determinacy of the Games

In this section, we consider Σ^1_1-IDTR and the corresponding determinacy of the game.

We first see the next theorem.

Theorem 2.10. Over RCA_0, the class $\Delta((\Sigma^0_n)_{k+1})$ is equivalent to the class $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_k)$.

Using this theorem, we prove the main results of this paper which says over RCA_0 that Σ^1_1-IDTR and $\Delta((\Sigma^0_2)_2)$-Det are equivalent.

We give the outline of the proof for the next lemma, because this is the most difficult part, and furthermore the similar technique can be used when we prove $\mathcal{C}$-ID(TR) in assuming some determinacy of game above $\Sigma^0_2 \wedge \Pi^0_2$.

Lemma 2.11. $\mathsf{ACA}_0 \vdash \mathsf{Sep}(\Delta^0_2, \Sigma^0_2)\text{-Det} \rightarrow \Sigma^1_1\text{-IDTR}$.

We explain the idea of the proof. We construct a $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-game whose determinacy implies the existence of $\langle V^r : r \in \text{field}(\preceq)\rangle$ defined by Σ^1_1-IDTR. A play of the game starts with player I's choosing a pair (y^*, r^*) and arrising a question "$y^* \in F^{r^*}$?".

For this question, player II answers 1, which means "Yes", if he thinks $y^* \in F^{r^*}$. Then, he constructs $V^{r_0}, V^{r_1}, V^{r_2}, \ldots$ acoording to a given well-ordering $\preceq$. From now, the player who constructs the pre-wellorderings $\langle V^r : r \prec r^*\rangle$ is called Pro. Conversely, the other player, who tries to point out mistakes in the construction of Pro, is called Con.

Pro wins the game if he can construct the pre-wellorderings satisfying the all conditions of Σ^1_1-IDTR with $y^* \in F^{r^*}$.

As Pro constructs the pre-wellorderings, Con can points out (if any) unsuitable elements in the Pro's construction. Con wins if he finds out an infinite descending sequence through Pro's construction V^{r^*}. Or, Con may point out a possible error in $V^{r^{**}} (r^{**} \prec r^*)$. If so, they start a new stage where the player who asserts $y \in V^{r^{**}}$ becomes Pro and the other Con, and new Pro constructs $\langle V^r : r \prec r^{**}\rangle$.

We note that if Con points out only finitely many times in V^{r^*}, it means that Pro and Con agree on the initial segment of V^{r^*} below the last challenge, and Con wins if his point-out is really suitable.

Easily, we can show that player II always has a winning strategy τ in this game. We define W^r for each $r \in \mathsf{field}(\preceq)$ by using the winning starategy τ. Then, $\langle W^r : r \in \mathsf{field}(\preceq)\rangle$ is the desired sequence of pre-wellorderings.

Lemma 2.12. $\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \Sigma^1_1\text{-}\mathsf{IDTR} \rightarrow \mathsf{Sep}(\Delta^0_2, \Sigma^0_2)\text{-}\mathsf{Det}$.

Then we have the following theorem.

Theorem 2.13. Over RCA_0, the following are equivalent.

- Σ^1_1-IDTR,
- Σ^1_1-MITR,
- $\Delta((\Sigma^0_2)_2)$-Det.

2.5 $[\Sigma^1_1]^k$-IDTR and Determinacy of Games

We have seen the cases where only one operator is used. However, in [2], inductive definitions with many operators are introduced.

For the simplicity, we here consider only the case $k = 2$.

We note that the proof of theorem 2.15 and theorem 2.17 are proved by using basically same technique as we introduced in theorem 2.11.

Definition 2.14. Let S_0 and S_1 be sets of operators. The axiom $[S_0, S_1]$-ID is the following scheme.
For any operators $\Gamma_0 \in S_0, \Gamma_1 \in S_1$, there exist $W \subset \mathbb{N} \times \mathbb{N}$, $V^\infty, \langle V^x \subset \mathbb{N} \times \mathbb{N} \mid x \in F = \mathsf{field}(W)\rangle$ such that the following are all satisfied.

1. W is pre-wellordering on its field F,
2. $\forall x \in F \cup \{\infty\}$
    - V^x is pre-wellordering,
    - $\forall y \in F^x(V^x_y = \Gamma_0^{W_{<x}}(V^x_{<y}) \cup V^x_{<y})$,
    - $W_x = \Gamma_1(F^x) \cup W_{<x}$,
    - $\Gamma_0^{W_{<x}}(F^x) \subset F^x$,
3. $W_\infty = W_{<\infty} = F$.

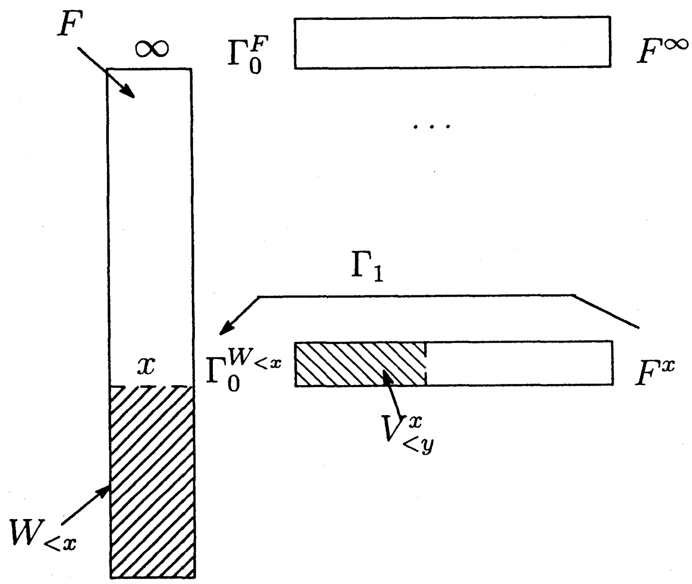

Figure 3: $[\mathcal{C}]^2$-ID

From this definition, the next theorem was proved in [2].

Theorem 2.15 (Medsalem, Tanaka [2]). Assume that $0 < k < \mathbb{N}$. Over RCA_0, the following are equivalent.

- $(\Sigma^0_2)_k$-Det,
- $[\Sigma^1_1]^k$-ID.

Combining the idea of iteration of inductive definition and the above definition with finitely many Σ^1_1 operators, we define an axiom scheme $[\Sigma^1_1]^k$-IDTR, and we prove that it is equivalent to $\Delta((\Sigma^0_2)_{k+1})$-Det.

We here also see the definition in the case $k = 2$.

Definition 2.16. The axiom scheme $[S_0, S_1]$-IDTR asserts the following. For any well-ordering $\preceq$ and any $\Gamma_0 \in S_0, \Gamma_1 \in S_1$, there exist $\langle W^r : r \in \mathrm{field}(\preceq)\rangle$, $V^{r,\infty}$, $\langle V^{r,x} : r \in \mathrm{field}(\preceq), x \in F^r_1\rangle$ such that the following are all satisfied.

1. W^r is pre-wellordering on its field F^r_1,
2. $\forall x \in F^r_1 \cup \{\infty\}$
   - $V^{r,x}$ is pre-wellordering on its field $F^{r,x}_0$,
   - $V^{r,x}_y = \Gamma_0^{F_1^{\prec r} \oplus W^r_{<x}}(V^{r,x}_{<y}) \cup V^{r,x}_{<y}$ for all y $\in F^{r,x}_0$
   - $W^r_x = \Gamma_1^{F_1^{\prec r}}(F^{r,x}_0) \cup W^r_{<x}$,

- $\Gamma_0^{F_1^{\prec r} \oplus W^r_{<x}}(F_0^{r,x}) \subset F_0^{r,x}$.

3. $W^r_\infty = W^r_{<\infty} = F_1^r$

where $F_1^{\prec r} = \oplus\{F_1^{r_i} : r_i \prec r\}$.

Theorem 2.17. Over RCA_0, the following are equivalent.

- $\Delta((\Sigma^0_2)_{k+1})$-$\mathsf{Det}$,
- $[\Sigma^1_1]^k$-IDTR.

3 Determincay of the Games in Cantor Space

To investigate the relationships between the determinacies of the games in Cantor space and Baire space, a technique which translates the games in Baire space into the games in Cantor space was introduced in [5]. *Translate* here means that, for a game $\phi(f)$ in Baire space, the game $\phi^*(f)$ in Cantor space is constructed so that the same player with the winning strategy in $\phi(f)$ has a winning strategy.

By constructing this king of game $\phi^*(f)$, if we assume that $\phi \in \mathcal{C}$ and $\phi^* \in \mathcal{C}'$, then we can prove over appropriate system that $\mathcal{C}'$-Det^* is provable from $\mathcal{C}$-Det.

We obtain the following.

Theorem 3.1. Over RCA_0 the following is equivalent.

- Σ^1_1-MITR,
- $\Delta((\Sigma^0_2)_3)$-Det^*.

Theorem 3.2. Assume that $k \geq 1$. Over RCA_0, the following are equivalent.

- $[\Sigma^1_1]^k$-IDTR,
- $\Delta((\Sigma^0_2)_{k+2})$-$\mathsf{Det}^*$.

References

[1] M. O . Medsalem and K. Tanaka, Δ^0_3-determinacy, comprehension and induction, *Journal of Symbolic Logic*, **72**, 452-462 (2007).

[2] M. O. Medsalem and K. Tanaka, Weak determinacy and iterations of inductive definitions, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 15, *World Sci. Publ., Hackensack, NJ*, (2008).

[3] M. Montalbán and R.A. Shore, The Limits of determinacy in second order arithmetic, preprint.

[4] T. Nemoto, Determinacy of Wadge classes and subsystems of second order arithmetic, *Mathematical Logic Quarterly*, **55**, 154-176 (2009).

[5] T. Nemoto, M. O. Medsalem and K. Tanaka, Infinite games in the Cantor space and subsystems of second order arithmetic, *Mathematical Logic Qurterly*, **53**, 226-236 (2007).

[6] S. G. Simpson, *Subsystems of Second Order Arithmetic*, Springer (1999).

[7] K. Tanaka, Weak axioms of determinacy and subsystems of analysis I (Δ^0_2 games), *Z. Math. Logik Grundlag. Math.*, **36**, 481-491 (1990).

[8] K. Tanaka, Weak axioms of determinacy and subsystems of analysis II (Σ^0_2 games), *Ann. Pure Appl. Logic* **52**, 181-193 (1991).