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Nonstandard arguments and recursive arguments

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Abstract
We give a new nonstandard method for conservation proofs over $\mathrm{B}\Sigma^0_2$ using a combination of recursion theory and nonstandard analysis.

1 Introduction

Techniques from nonstandard analysis play an important role in Reverse Mathematics. In [12, 13], Keisler gives nonstandard characterizations for the big five subsystems of second-order arithmetic. In [16, 20, 21, 22], several nonstandard techniques for analysis in second-order arithmetic are developed, and in [8, 17], Impens and Sanders show that several theorems of nonstandard analysis are equivalent to the $\Pi_1$-transfer principle. Also, combinatorics is an important topic in Reverse Mathematics (see, e.g., [18, 19]). Especially, Ramsey’s theorem for pairs ($\mathrm{RT}^2_2$) plays an important role in Reverse Mathematics as an intermediate axiom between $\mathrm{RCA}_0$ and $\mathrm{ACA}_0$. There are many theorems of combinatorics and model theory that are provable from $\mathrm{RT}^2_2$ (see, e.g., [3, 5, 6]). Thus, determining the exact strength of $\mathrm{RT}^2_2$ is very important. It is well-known that $\mathrm{RT}^2_2$ implies $\mathrm{B}\Sigma^0_2$.

On the other hand, Cholak, Jockusch and Slaman ([1]) show that $\mathrm{RCA}_0 + \mathrm{RT}^2_2 + \mathrm{I}\Sigma^0_2$ is a $\Pi_1^1$-conservative extension of $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$, i.e., the first-order part of $\mathrm{RT}^2_2$ is not stronger than $\mathrm{I}\Sigma^0_2$. Then, the question arises: is $\mathrm{RCA}_0 + \mathrm{RT}^2_2 + \mathrm{B}\Sigma^0_2$ a $\Pi_1^1$-conservative extension of $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$? A partial answer to this question is given by Slaman, Chong and Yang ([2]). They showed that $\mathrm{RCA}_0 + \mathrm{COH} + \mathrm{B}\Sigma^0_2$, $\mathrm{RCA}_0 + \mathrm{ADS} + \mathrm{B}\Sigma^0_2$ and $\mathrm{RCA}_0 + \mathrm{CAC} + \mathrm{B}\Sigma^0_2$ are $\Pi_1^1$-conservative extensions of $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$. Here, $\mathrm{COH}$, $\mathrm{ADS}$ and $\mathrm{CAC}$ are all combinatorial principles weaker than $\mathrm{RT}^2_2$.

In this paper, we will introduce a new approach for conservation proofs over $\mathrm{B}\Sigma^0_2$. We will show how to use recursion-theoretic arguments within nonstandard arithmetic and give new proofs of the conservation theorems for WKŁ and COH over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ (see [4] and [2] for the original proofs, respectively). It is well-known that the nonstandard approach works well for combinatorics (see, e.g., [7]). For Ramsey’s theorem, the nonstandard proof of $\mathrm{ACA}_0$ implies $\mathrm{RT}(k)$ is known [14, Theorem 2.2.16]. This proof can be formalized in the system of non-standard second-order arithmetic corresponding to $\mathrm{ACA}_0$ introduced in [23]. In this proof, the $\Pi_1^0$-transfer principle is the key element. In the nonstandard arithmetic, the $\Pi_1^0$-transfer principle is conservative over $\mathrm{B}\Sigma^0_2$, and this fact plays a key role for the conservation proofs in this paper.

Nonstandard arithmetic

Let $\mathcal{L}$ be the language of first-order arithmetic, and let $\mathcal{L}_2$ be the language of second-order arithmetic. For a finite set of unary predicates $\hat{A}$, an $\mathcal{L} \cup \hat{A}$-structure is a pair $M = (M; \hat{A}^M)$.
where $A^M \subseteq M$ for any $A \in \bar{A}$. Let $\mathcal{L}^*$ be the language of nonstandard arithmetic, i.e., $\mathcal{L}^* = \mathcal{L} \cup \{V^*, \sqrt{}\}$ where $V^*$ and $\sqrt{}$ are unary predicate symbols denoting the standard and nonstandard universe respectively, and $\sqrt{}$ is a function symbol denoting the embedding from the standard universe into the nonstandard universe. An $\mathcal{L}^* \cup \bar{A}$-structure is a triple $\mathfrak{M} = (M, M^*, \sqrt{})$ such that $M = \{(x \mid \mathfrak{M} \models x \in V^*)\}$ and $M^* = \{(x \mid \mathfrak{M} \models x \in V^*)\}$ are $\mathcal{L} \cup \bar{A}$-structures and $\sqrt{}$ is a mapping from $M$ to $M^*$. We usually use the identification $M \cong \sqrt{(M) \subseteq M^*}$, i.e., identify $a \in M$ with $\sqrt{(a)} \in M^*$.

An $\mathcal{L} \cup \bar{A}$-structure $M$ is said to be a model of $\Sigma^0_n$ (resp. $\text{BS}_{2}^0$) if $(M, \bar{A}) \models \Sigma^0_n$ (resp. $\text{BS}_{2}^0$) as a second order structure. In other words, $(M; \bar{A})$ satisfies the induction axioms (resp. bounding axioms) for $\Sigma^0_n$ formulas.

**Definition 1.1.** For a finite set of unary predicates $\bar{A}$, we define axioms for $\mathcal{L}^* \cup \bar{A}$ as follows:

- $\text{BNS}$ consists of the following:
  - $\sqrt{}$ is an embedding (with respect to $+, \times, \bar{A}$-structures) from $V^*$ to $V^*$,
  - $V^*$ is an end extension of $\sqrt{(V^*)}$,
  - $V^* \models \Sigma^0_1$ and $V^* \models \Sigma^0_1$.

- $\Pi^0_n$ TP: $\forall x \in V^*(V^* \models \varphi(x, \bar{A}) \leftrightarrow V^* \models \varphi(x, \bar{A}))$ for any $\varphi \in \Pi^0_n$ formulas.

Note that we can easily show that BNS implies $\Pi^0_0$ TP.

## 2 $\text{BS}_{2}^0$ and $\Pi^0_0$ TP

In this section, we prove that $\text{BNS} + \Pi^0_0$ TP is a (first-order) conservative extension of $\text{BS}_{2}^0$. To prove this, we use a version of Friedman’s self-embedding theorem.

From now on, we identify an $\mathcal{L} \cup \bar{A}$ formula $\varphi$ with an $\mathcal{L}^* \cup \bar{A}$ formula $\varphi^*$, where $\varphi^*$ is a formula constructed by replacing $\forall x$ (resp. $\exists x$) in $\varphi$ into $\forall x \in V^*$ (resp. $\exists x \in V^*$).

**Theorem 2.1.** Let $n \geq 1$. Then, $\text{BNS} + \Pi^0_0$ TP $+(V^*, V^* \models \Sigma^0_{n-1})$ proves $\text{BS}_{n+1}^0$. In other words, for any finite set of unary predicates $\bar{A}$, if $M = (M; \bar{A}^M)$ and $M^* = (M^*; \bar{A}^M)$ are models of $\Sigma^0_{n-1}$ such that $M^*$ is an elementary end extension of $M$ with respect to $\Pi^0_n$ formulas, then $M$ is a model of $\Sigma^0_{n+1}$.

**Proof.** This proof is essentially due to Theorem B of [15]. Let $\theta(x, y) \equiv \forall z \theta_0(x, y, z)$ be a $\Pi^0_1$ formula, and let $a \in M$ such that $M \models \forall x < a \exists y \theta(x, y)$. We will show that there exists $b \in M$ such that $M \models \forall x < a \exists y < b \theta(x, y)$. By $\Pi^0_0$ TP, for any $c \in M^* \setminus M$, we have $M^* \models \forall x < a \exists y < c \theta(x, y).$ Take $d \in M^* \setminus M$. Then, for any $c \in M^* \setminus M$, we have $M^* \models \forall x < a \exists y < c \forall z < d \theta_0(x, y, z)$. Then, there exists $b \in M$ such that $M^* \models \forall x < a \exists y < b \forall z < d \theta_0(x, y, z)$ by underspill for $\Sigma^0_{n-1}$ formula, which is available from $M^* \models \Sigma^0_{n-1}$. (Note that $\forall x < a \exists y < b \forall z < d \theta_0(x, y, z)$ is equivalent to a $\Sigma^0_{n-1}$ formula since $M^* \models \Sigma^0_{n-1}$.) Thus, we have $M \models \forall x < a \exists y < b \theta(x, y).$ This means that $M$ satisfies $\Pi^0_n$, which is equivalent to $\text{BS}^0_{n+1}$. \hfill \Box

The following lemma is a modification of a version of Friedman’s self-embedding theorem. See also [11, page 166, Exercise 12.2]
Lemma 2.2. Let $M$ and $N$ be countable recursively saturated models of $\text{BSy}_{n+1}$ such that $\text{SSy}(M) = \text{SSy}(N)$. Let $a \in M$ and $b, c \in N$ such that $M \models \exists \psi(x, a)$ implies $N \models \exists x < bv(x, c)$ for any $\Pi_n$ formulas $\psi(x, y)$. Then, there exists an embedding $f : M \to N$ such that $f(M) \subseteq N$, $f(M) < b$, $f(a) = c$ and $f$ is an elementary embedding with respect to $\Pi_n$ formulas.

Proof. We will construct sequences $\{a_i\}_{i \in \omega} = M$ and $\{c_i\}_{i \in \omega} \subseteq N$ such that $a_0 = a$, $c_0 = c$ and $M \models \exists x \psi(x, a_i)$ implies $N \models \exists x < bv(x, c_i)$ for any $\Pi_n$ formulas by a back and forth argument, where $a_i = (a_0, \ldots, a_i)$ and $c_i = (c_0, \ldots, c_i)$. We fix enumerations $M = \{p_k\}_{k \in \omega}$ and $N = \{q_k\}_{k \in \omega}$ such that each element of $d \in N$ occurs infinitely often in $\{q_k\}_{k \in \omega}$.

Assume that we have already constructed $\{a_j\}_{j < i}$ and $\{c_j\}_{j < i}$ which satisfy the desired conditions. If $i = 2k + 1$, put $a_i = p_k$. By recursive saturation, there exists $a \in M$ such that for any $\theta(x) \in \Pi_n$, $\theta(x) \in \text{code}(a) \leftrightarrow \exists z \theta(\langle a_i, z \rangle)$. Since $\text{SSy}(M) = \text{SSy}(N)$, there exists $\beta \in N$ such that $\text{SSy}(\alpha) = \text{SSy}(\beta)$. Then, $q(y) = \{\theta(x) \in \text{code}(\beta) \rightarrow \exists z \theta(\langle a_i, z \rangle) \land y < b \mid \theta(x) \in \Pi_n\}$ is a recursive type over $N$ (which we can easily check that $q(y)$ is finitely satisfiable). Take a solution $c'$ of $q(y)$ and define $c_i = c'$. Then $\{a_j\}_{j \leq i}$ and $\{c_j\}_{j \leq i}$ satisfy the desired conditions.

If $i = 2k + 2$ and $q_k > \max\{a_i, c_i\}$, then $a_i = c_0$ and $a_i = a_0$. If $i = 2k + 2$ and $q_k \leq \max\{a_i - 1, c_i - 1\}$, put $a_i = c_0$ and $c_i = a_0$. By recursive saturation, there exists $\alpha \in N$ such that for any $\theta(x) \in \Sigma_n$, $\theta(x) \in \text{code}(\alpha) \leftrightarrow \exists z \theta(\langle a_i, z \rangle)$. Since $\text{SSy}(N) = \text{SSy}(M)$, there exists $\alpha \in M$ such that $\text{SSy}(\alpha) = \text{SSy}(\beta)$. Then, $p(x) = \{\theta(x) \in \text{code}(\alpha) \rightarrow \exists z \theta(\langle a_i, x, z \rangle) \mid \theta(x) \in \Sigma_n\}$ is a recursive type over $M$. To show that $p(x)$ is finitely satisfiable, let $\theta_0(x), \ldots, \theta_{i-1}(x) \in \Sigma_n$ such that $N \models \bigwedge_{k \leq i} \forall z < \beta k(\langle a_i, z \rangle)$. Then, $N \models \forall y < b \exists x \leq \max\{a_i - 1\} \bigwedge_{k \leq i} \forall z \leq y \theta_k(\langle a_i, x, z \rangle)$ (note that there is a $\Sigma_n$ formula which is equivalent to $\exists x \leq \max\{a_i - 1\} \bigwedge_{k \leq i} \forall z \leq y \theta_k(\langle a_i, x, z \rangle)$ over $\text{BSy}_{n+1}^0$). Then, by $M \models \text{BSy}_{n+1}^0$, we have $M \models \exists x \leq \max\{a_i - 1\} \forall y \forall y' > y \bigwedge_{k \leq i} \forall z \leq y' \theta_k(\langle a_i - 1, x, z \rangle)$. Thus, $M \models \exists x \leq \max\{a_i - 1\} \bigwedge_{k \leq i} \forall z \theta_k(\langle a_i - 1, x, z \rangle)$, which means that $p(x)$ is finitely satisfiable. Take a solution $\alpha'$ of $p(x)$, and define $a_i = a'$. Then $\{a_j\}_{j \leq i}$ and $\{c_j\}_{j \leq i}$ satisfy the desired conditions.

Define a function $f : M \to N$ as $f(a_i) = c_i$. Then, we can easily check that $f$ is the desired embedding. \qed

Note that in the previous proof, we only used $M \models \text{BSy}_{n+1}^0$ and $N \models \text{BSy}_{n}^0$.

Theorem 2.3. Let $M$ be a countable recursively saturated model of $\text{BSy}_{n+1}$. Then, there exists a self-embedding $f : M \to M$ such that $f(M) \subseteq M$ and $f$ is an elementary embedding with respect to $\Pi_n$ formulas.

Proof. Let $M$ be a countable recursively saturated model of $\text{BSy}_{n+1}$, and let $N$ be a copy of $M$, i.e., $M \cong N$. Define a recursive type $p(x)$ over $M$ as $p(x) = \{\exists y \theta(y) \rightarrow \exists y < x \theta(y) \mid \theta \in \Pi_n\}$. Then, there exists $b \in N$ such that $N \models p(b)$. Define $a = 0 \in M$ and $c = 0 \in N$, then, $M, N, a, b, c$ enjoy the requirements of the previous lemma. \qed

Theorem 2.4. Let $\bar{A}$ be a finite set of unary predicates, and let $M = (M; \bar{A}^M)$ be a countable recursively saturated model of $\text{BSy}_{n+1}$. Then, $M \models \text{BSy}_{n+1}^0$ if and only if there exists a self-embedding $f : M \to M$ such that $f(M) \subseteq M$ and $f$ is an elementary embedding with respect to $\Pi_n^{\bar{A}}$ formulas.

Proof. The proof of the forward direction is an easy generalization of the previous lemma and theorem. We will prove the reverse direction by induction on $n$. Assume that there exists a self-embedding $f : M \to M$ such that $f(M) \subseteq M$ and $f$ is an elementary embedding with respect to self-
$\Pi^A_n$ formulas. By induction hypothesis, we have $M \models B\Sigma^0_n$. Then, the triple $(M, M, f)$ is a model of $BNS + \Pi^0_n TP + (V^*, V^* \models I\Sigma^0_{n-1})$. Thus, we have $M \models B\Sigma^0_{n+1}$ by Theorem 2.1.

**Corollary 2.5.** $BNS + \Pi^0_n TP + (V^*, V^* \models I\Sigma^0_{n-1})$ and $BNS + \Pi^0_n TP + (V^*, V^* \models B\Sigma^0_{n+1})$ are conservative extensions of $B\Sigma^0_{n+1}$ (with respect to $\mathcal{L} \cup \overline{A}$-sentences). In other words, for any $\mathcal{L} \cup \overline{A}$-sentence $\varphi$, the following are equivalent.

1. $B\Sigma^0_{n+1} \vdash \varphi$.
2. $BNS + \Pi^0_n TP + (V^*, V^* \models I\Sigma^0_{n-1}) \vdash (V^* \models \varphi)$.
3. $BNS + \Pi^0_n TP + (V^*, V^* \models B\Sigma^0_{n+1}) \vdash (V^* \models \varphi)$.

**Proof.** We have proved $1 \rightarrow 2$ in Theorem 2.1, and $2 \rightarrow 3$ is trivial. We will show $\neg 1 \rightarrow \neg 3$.

Let $\varphi$ be an $\mathcal{L} \cup \overline{A}$-sentence such that $B\Sigma^0_{n+1} \not\models \varphi$. Then, there exists a countable model $M_0 \models B\Sigma^0_{n+1} \not\models \varphi$. We can easily construct an elementary extension $M \supseteq M_0$ such that $M$ is recursively saturated. By the previous lemma, there exists a $\Pi^A_n$ elementary embedding $f : M \rightarrow M$. Then, the triple $(M, M, f)$ is a model of $BNS + \Pi^0_n TP + (V^*, V^* \models B\Sigma^0_{n+1}) + (V^* \models \neg \varphi)$. Thus, $BNS + \Pi^0_n TP + (V^*, V^* \models B\Sigma^0_{n+1}) \not\models (V^* \models \varphi)$.

Note that the previous corollary implies that $BNS + \Pi^0_n TP + (V^*, V^* \models B\Sigma^0_{n+1})$ (as a system of nonstandard second-order arithmetic) is a $\Pi^1_1$ conservative extension of $B\Sigma^0_{n+1}$ as a second-order theory. In fact, $BNS + \Pi^0_1 TP + (V^*, V^* \models WK\mathcal{L}_0 + B\Sigma^0)$ is a (full second-order) conservative extension of $WK\mathcal{L}_0 + B\Sigma^0$. In general, it is not known whether $BNS + \Pi^0_n TP + (V^*, V^* \models B\Sigma^0_{n+1})$ is a full second-order conservative extension of $B\Sigma^0_{n+1}$ or not. Tin Lok Wong kindly informed the author that by Theorem B of [15], we have $BNS + \Pi^0_n TP$ is a full second-order conservative extension of $B\Sigma^0^0$.

## 3 First jump control and $\Pi^0_1 TP$

In this section we will show that several conservation results over $B\Sigma^0_2$ can be proved by combining some well-known first jump control arguments from the recursion theory, such as a version of the finite injury priority argument, with the transfer principle. In a model $\mathcal{M} = (M, M^*, id_M)$ of $BNS + \Pi^0_1 TP$, we can use methods of nonstandard analysis by considering $M$ as the standard universe and $M^*$ as the nonstandard universe which satisfies the restricted transfer principle.

The following notion of resplendence plays a key role to use our constructions in Subsections 3.1 and 3.2 repeatedly.

**Definition 3.1** (Resplendence). Let $\mathcal{L}_0$ be a first-order language, and let $M$ be an $\mathcal{L}_0$-structure. Then, $M$ is said to be resplendent if for every $\bar{a} \in M$, for every new unary predicate symbol $A$ and for every $\mathcal{L}_0 \cup \{A\}$-formula $\psi(\bar{x}, A)$ such that $Th(M; \mathcal{L}_0 \cup M) \cup \{\psi(\bar{a}, A)\}$ is consistent, $M$ can be expanded into $\mathcal{L}_0 \cup \{A\}$-structure $(M; A^M)$ such that $(M; A^M) \models \psi(\bar{a}, A)$.

$M$ is said to be chronically resplendent if for every $\bar{a} \in M$, for every new unary predicate symbol $A$ and for every $\mathcal{L}_0 \cup \{A\}$-formula $\psi(\bar{x}, A)$ such that $Th(M; \mathcal{L}_0 \cup M) \cup \{\psi(\bar{a}, A)\}$ is consistent, $M$ can be expanded into $\mathcal{L}_0 \cup \{A\}$-structure $(M; A^M)$ such that $(M; A^M) \models \psi(\bar{a}, A)$ and $(M; A^M)$ is resplendent.

**Theorem 3.1** (Chronical resplendence and recursive saturation [11, 14]). Let $\mathfrak{A}$ be a first-order structure with a finite language. Then, the following are equivalent.
1. $\mathcal{A}$ is recursively saturated.

2. $\mathcal{A}$ is resplendent.

3. $\mathcal{A}$ is chronically resplendent.

Proof. See [11, Theorem 15.7, Corollary 15.13] and [14, Propositions 1.9.2, 1.9.3, 1.9.4].

We next define the fix notation $\Phi_{e,s}^{*}$ to simulate recursive arguments using oracles in nonstandard arithmetic. Let $\mathcal{A}$ be a finite set of predicates. We fix a universal $\Pi_{0}^{1}$ formula $\Phi(e,x,\bar{X},Y) \equiv \forall n \Theta(n,x,\bar{X}[n],Y[n])$, i.e., for any $\Pi_{0}^{1}$ formula $\varphi(x,\bar{X},Y)$, there exists $e < \omega$ such that $\Sigma_{1}^{2} \vdash \Phi(e,x,\bar{X},Y) \leftrightarrow \varphi(x,\bar{X},Y)$.

Within $M = (M,\bar{A}^{M}) \models \Pi_{0}^{1}$, given $s,e = (e',a) \in M$ and $\tau \in 2^{<M}$ such that $lh(\tau) \geq s$, we write $\Phi_{e,s}^{\mathcal{A},\tau} \uparrow$ for $\forall n \leq s \Theta(e',a,\bar{A}^{M}[n],\tau \restriction n)$, and we write $\Phi_{e,s}^{\mathcal{A},\tau} \downarrow$ for $\neg(\Phi_{e,s}^{\mathcal{A},\tau} \uparrow)$. We often omit $\mathcal{A}$ and write $\Phi_{e,s}^{\mathcal{A}} \uparrow$ if the oracle $\mathcal{A}$ is fixed. Then, for any $\Pi_{0}^{1}$ formula $\varphi(x,\bar{X},Y)$ for any $a \in M$ and for any $G^{M} \subseteq M$, there exists $e' < \omega$ such that $M^{G} = (M;G^{M}) \models \varphi(a,\bar{A}^{M},G^{M}) \leftrightarrow M^{G} \models \forall \Phi_{e,s}^{\mathcal{A}}[a] \uparrow$.

The next lemma shows that controlling the first jump implies controlling $\Pi_{1}$ transfer principle.

Lemma 3.2. Let $\mathcal{A}$ be a finite set of unary predicates, and let $M = (M,\bar{A}^{M})$ and $M^{*} = (M^{*},\bar{A}^{M^{*}})(\supseteq M)$ be $\mathcal{L} \cup \mathcal{A}$ structures such that $\mathfrak{M} = (M,M^{*},id_{M}) \models \text{BNS} + \Pi_{1}^{0}\text{TP}$. Let $G$ be a new unary predicate, and let $G^{*} \subseteq M^{*}$, $G^{M} \subseteq M$ such that $G^{M} = M \cap G^{M^{*}}$. Define expansion of $M$ and $M^{*}$ as $M^{G} = (M;G^{M})$ and $M^{*G} = (M^{*};G^{M^{*}})$. Then, the following are equivalent.

1. For any $e \in M$, either $(\exists s \in M \ M^{*G} \models \Phi_{e,s}^{\mathcal{A},G^{M^{*}}}[s] \downarrow)$ or $(M^{*G} \models \forall s \Phi_{e,s}^{\mathcal{A},G^{M^{*}}}[s] \uparrow)$ holds.

2. $\mathfrak{M}^{G} = (M^{G},M^{*G},id_{M}) \models \text{BNS} + \Pi_{1}^{0}\text{TP}$ as an $\mathcal{L}^{*} \cup \mathcal{A} \cup \{G\}$-structure.

Proof. In this proof, we omit $\mathcal{A}$ for $\Phi$. The implication $2 \rightarrow 1$ is trivial. Note that for any $e \in M$, the assertion $(\exists s \in M \ M^{*G} \models \Phi_{e,s}^{\mathcal{A},G^{M^{*}}}[s] \downarrow)$ is equivalent to $(M^{G} \models \exists s \Phi_{e,s}^{\mathcal{A},G^{M^{*}}}[s] \downarrow)$ since $G^{M^{*}}[s] = G^{M}[s]$ for any $s \in M$.

To show $1 \rightarrow 2$, we only need to show that for any $\Pi_{0}^{1}$ formula $\forall n \varphi(n,x,\bar{X},Y)$ and $a \in M$, $M^{G} \models \forall n \varphi(n,a,\bar{A}^{M},G^{M})$ implies $M^{*G} \models \forall n \varphi(n,a,\bar{A}^{M^{*}},G^{M^{*}})$. Let $\forall n \varphi(n,x,\bar{X},Y)$ be a $\Pi_{0}^{1}$ formula, and let $a \in M$. Then, there exists $e' < \omega$ such that $\Sigma_{1}^{0} \models \forall n \varphi(n,x,\bar{X},Y) \leftrightarrow \forall s(\Phi_{(e',x),s}^{\mathcal{A}} \uparrow)$. Let $e = (e',a) \in M$. Then $\exists s \in M \ M^{*G} \models \Phi_{e,s}^{\mathcal{A},G^{M^{*}}}[s] \downarrow$ means that $M^{G} \models \exists n \varphi(n,a,\bar{A}^{M},G^{M})$, and $M^{*G} \models \forall s \Phi_{e,s}^{\mathcal{A},G^{M^{*}}}[s] \uparrow$ means that $M^{*G} \models \forall n \varphi(n,a,\bar{A}^{M^{*}},G^{M^{*}})$. This completes the proof.

Finally, we prepare a basic property for $\Delta_{0}^{1}$ definable sets.

Lemma 3.3. Let $\mathcal{A}$ be a finite set of unary predicates. Let $M = (M;\bar{A}^{M})$ be a model of $\text{B}\Sigma_{0}^{1}$, and let $B^{M} \in \Delta_{0}^{1}(M,\bar{A}^{M})$. Then, $(M;\bar{A}^{M} \cup \{B^{M}\})$ is a model of $\text{B}\Sigma_{0}^{1}$. Moreover, if $M = (M;\bar{A}^{M})$ is recursively saturated, then $(M;\bar{A}^{M} \cup \{B^{M}\})$ is recursively saturated.

Proof. We can easily show that for any $\Sigma_{1}^{0}(B)$ formula $\varphi$, there exists a $\Sigma_{1}^{0}$ formula $\psi$ such that $(M;\bar{A}^{M} \cup \{B^{M}\}) \models \varphi \leftrightarrow \psi$.

$\square$
3.1 Conservation proof for WKL

In this part, we will prove that $\text{WKL}_0 + \Sigma^0_2$ is a $\Pi^1_1$ conservative extension of $\text{RCA}_0 + \Sigma^0_2$. We will combine the proof of the low basis theorem for binary trees with the previous nonstandard arguments.

Lemma 3.4. Let $\bar{A}$ be a finite set of unary predicates. Let $M = (M; \bar{A}^M)$ be a countable recursively saturated model of $\Sigma^0_2$ and let $T \in \bar{A}^M$ be an infinite binary tree in $M$. Then, there exists $G \subseteq M$ such that $(M; \bar{A} \cup \{G\})$ is recursively saturated and
\[ (\dagger) \quad (M; \bar{A}^M \cup \{G\}) \models \Sigma^0_2 + (G \text{ is a path of } T). \]

Proof. By Theorem 3.1, if we find $G^M \subseteq M$ which satisfies $(\dagger)$, then we can redefine $G$ such that $(M; \bar{A}^M \cup \{G\})$ is recursively saturated and $G$ satisfies $(\dagger)$ again. Thus, we only need to construct $G^M \subseteq M$ which satisfies $(\dagger)$.

By Theorem 2.4, take a $\Pi^A_1$-elementary end extension $M^* = (M^*; \bar{A}^{M^*}) \models \Sigma^0_1$ of $M$. Then, $\mathcal{M} = (M, M^*, \text{id}_M) \models \text{BNS + } \Pi^A_1\text{TP}$. We write $T^*$ for a set $\{a \in M^* \mid M^* \models a \in A_T\}$ where $A_T \in \bar{A}$ such that $T = A_T^M$. We will imitate the first jump control construction to take a path of $T^*$ which is low within $M^* = (M^*, \Delta^0_0(M^*; \bar{A}^{M^*})) \models \text{RCA}_0$. In $M^*$, we can construct a sequence $\langle \eta(e, s) \in 2 \mid e < s, s \in M^* \rangle$ which satisfies the following:

For any $s$,

- if there exists $e < s$ such that
  \[ \eta(e, s) = 0 \land \neg(\exists \tau \in T^* \mid |\tau| = s \land \forall i < e (\eta(i, s) = 0 \Rightarrow \Phi_{i, s}^{\tau} \uparrow)), \]  
  \[ \eta(i, s + 1) = \begin{cases} 
  \eta(i, s) & i < e_0 \\
  1 & i = e_0 \\
  0 & e_0 < i \leq s,
  \end{cases} \tag{1} \]

then, $e_0 = \min\{e < s \mid e \text{ satisfies (1)}\}$ and

- otherwise,
  \[ \eta(i, s + 1) = \begin{cases} 
  \eta(i, s) & i < s \\
  0 & i = s.
  \end{cases} \]

Let $\eta^e := \langle \eta(i, s) \mid i \leq e \rangle \in 2^{e+1}$, and let $I_e := \{\eta \in 2^{e+1} \mid \exists s \in M^* \eta = \eta^s\}$. Define $\bar{\eta}^e := \max I_e$ as the lexicographic order on $I_e$, and $s_e := \min\{s \in M^* \mid \eta^s = \bar{\eta}^e\}$. Then, by $\Pi^A_1\text{TP}$, $e \in M$ implies $s_e \in M$ since $\bar{\eta}^e \in M$ and $\langle \exists \eta^s = \bar{\eta}^e \rangle$ can be expressed by a $\Sigma^A_1$ formula within $M^*$. We can easily check the following:

- $i \leq j$ implies $s_i \leq s_j$ and $\bar{\eta}^i \subseteq \bar{\eta}^j$.
- $s_e \leq t$ implies $\bar{\eta}^e = \bar{\eta}^s$.
- $T^* = \{\tau \in T^* \mid \forall i \leq e(\eta(i, s_e) = 0 \Rightarrow \Phi_{i, s_e}^{\tau} \uparrow}\}$ is infinite as a subset of $M^*$.
- $i \leq j$ implies $T_i \subseteq T_j$.
- If $\eta(e, s_e) = 1$, $\tau \in T_e$ and $|\tau| > s_e$, then $\Phi_{e, s_e}^{\tau} \downarrow$. 


Let $\alpha \in M^* \setminus M$. By Harrington's forcing argument for $M^*$, there exists $G^{M^*} \subseteq M^*$ such that $(M^*; \bar{A}M^* \cup \{G^{M^*}\}) \models \Sigma_0^0$ and $G^{M^*}$ is a path of $T^\alpha$. Define $G^M := G^{M^*} \cap M$, and define $L \cup \bar{A} \cup \{G\}$-structures $M^G$ and $M^{G^*}$ as $M^G = (M; \bar{A}M \cup \{G\})$ and $M^{G^*} = (M^*; \bar{A}M^* \cup \{G^{M^*}\})$. Then, for any $n \in M$, we have $G^M[n] = G^{M^*}[n]$ which is in $T^\alpha \cap M \subseteq T$. Thus, $G^M$ is a path of $T$.

Finally, we show that $\mathfrak{M}^G = (M^G, M^{G^*}, \text{id}_M) \models \Pi_1^0 \text{TP}$, which implies $(M; \bar{A}M \cup \{G,M^G\}) \models \Sigma_2^0$ by Theorem 2.1. Note that for any $e \in M$ and for any $n \in M^*$, we have $G^{M^*}[n] \in T_e$ since $\alpha > s_e \in M$ and $T_\alpha \subseteq T_e$. Then, for any $e \in M$, we have $\Phi^G_{e,s}[s] \downarrow$ if $\eta(e, s_e) = 1$, and we have $\Phi^G_{e,s}[s] \uparrow$ for any $s \in M^*$ if $\eta(e, s_e) = 0$. Thus, by Lemma 3.2, we have $\mathfrak{M}^G = (M^G, M^{G^*}, \text{id}_M) \models \Pi_1^0 \text{TP}$. This completes the proof. \hfill \Box

**Theorem 3.5.** $\text{WKL}_0 + \Sigma_2^0$ is a $\Pi_1^1$ conservative extension of $\text{RCA}_0 + \Sigma_2^0$.

**Proof.** Let $\varphi(X)$ be an arithmetical formula such that $\text{RCA}_0 + \Sigma_2^0 \not\models \forall X \varphi(X)$. Then there exists a countable recursively saturated model $(M, S)$ and $A_0 \in S$ such that $(M, S) \models \text{RCA}_0 + \Sigma_2^0 + \neg \varphi(A_0)$. Starting from a first-order countable recursively saturated model $(M; A_0)$, we use Lemma 3.3 and Lemma 3.4 $\omega$-times and construct a sequence $(A_i \subseteq M)_{i<\omega}$ such that for each $N < \omega, (M, \{A_i\}_{i<N})$ is recursively saturated and satisfies $\Sigma_2^0$ and $(M, \{A_i\}_{i<\omega}) \models \text{WKL}_0$. Then, we have $(M, \{A_i\}_{i<\omega}) \models \text{WKL}_0 + \Sigma_2^0 + \neg \varphi(A_0)$, which means that $\text{WKL}_0 + \Sigma_2^0 \not\models \forall X \varphi(X)$. \hfill \Box

### 3.2 Conservation proof for COH

In this part, we will prove that $\text{RCA}_0 + \text{COH} + \Sigma_2^0$ is a $\Pi_1^1$ conservative extension of $\text{RCA}_0 + \Sigma_2^0$. For this, we will imitate the first jump control construction for a low$_2$ cohesive set in [1] with the nonstandard arguments. (Jockusch and Stephan first constructed a low$_2$ cohesive set in [9]. See also [10]).

We first define the notion of cohesiveness. Let $R \subseteq M$ and $M = (M; R) \models \Sigma_0^0$. For $i \in M$, define $R_i = \{x \in M \mid (x, i) \in R\}$. For $X, Y \subseteq M$, we write $X \subseteq_{\text{al}} Y$ if $M \models \exists x \forall y \geq x (y \in X \rightarrow y \in Y)$. Then, $G \subseteq M$ is said to be $R$-cohesive if $M \models \forall! (G \subseteq_{\text{al}} R_i \lor G \subseteq_{\text{al}} R_i^c)$. The axiom COH of second-order arithmetic asserts that $\forall X \exists Y (Y$ is $X$-cohesive).

**Lemma 3.6.** Let $\bar{A}$ be a finite set of unary predicates. Let $M = (M; \bar{M})$ be a countable recursively saturated model of $\Sigma_2^0$ and let $R \in \bar{A}$. Then, there exists $G \subseteq M$ such that $(M; \bar{M} \cup \{G\})$ is recursively saturated and

$$
(1) \quad (M; \bar{M} \cup \{G\}) \models \Sigma_2^0 + (G$ is $R$-cohesive).
$$

**Proof.** By Theorem 3.1, if we find $G^M \subseteq M$ which enjoys $(1)$, then we can redefines $G$ such that $(M; \bar{M} \cup \{G\})$ is recursively saturated and $G$ enjoys $(1)$ again. Thus, we only need to construct $G^M \subseteq M$ which enjoys $(1)$.

By Theorem 2.4, take a $\Pi_1^1$-elementary end extension $(M^*; \bar{A}M^*) \models \Sigma_0^0$ of $M$. Then, $\mathfrak{M} = (M, M^*, \text{id}_M) \models \text{BNS} + \Pi_1^0 \text{TP}$. We write $R^*$ for a set $\{a \in M^* \mid M^* \models a \in A_R\}$ where $A_R \in \bar{A}$ such that $R = A_R M^*$. Note that $R_i = M \cap R_i^*$ for any $i \in M$. Take $\alpha \in M^* \setminus M$, and define a sequence $\sigma \in 2^\alpha$ as $\sigma(i) = 1 \leftrightarrow \alpha \in R_i^*$. For $p \in 2^{\leq \alpha}$, define $R_p^*$ as

$$
R_p^* = \left( \bigcap_{\rho(i)=1,i<|\rho|} R_i^* \right) \cap \left( \bigcap_{\rho(i)=0,i<|\rho|} R_i^{c*} \right).
$$
Then, for any $n \in M$, $R_{\sigma|n} = R_{\sigma|n}^* \cap M$ is unbounded in $M$. This can be proved by $\alpha \in R_{\sigma|n}^*$ and $\Pi_1^0$TP. We will do the first jump control construction using a nonstandard oracle $\sigma$ to take an $R$-cohesive set within $M^* = (M^*, \Delta_0^0(M^*; \mathcal{A}^{M^*})) \models \text{RCA}_0$. The idea of the following construction is essentially due to Theorem 4.3 of [1].

For $\tau \in 2^{<M^*}$, define $\text{card}(\tau) := \text{card}\{i < |\tau| \mid \tau(i) = 1\}$. For $\tau, \tau' \in 2^{<M^*}$ and $X \subseteq M^*$, we write $\tau' \in (\tau, X)$ if $\tau' \subseteq \tau$ and $\forall i < |\tau'|(|\tau'(i) = 0 \lor i < |\tau| \lor i \in X)$. In $M^*$, we construct sequences $(\eta(e, s) \in 3 \mid e < s, s \in M^*)$ and $(\tau(e, s) \in 2^{<s} \mid e < s, s \in M^*)$ as follows:

(\dagger\dagger) Let $\tau(-1, 0) = \emptyset$. For each $s$, we do one of the following.

(I) If there exists $e < \min\{s, |\sigma|\}$ such that

$$\eta(e, s) = 1 \land \forall e' < e \eta(e', s) \neq 0 \lor \exists \tau \in (\tau(e, s), R_{\sigma|e+1}^*)(|\tau| \leq s \land \phi_{\tau, s} \downarrow),$$

then, let $e_0 = \min\{e < s \mid e \text{ satisfies (2)}\}$, $\tau_0 = \min\{\tau \in (\tau(e, s), R_{\sigma|e+1}^*) \mid \phi_{\tau, s} \downarrow\}$ and define

$$\eta(i, s+1) = \begin{cases} 
\eta(i, s) & i < e_0 \\
2 & i = e_0 \\
0 & e_0 < i \leq s, 
\end{cases}$$

$$\tau(i, s+1) = \begin{cases} 
\tau(i, s) & i < e_0 \\
\tau_0 & e_0 \leq i \leq s, 
\end{cases}$$

(II) If (I) is false case and there exists $e < \min\{s, |\sigma|\}$ such that

$$\eta(e, s) = 0 \land \forall e' < e \eta(e', s) \neq 0 \lor \exists \tau \in (\tau(e, s), R_{\sigma|e+1}^*)(|\tau| \leq s \land \text{card}(\tau) \geq e),$$

then, let $e_0 = \min\{e < s \mid e \text{ satisfies (3)}\}$, $\tau_0 = \min\{\tau \in (\tau(e, s), R_{\sigma|e+1}^*) \mid \text{card}(\tau) \geq e\}$ and define

$$\eta(i, s+1) = \begin{cases} 
\eta(i, s) & i < e_0 \\
1 & i = e_0 \\
0 & e_0 < i \leq s, 
\end{cases}$$

$$\tau(i, s+1) = \begin{cases} 
\tau(i, s) & i < e_0 \\
\tau_0 & e_0 \leq i \leq s, 
\end{cases}$$

(III) Otherwise, we define

$$\eta(i, s+1) = \begin{cases} 
\eta(i, s) & i < e_0 \\
0 & i = e_0 \\
0 & e_0 < i \leq s, 
\end{cases}$$

$$\tau(i, s+1) = \begin{cases} 
\tau(i, s) & i < s \\
\tau(s-1, s) & e_0 \leq i \leq s. 
\end{cases}$$

Let $\eta^e := (\eta(i, s) \mid i \leq e) \in 3^{e+1}$, and let $I_e := \{\eta \in 3^{e+1} \mid \exists s \in M^* \eta = \eta^e\}$. Define $\eta^e := \max I_e$ as the lexicographic order on $I_e$, $s_e := \min\{s \in M^* \mid \eta^e = \eta^e\}$, and $r^e := \tau(e, s_e)$.

We will show that $e \in M$ implies $s_e \in M$. Fix $e \in M$. Define $*e = e + 1 \in M$, and do the construction (\dagger\dagger) by replacing $\sigma$ with $*\sigma$. Let $*\eta(i, s), *\tau(i, s), *s_i, \ldots$ be the results of this construction. By $\Sigma_0^0$ in $M^*$, we can easily show that $\forall i \leq e(\eta(i, s) = *\eta(i, s) \land \tau(i, s) = *\tau(i, s))$ for any $s \in M^*$. Thus, for $i \leq e$, we have $*s_i = \min\{s \in M^* \mid *\eta^i = *\eta^i\} = s_i$. Then, by $\Pi_1^0$TP, $s_i = *s_i \in M$ for $i \leq e$ since "$3s *\eta^i = *\eta^i"$ can be expressed by a $\Sigma_1^1$ formula within $M^*$.

We can easily check the following:

- $|r^e| \leq s_e$
- $i \leq j$ implies $s_i \leq s_j$, $\eta^i \subseteq \eta^j$ and $r^j \subseteq r^j$. 
\begin{itemize}
\item $s_e \leq t$ implies $\bar{\eta}^e = \eta^e_t$ and $\bar{\tau}^e = \tau(e, t)$.
\item If $\eta(e, s_e) \geq 1$, then $\text{card}(\bar{\tau}^e) \geq e$.
\item If $\eta(e, s_e) = 2$ and $i \geq e$, then $\phi_{e, s_e}^{i+1}$ \downarrow.
\item If $\eta(e, s_e) = 1$, then $\forall r' \in (\bar{\tau}^e, R^*_e \bar{\tau}^e \beta+1) \phi_{e, |r'|} \downarrow$.
\end{itemize}

Let $\beta = \min\{e \mid \eta(e, s_e) = 0\} \cup \{\alpha\}$. We will show that $\beta \in M^* \setminus M$ by way of contradiction. Assume $\beta \in M$. Then, we have $|\bar{\tau}^\beta| \leq s_\beta \in M$, $\text{card}(\bar{\tau}^\beta) \geq \text{card}(\bar{\tau}^{\beta-1}) \geq \beta - 1$, and $\forall \tau' \in (\bar{\tau}^\beta, R^*_e \bar{\tau}^\beta \beta+1) \text{card}(\tau') < \beta$. Therefore, for any $n \in R^*_e \bar{\tau}^\beta \beta+1$, we have $n \leq s_\beta$. This contradicts the fact that $M \cap R^*_e \beta+1$ is unbounded in $M$.

Finally, we will define $\mathcal{L} \cup \mathcal{A} \cup \{G\}$-structures $M^G = (M, \bar{A}^M \cup \{G^M\})$ and $M^*G = (M^*; \bar{A}^{M^*} \cup \{G^M\})$, and show that $G^M$ is $R$-cohesive and $\mathfrak{M}^G = (M, M^{*G}, \text{id}_M) \models \Pi^0_2 \text{TP}$. Let $G^M = \{n \in M^* \mid n < |\bar{\tau}^\beta| \land \bar{\tau}^\beta(n) = 1\}$, and let $G^M = G^M \cap M$. Then, $G^M$ is unbounded in $M$ since $G^M[s_e] \supseteq \bar{\tau}^e$ and $\text{card}(\bar{\tau}^e) \geq e$ for any $e \in M$. For any $e \in M$ and for any $t \in M^*$ such that $t \geq e$, we have $G^M[t] \in (\bar{\tau}^e, R^*_e \bar{\tau}^e \beta+1)$ since $\bar{\tau}^e \in (\bar{\tau}^\beta, R^*_e \bar{\tau}^\beta \beta+1)$. This implies $G^M \models G^M \subseteq R_t \lor G^M \subseteq \forall e \in M$. This means that $G^M$ is $R$-cohesive in $M^G$, and we also have $M^*G \models \forall s \Phi^G_{e, s} \downarrow$ for any $e \in M$ such that $\eta(e, s_e) = 1$. On the other hand, if $e \in M$ and $\eta(e, s_e) = 2$, then $M^*G \models \forall s (\Phi^G_{e, s_e} \downarrow)$. Thus, we have $\forall \mathfrak{M} = \Pi^0_2 \text{TP}$ by Theorem 3.2, which implies $(M; \bar{A}^M \cup \{G^M\}) \models \Sigma^0_2$ by Theorem 2.1. This completes the proof. \hfill \Box

**Theorem 3.7.** $\text{RCA}_0 + \text{COH} + \Sigma^0_2$ is a $\Pi^1_1$ conservative extension of $\text{RCA}_0 + \Sigma^0_2$.

**Proof.** Let $\varphi(X)$ be an arithmetical formula such that $\text{RCA}_0 + \Sigma^0_2 \not\vdash \forall X \varphi(X)$. Then there exists a countable recursively saturated model $(M, S)$ and $A_0 \in S$ such that $(M, S) \models \text{RCA}_0 + \Sigma^0_2 + \neg \varphi(A_0)$. Starting from a first-order countable recursively saturated model $(M, A_0)$, we use Lemma 3.3 and Lemma 3.6 $\omega$-times and construct a sequence $\{A_i \subseteq M\}_{i<\omega}$ such that for each $N < \omega$, $(M; \{A_i\}_{i<N})$ is recursively saturated and satisfies $\Sigma^0_2$ and $(M, \{A_i\}_{i<\omega}) \models \text{RCA}_0 + \text{COH}$. Then, we have $(M, \{A_i\}_{i<\omega}) \models \text{RCA}_0 + \text{COH} + \Sigma^0_2 \not\vdash \forall X \varphi(X)$, which means that $\text{RCA}_0 + \text{COH} + \Sigma^0_2 \not\vdash \forall X \varphi(X)$. \hfill \Box

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**References**


