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On a Motion of a Vortex Filament in the Half Space

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Abstract

A model equation for the motion of a vortex filament immersed in three dimensional, incompressible and inviscid fluid is investigated as a humble attempt to model the motion of a tornado. We solve an initial-boundary value problem in the half space where we impose a boundary condition in which the vortex filament is allowed to move on the boundary.

1 Introduction

Many researchers have studied tornadoes from several perspectives. A systematic research and observation of tornadoes is difficult mainly because of two reasons: a precise prediction of tornado formation is not yet possible, and the life-span of a tornado is very short, giving only short openings for any kind of measurements. Many aspects of tornadoes are still unknown.

In 1971, Fujita [3] gave a systematic categorization of tornadoes. He proposed the so-called Fujita scale in which tornadoes are classified according to the damage that it dealt to buildings and other surroundings. The scale provides a correlation between ranges of wind speed and the damage that it causes. The enhanced version, called the Enhanced Fujita Scale, is used to classify tornadoes to date. McDonald [10] gives a review of Fujita's contributions to tornado research.

Since then, due to the advancement of technology, more accurate and thorough observations and simulations have become possible, and theories for the formation and motion of tornadoes have developed. Klemp [5] and the references within give an extensive review on the known dynamics of tornadoes.

Motivated by these works, we investigate the motion of a vortex filament. The vortex filament equation, also called the Localized Induction Equation (LIE) models the movement of a vortex filament, which is a space curve where the vorticity of the fluid is concentrated, and is described by

$$\boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, \tag{1.1}$$

where $\boldsymbol{x}(s,t) = (x^1(s,t),\ x^2(s,t),\ x^3(s,t))$ is the position of the vortex filament parameterized by the arc length s at time t, $\times$ denotes the exterior product, and the subscripts denote differentiation with respect to that variable. We also use ∂_s and ∂_t for partial differentiation with respect to the corresponding variables.

The LIE was first derived by Da Rios [2] and re-examined by Arms and Hama [1]. Since then, many authors have worked with the equation. Nishiyama and Tani [7, 8] gave the unique solvability of the initial value problem for the LIE in Sobolev spaces. A different approach was taken by Hasimoto [4]. He used the so-called Hasimoto transformation to transform (1.1) into a nonlinear Schrödinger equation:

$$\frac{1}{\mathrm{i}}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2}|\psi|^2 \psi,$$

where ψ is given by

$$\psi = \kappa \exp\left(\mathrm{i}\int_0^s \tau \,\mathrm{d}s\right),$$

κ is the curvature, and τ is the torsion of the filament. Even though this expression is undefined at points of the filament where the curvature vanishes, Koiso [6] proved that a modified Hasimoto transformation is well-defined in the class of C^∞ functions. He used a geometrical approach to define the modified Hasimoto transformation and showed the unique solvability of the initial value problem in the class of C^∞ functions.

Regarding initial-boundary value problems, the only known result that the authors know is by Nishiyama and Tani [8]. The boundary condition imposed there necessarily fixes the end point of the vortex filament and does not allow it to move on the boundary. From the physical point of view, the vortex filament must be closed, extend to the spatial infinity, or end on boundaries of the fluid region. In the last case, we have to impose an appropriate boundary condition to show the well-posedness of the problem. Since it is hard to find what kind of boundary condition is physically reasonable if we begin our analysis from the Schrödinger equation, we chose to work with the original vortex filament equation (1.1).

In light of modeling the motion of a tornado, we consider (1.1) in a framework in which the end of the vortex filament is allowed to move on the boundary. We do this by setting a different boundary condition than that of [8].

The contents of this paper are as follows. In section 2, we formulate our problem and give basic notations. In section 3, we derive compatibility conditions for our initial-boundary value problem. Sections 4 and 5 are concerned with constructing the solution. Finally, we give a remark on the nonlinear Schrödinger equation related to our initial-boundary value problem.

2 Setting of the Problem

We consider the initial-boundary value problem for the motion of a vortex filament in the half-space in which the filament is allowed to move on the boundary:

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, & s > 0,\ t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{e}_3, & t > 0, \end{cases} \tag{2.1}$$

where $\boldsymbol{e}_3 = (0,0,1)$. We assume that

$$|\boldsymbol{x}_{0s}(s)| = 1 \quad \text{for} \quad s \geq 0, \qquad x_0^3(0) = 0, \tag{2.2}$$

for the initial datum. The first condition states that the initial vortex filament is parameterized by the arc length and the second condition just states that the curve is parameterized starting from the boundary. Here we observe that by taking the inner product of $\boldsymbol{e}_3$ with the equation, taking the trace at $s = 0$, and noting the boundary condition we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_3 \cdot \boldsymbol{x})\,|_{s=0} &= \boldsymbol{e}_3 \cdot (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})|_{s=0} \\ &= \boldsymbol{x}_s \cdot (\boldsymbol{x}_s \times \boldsymbol{x}_{ss})|_{s=0} \\ &= 0, \end{aligned}$$

where " $\cdot$ " denotes the inner product and $|_{s=0}$ denotes the trace at $s = 0$. This means that if the end of the vortex filament is on the boundary initially, then it will stay on the boundary, but is not necessarily fixed. This is our reason for the notion "allowed to move on the boundary".

By introducing new variables $\boldsymbol{v}(s,t) := \boldsymbol{x}_s(s,t)$ and $\boldsymbol{v}_0(s) := \boldsymbol{x}_{0s}(s)$, (2.1) and (2.2) become

$$\begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss}, & s > 0,\ t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \end{cases} \tag{2.3}$$

$$|\boldsymbol{v}_0(s)| = 1, \quad s \geq 0. \tag{2.4}$$

Once we solve (2.3), the solution $\boldsymbol{x}$ of (2.1) and (2.2) can be constructed by

$$\boldsymbol{x}(s,t) = \boldsymbol{x}_0(s) + \int_0^t \boldsymbol{v}(s,\tau) \times \boldsymbol{v}_s(s,\tau)\,\mathrm{d}\tau.$$

So from now on, we concentrate on the initial-boundary value problem (2.3) under the condition (2.4). Note that if the initial datum satisfies (2.4), then any smooth solution $\boldsymbol{v}$ of (2.3) satisfies

$$|\boldsymbol{v}(s,t)| = 1, \quad s \geq 0,\ t \geq 0. \tag{2.5}$$

This can be confirmed by taking the inner product of the equation with $\boldsymbol{v}$.

We define basic notations that we will use throughout this paper.

For a domain Ω, a non-negative integer m, and $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is the Sobolev space containing all real-valued functions that have derivatives in the sense of distribution up to order m belonging to $L^p(\Omega)$. We set $H^m(\Omega) = W^{m,2}(\Omega)$ as the Sobolev space equipped with the usual inner product, and $H^0(\Omega) = L^2(\Omega)$. We will particularly use the cases $\Omega = \mathbf{R}$ and $\Omega = \mathbf{R}_+$, where $\mathbf{R}_+ = \{s \in \mathbf{R}; s > 0\}$. The norm in $H^m(\Omega)$ is denoted by $||\cdot||_m$ and we simply write $||\cdot||$ for $||\cdot||_0$. We do not indicate the domain in the symbol for the norms since we use it in a way where there is no risk of confusion.

For a Banach space X, $C^m([0,T];X)$ denotes the space of functions that are m times continuously differentiable in t with respect to the topology of X.

For any function space described above, we say that a vector valued function belongs to the function space if each of its components does.

3 Compatibility Conditions

We derive necessary conditions for a smooth solution to exist for (2.3) with (2.4).

Suppose that $\boldsymbol{v}(s,t)$ is a smooth solution of (2.3) with (2.4) defined in $\mathbf{R}_+ \times [0,T]$ for some positive T. We have already seen that for all $(s,t) \in \mathbf{R}_+ \times [0,T]$

$$|\boldsymbol{v}(s,t)|^2 = 1. \tag{3.1}$$

By differentiating the boundary condition with respect to t we see that

$$\partial_t^n \boldsymbol{v}\big|_{s=0} = \mathbf{0} \quad \text{for} \quad n \in \mathbf{N},\ t > 0. \tag{$(B)_n$}$$

We next show

Lemma 3.1 *For a smooth solution $\boldsymbol{v}(s,t)$ under consideration, it holds that*

$$\boldsymbol{v} \times \partial_s^{2n} \boldsymbol{v}\big|_{s=0} = \mathbf{0}, \tag{$(C)_n$}$$

$$\partial_s^j \boldsymbol{v} \cdot \partial_s^l \boldsymbol{v}\big|_{s=0} = 0 \quad \text{for} \quad j + l = 2n+1. \tag{$(D)_n$}$$

Proof. We prove them by induction. From $(B)_1$ and by taking the trace of the equation we see that

$$\mathbf{0} = \boldsymbol{v}_t \,|_{s=0} = \boldsymbol{v} \times \boldsymbol{v}_{ss} \,|_{s=0},$$

thus, $(C)_1$ holds. By taking the exterior product of $\boldsymbol{v}_s$ and $(C)_1$ we have

$$\{(\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\,\boldsymbol{v} - (\boldsymbol{v}_s \cdot \boldsymbol{v})\,\boldsymbol{v}_{ss}\} \,|_{s=0} = \mathbf{0}.$$

On the other hand, by differentiating (3.1) with respect to s we have $\boldsymbol{v} \cdot \boldsymbol{v}_s = 0$. Combining these two and the fact that $\boldsymbol{v}$ is a non-zero vector, we arrive at

$$\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss} \,|_{s=0} = 0.$$

Finally, by differentiating (3.1) with respect to s three times and setting $s = 0$, we have

$$0 = 2\,(\boldsymbol{v} \cdot \boldsymbol{v}_{sss} + 3\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss}) \,|_{s=0} = 2\boldsymbol{v} \cdot \boldsymbol{v}_{sss} \,|_{s=0},$$

so, $(D)_1$ holds.

Suppose that the statements hold up to $n-1$ for some $n \geq 2$. By differentiating $(C)_{n-1}$ with respect to t we have

$$\boldsymbol{v} \times \left(\partial_s^{2(n-1)} \boldsymbol{v}_t\right)\Big|_{s=0} = \mathbf{0},$$

where we have used $(B)_1$. We see that

$$\partial_s^{2(n-1)} \boldsymbol{v}_t = \partial_s^{2(n-1)} (\boldsymbol{v} \times \boldsymbol{v}_{ss}) = \sum_{k=0}^{2(n-1)} \binom{2(n-1)}{k} \left(\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v}\right),$$

where $\binom{2(n-1)}{k}$ is the binomial coefficient. So we have

$$\sum_{k=0}^{2(n-1)} \binom{2(n-1)}{k} \left\{\boldsymbol{v} \times \left(\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v}\right)\right\}\Big|_{s=0} = \mathbf{0}. \tag{3.2}$$

We examine each term in the summation. When $2 \leq k \leq 2(n-1)$ is even, we see from the assumptions of induction $(C)_{k/2}$ and $(C)_{(2(n-1)-k+2)/2}$ that both $\partial_s^k \boldsymbol{v}$ and $\partial_s^{2(n-1)-k+2} \boldsymbol{v}$ are parallel to $\boldsymbol{v}$, so that

$$\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v}\Big|_{s=0} = \mathbf{0}.$$

When $1 \leq k \leq 2(n-1)$ is odd, we rewrite the exterior product in (3.2) as

$$\boldsymbol{v} \times \left(\partial_s^k \boldsymbol{v} \times \partial_s^{2(n-1)-k+2} \boldsymbol{v}\right) = \left(\boldsymbol{v} \cdot \partial_s^{2(n-1)-k+2} \boldsymbol{v}\right) \partial_s^k \boldsymbol{v} - \left(\boldsymbol{v} \cdot \partial_s^k \boldsymbol{v}\right) \partial_s^{2(n-1)-k+2} \boldsymbol{v}.$$

Since $2(n-1)-k+2$ is also odd, by $(D)_{(k-1)/2}$ and $(D)_{(2(n-1)-k+1)/2}$ we have

$$\boldsymbol{v}\cdot\partial_s^k\boldsymbol{v}\big|_{s=0}=\boldsymbol{v}\cdot\partial_s^{2(n-1)-k+2}\boldsymbol{v}\big|_{s=0}=0.$$

Thus, only the term with $k=0$ remains and we get

$$\boldsymbol{v}\times\left(\boldsymbol{v}\times\partial_s^{2n}\boldsymbol{v}\right)\big|_{s=0}=\mathbf{0}.$$

Here, we note that

$$\boldsymbol{v}\times(\boldsymbol{v}\times\partial_s^{2n}\boldsymbol{v})=(\boldsymbol{v}\cdot\partial_s^{2n}\boldsymbol{v})\boldsymbol{v}-\partial_s^{2n}\boldsymbol{v},$$

where we used (3.1). Taking the exterior product of this with $\boldsymbol{v}$ we see that $(C)_n$ holds. Taking the exterior product of $\partial_s^{2n+1-2k}\boldsymbol{v}$ with $(C)_k$ and using $(D)_{n-k}$ for $1\le k\le n$ yields

$$\left(\partial_s^{2k}\boldsymbol{v}\cdot\partial_s^{2n+1-2k}\boldsymbol{v}\right)\boldsymbol{v}\big|_{s=0}=\mathbf{0}.$$

Since $\boldsymbol{v}$ is non zero, we have for $1\le k\le n$

$$\partial_s^{2k}\boldsymbol{v}\cdot\partial_s^{2n+1-2k}\boldsymbol{v}\big|_{s=0}=0. \tag{3.3}$$

Finally, by differentiating (3.1) with respect to s $(2n+1)$ times, we have

$$\sum_{j=0}^{2n+1}\binom{2n+1}{j}\left(\partial_s^j\boldsymbol{v}\cdot\partial_s^{2n+1-j}\boldsymbol{v}\right)\Big|_{s=0}=0.$$

Since every term except $j=0,\ 2n+1$ is of the form (3.3), we see that

$$\boldsymbol{v}\cdot\partial_s^{2n+1}\boldsymbol{v}\big|_{s=0}=0,$$

which, together with (3.3), finishes the proof of $(D)_n$. □

Worth noting are the following two properties which will be used in later parts of this paper. For integers n,

$$\boldsymbol{e}_3\times\partial_s^{2n}\boldsymbol{v}\big|_{s=0}=\mathbf{0},\quad \boldsymbol{e}_3\cdot\partial_s^{2n+1}\boldsymbol{v}\big|_{s=0}=0.$$

These are special cases of $(C)_n$ and $(D)_n$ with the boundary condition substituted in.

By taking the limit $t\to 0$ in $(C)_n$, we derive a necessary condition for the initial datum.

Definition 3.2 *For $n\in\mathbf{N}\cup\{0\}$, we say that the initial datum $\boldsymbol{v}_0$ satisfies the compatibility condition $(A)_n$ if the following condition is satisfied for $0\le k\le n$*

$$\begin{cases}\boldsymbol{v}_0|_{s=0}=\boldsymbol{e}_3, & k=0,\\ \left(\boldsymbol{v}_0\times\partial_s^{2k}\boldsymbol{v}_0\right)\big|_{s=0}=\mathbf{0}, & k\in\mathbf{N}.\end{cases}$$

From the proof of Lemma 3.1, we see that if $\boldsymbol{v}_0$ satisfies (2.4) and the compatibility condition $(A)_n$, then $\boldsymbol{v}_0$ also satisfies $(D)_k$ for $0\le k\le n$ with $\boldsymbol{v}$ replaced by $\boldsymbol{v}_0$ as long as the trace exists.

4 Extension of the Initial Datum

For the initial datum $\boldsymbol{v}_0$ defined on the half-line, we extend it to the whole line by

$$\widetilde{\boldsymbol{v}}_0(s) = \begin{cases} \boldsymbol{v}_0(s), & s \geq 0, \\ -\overline{\boldsymbol{v}}_0(-s), & s < 0, \end{cases} \tag{4.1}$$

where $\overline{\boldsymbol{v}} = (v^1, v^2, -v^3)$ for $\boldsymbol{v} = (v^1, v^2, v^3) \in \mathbf{R}^3$.

Proposition 4.1 *For any integer $m \geq 2$, if $\boldsymbol{v}_{0s} \in H^m(\mathbf{R}_+)$ and $\boldsymbol{v}_0$ satisfies (2.4) and the compatibility condition $(A)_{[\frac{m}{2}]}$, then $\widetilde{\boldsymbol{v}}_{0s} \in H^m(\mathbf{R})$. Here, $[\frac{m}{2}]$ indicates the largest integer not exceeding $\frac{m}{2}$.*

Proof. Fix an arbitrary integer $m \geq 2$. We will prove by induction on k that $\partial_s^{k+1}\widetilde{\boldsymbol{v}}_0 \in L^2(\mathbf{R})$ for any $0 \leq k \leq m$. Specifically we show that the derivatives of $\widetilde{\boldsymbol{v}}_0$ in the distribution sense on the whole line $\mathbf{R}$ up to order $m+1$ have the form

$$(\partial_s^{k+1}\widetilde{\boldsymbol{v}}_0)(s) = \begin{cases} \left(\partial_s^{k+1}\boldsymbol{v}_0\right)(s), & s > 0, \\ -(-1)^{k+1}\left(\overline{\partial_s^{k+1}\boldsymbol{v}_0}\right)(-s), & s < 0, \end{cases} \tag{4.2}$$

for $0 \leq k \leq m$.

Since $\boldsymbol{v}_0 \in L^\infty(\mathbf{R}_+)$ and $\boldsymbol{v}_{0s} \in H^2(\mathbf{R}_+)$, Sobolev's embedding theorem states $\boldsymbol{v}_{0s} \in L^\infty(\mathbf{R}_+)$ and thus $\boldsymbol{v}_0 \in W^{1,\infty}(\mathbf{R}_+)$, so that the trace $\boldsymbol{v}_0(0)$ exists. For any vector valued $\boldsymbol{\varphi} \in C_0^\infty(\mathbf{R})$,

$$\begin{aligned} \int_{\mathbf{R}} \tilde{\boldsymbol{v}}_0 \cdot \partial_s\boldsymbol{\varphi}\, \mathrm{d}s &= \int_0^\infty \boldsymbol{v}_0(s) \cdot \partial_s\boldsymbol{\varphi}(s)\, \mathrm{d}s - \int_{-\infty}^0 \overline{\boldsymbol{v}}_0(-s) \cdot \partial_s\boldsymbol{\varphi}(s)\, \mathrm{d}s \\ &= -\int_0^\infty \partial_s\boldsymbol{v}_0(s) \cdot \boldsymbol{\varphi}(s)\, \mathrm{d}s + \int_{-\infty}^0 (-1)\partial_s\overline{\boldsymbol{v}}_0(-s) \cdot \boldsymbol{\varphi}(s)\, \mathrm{d}s \\ &\quad - \boldsymbol{v}_0 \cdot \boldsymbol{\varphi}\big|_{s=0} - \overline{\boldsymbol{v}}_0 \cdot \boldsymbol{\varphi}\big|_{s=0} . \end{aligned}$$

By definition we have

$$- v_0 \cdot \boldsymbol{\varphi}\big|_{s=0} - \overline{v}_0 \cdot \boldsymbol{\varphi}\big|_{s=0} = -2\left(v_0^1(0),\ v_0^2(0),\ 0\right) \cdot \boldsymbol{\varphi}(0),$$

but from $(A)_0$, $v_0^1(0) = v_0^2(0) = 0$, so the trace term is zero and the case $k = 0$ is proved.

Suppose that (4.2) with $k+1$ replaced by k holds for some $k \in \{1, 2, \ldots, m\}$. We check that the derivative $\partial_s^k\widetilde{\boldsymbol{v}}_0$ does not have a jump discontinuity at $s = 0$. We similarly calculate

$$\begin{aligned} \int_{\mathbf{R}} \partial_s^k\tilde{\boldsymbol{v}}_0 \cdot \partial_s\boldsymbol{\varphi}\, \mathrm{d}s &= \int_0^\infty \partial_s^k\boldsymbol{v}_0(s) \cdot \partial_s\boldsymbol{\varphi}(s)\, \mathrm{d}s - \int_{-\infty}^0 (-1)^k\partial_s^k\overline{\boldsymbol{v}}_0(-s) \cdot \partial_s\boldsymbol{\varphi}(s)\, \mathrm{d}s \\ &= -\int_0^\infty \partial_s^{k+1}\boldsymbol{v}_0(s) \cdot \boldsymbol{\varphi}(s)\, \mathrm{d}s + \int_{-\infty}^0 (-1)^{k+1}\overline{\partial_s^{k+1}\boldsymbol{v}_0}(s) \cdot \boldsymbol{\varphi}(s)\, \mathrm{d}s \\ &\quad - \partial_s^k\boldsymbol{v}_0 \cdot \boldsymbol{\varphi}\big|_{s=0} - (-1)^k\overline{\partial_s^k\boldsymbol{v}_0} \cdot \boldsymbol{\varphi}\Big|_{s=0} . \end{aligned}$$

When k is even, from the definition of $\overline{\partial_s^k \boldsymbol{v}_0}$,

$$-\left.\partial_s^k \boldsymbol{v}_0 \cdot \boldsymbol{\varphi}\right|_{s=0} - (-1)^k \left.\overline{\partial_s^k \boldsymbol{v}_0} \cdot \boldsymbol{\varphi}\right|_{s=0} = -2\left(\partial_s^k v_0^1(0),\ \partial_s^k v_0^2(0),\ 0\right) \cdot \boldsymbol{\varphi}(0),$$

but from $(A)_{\frac{k}{2}}$ we have

$$\mathbf{0} = \boldsymbol{v}_0 \times \left.\partial_s^k \boldsymbol{v}_0\right|_{s=0} = \boldsymbol{e}_3 \times \partial_s^k \boldsymbol{v}_0(0),$$

which means that $\partial_s^k \boldsymbol{v}_0(0)$ is parallel to $\boldsymbol{e}_3$ and that the first and second components are zero. When k is odd,

$$-\left.\partial_s^k \boldsymbol{v}_0 \cdot \boldsymbol{\varphi}\right|_{s=0} - (-1)^k \left.\overline{\partial_s^k \boldsymbol{v}_0} \cdot \boldsymbol{\varphi}\right|_{s=0} = -2\left(0,\ 0,\ \partial_s^k v^3(0)\right) \cdot \boldsymbol{\varphi}(0),$$

but $(A)_{[\frac{k}{2}]}$ implies $(D)_{[\frac{k}{2}]}$ and particularly

$$0 = \boldsymbol{v}_0 \cdot \left.\partial_s^k \boldsymbol{v}_0\right|_{s=0} = \boldsymbol{e}_3 \cdot \partial_s^k \boldsymbol{v}_0(0) = (\partial_s^k v_0^3)(0),$$

so the third component is zero. In both cases the term with the trace is zero and we have

$$\begin{aligned}\int_{\mathbf{R}} \partial_s^k \tilde{\boldsymbol{v}}_0 \cdot \partial_s \boldsymbol{\varphi}\ ds &= -\int_0^\infty \partial_s^{k+1} \boldsymbol{v}_0(s) \cdot \boldsymbol{\varphi}(s)\ ds + \int_{-\infty}^0 (-1)^{k+1} \overline{\partial_s^{k+1} \boldsymbol{v}_0}(s) \cdot \boldsymbol{\varphi}(s)\ ds \\ &= -\left\{\int_0^\infty \partial_s^{k+1} \boldsymbol{v}_0(s) \cdot \boldsymbol{\varphi}(s)\ ds - \int_{-\infty}^0 (-1)^{k+1} \overline{\partial_s^{k+1} \boldsymbol{v}_0}(s) \cdot \boldsymbol{\varphi}(s)\ ds\right\}.\end{aligned}$$

This verifies (4.2) and finishes the proof of the proposition. □

5 Existence and Uniqueness of Solution

Using $\widetilde{\boldsymbol{v}}_0$, we consider the following initial value problem:

$$\boldsymbol{u}_t = \boldsymbol{u} \times \boldsymbol{u}_{ss}, \qquad s \in \mathbf{R},\ t > 0, \tag{5.1}$$

$$\boldsymbol{u}(s,0) = \widetilde{\boldsymbol{v}}_0(s), \qquad s \in \mathbf{R}. \tag{5.2}$$

By Proposition 4.1, the existence and uniqueness theorem (cf. Nishiyama [9]) of a strong solution $\boldsymbol{u}$ is applicable. Specifically we use the following theorem.

Theorem 5.1 (Nishiyama [9]) *For a non-negative integer m, if $\widetilde{\boldsymbol{v}}_{0s} \in H^{2+m}(\mathbf{R})$ and $|\widetilde{\boldsymbol{v}}_0| = 1$, then the initial value problem* (5.1) *and* (5.2) *has a unique solution $\boldsymbol{u}$ such that*

$$\boldsymbol{u} - \widetilde{\boldsymbol{v}}_0 \in C\big([0,\infty); H^{3+m}(\mathbf{R})\big) \cap C^1\big([0,\infty); H^{1+m}(\mathbf{R})\big)$$

and $|\boldsymbol{u}| = 1$.

From Proposition 4.1, the assumptions of the theorem are satisfied if $\boldsymbol{v}_{0s} \in H^{2+m}(\mathbf{R}_+)$ and $\boldsymbol{v}_0$ satisfies the compatibility condition $(A)_{[\frac{2+m}{2}]}$ and (2.4).

Now we define the operator T by

$$(\mathrm{T}\boldsymbol{w})(s) = -\overline{\boldsymbol{w}}(-s),$$

for $\mathbf{R}^3$-valued functions $\boldsymbol{w}$ defined on $s \in \mathbf{R}$. By direct calculation, we can verify that $\mathrm{T}\widetilde{\boldsymbol{v}}_0 = \widetilde{\boldsymbol{v}}_0$ and that $\mathrm{T}(\boldsymbol{u} \times \boldsymbol{u}_{ss}) = (\mathrm{T}\boldsymbol{u}) \times (\mathrm{T}\boldsymbol{u})_{ss}$. Taking these into account and applying the operator T to (5.1) and (5.2), we have

$$\begin{cases} (\mathrm{T}\boldsymbol{u})_t = (\mathrm{T}\boldsymbol{u}) \times (\mathrm{T}\boldsymbol{u})_{ss}, & s \in \mathbf{R},\ t > 0, \\ (\mathrm{T}\boldsymbol{u})(s,0) = (\mathrm{T}\widetilde{\boldsymbol{v}}_0)(s) = \widetilde{\boldsymbol{v}}_0(s), & s \in \mathbf{R}, \end{cases}$$

which means that $\mathrm{T}\boldsymbol{u}$ is also a solution of (5.1) and (5.2). Thus we have $\mathrm{T}\boldsymbol{u} = \boldsymbol{u}$ by the uniqueness of the solution. Therefore, for any $t \in [0, T]$

$$\boldsymbol{u}(0,t) = (\mathrm{T}\boldsymbol{u})(0,t) = -\overline{\boldsymbol{u}}(0,t),$$

which is equivalent to $u^1(0,t) = u^2(0,t) = 0$. Therefore, it holds that $u^3(0,t) = -1$ or 1 because we have $|\boldsymbol{u}| = 1$. But in view of $\widetilde{\boldsymbol{v}}_0(0) = \boldsymbol{v}_0(0) = \boldsymbol{e}_3$, we obtain $\boldsymbol{u}(0,t) = \boldsymbol{e}_3$ by the continuity in t.

This shows that the restriction of $\boldsymbol{u}$ to $\mathbf{R}_+$ is a solution of our initial-boundary value problem. Using this function $\boldsymbol{v} := \boldsymbol{u}|_{\mathbf{R}_+}$, we can construct the solution $\boldsymbol{x}$ to the original equation as we stated in section 2. Thus we have

Theorem 5.2 *For a non-negative integer m, if $\boldsymbol{x}_{0ss} \in H^{2+m}(\mathbf{R}_+)$ and $\boldsymbol{x}_{0s}$ satisfies the compatibility condition $(A)_{[\frac{2+m}{2}]}$ and (2.2), then there exists a unique solution $\boldsymbol{x}$ of (2.1) such that*

$$\boldsymbol{x} - \boldsymbol{x}_0 \in C\big([0,\infty); H^{4+m}(\mathbf{R}_+)\big) \cap C^1\big([0,\infty); H^{2+m}(\mathbf{R}_+)\big),$$

and $|\boldsymbol{x}_s| = 1$.

Proof. The uniqueness is left to be proved. Suppose that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are solutions as in the theorem. Then, by extending $\boldsymbol{x}_i$ $(i = 1, 2)$ by

$$\widetilde{\boldsymbol{x}}_i(s,t) = \begin{cases} \boldsymbol{x}_i(s,t) & s \geq 0, t > 0, \\ \overline{\boldsymbol{x}}_i(-s,t) & s < 0, t > 0, \end{cases}$$

we see that $\widetilde{\boldsymbol{x}}_i$ are solutions of the vortex filament equation in the whole space. Thus $\boldsymbol{x}_1 = \boldsymbol{x}_2$ follows from the uniqueness of the solution in the whole space. □

6 Remark on the Schrödinger Equation

After choosing a reasonable boundary condition, we were able to show, somewhat formally, the equivalence of the vortex filament equation and the Schrödinger equation mentioned in the introduction, in the presence of a boundary. We reiterate the equations for convenience.

$$(6.1)\qquad \begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss}, & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \end{cases}$$

$$(6.2)\qquad \begin{cases} \mathrm{i}q_t = q_{ss} + \frac{1}{2}|q|^2 q, & s > 0, t > 0, \\ q(s,0) = q_0(s), & s > 0, \\ q_s(0,t) = 0, & t > 0. \end{cases}$$

Here, i is the imaginary unit, and we assume that $|v_0| = 1$ and the compatibility condition mentioned in section 4 is satisfied. We first derive a compatibility condition for (6.2).

Lemma 6.1 *The compatibility condition for* (6.2) *is that for* $n \in \mathbf{N} \cup \{0\}$,

$$\partial_s^{2n+1} q_0(0) = 0.$$

Proof. We prove that a smooth solution q of (6.2) satisfies $\partial_s^{2n+1} q(0,t) = 0$ for $t > 0$. It is obvious for $n = 0$. Assume it holds up to $n-1$ for some $n \geq 1$. By differentiating the equation by s $(2n-1)$ times we have

$$\mathrm{i}\partial_s^{2n-1} q_t = \partial_s^{2n+1} q + \frac{1}{2}\partial_s^{2n-1}\left\{|q|^2 q\right\}.$$

Since the last term always contains an odd order derivative less than or equal to $2n-1$, by setting $s = 0$ we get for any $t > 0$

$$\partial_s^{2n+1} q(0,t) = 0,$$

and taking the limit $t \to 0$ yields the desired compatibility condition. □

6.1 Vortex Filament Equation to the Nonlinear Schrödinger Equation

We first transform (6.1) to (6.2), so assume that we have a smooth solution $\boldsymbol{v}$ of (6.1) with an initial datum $\boldsymbol{v}_0$ satisfying the compatibility condition. The solution will necessarily satisfy $|\boldsymbol{v}(s,t)| = 1$. The idea is to construct a basis of the tangent space of the unit sphere, S^2, that is parallel to the curve $\boldsymbol{v}$ on the manifold S^2. So we first construct a vector $\boldsymbol{e}^1(s,t)$ orthogonal to $\boldsymbol{v}$ with unit length satisfying

$$\nabla_s \boldsymbol{e}^1 = \boldsymbol{0},$$

where ∇_s is the covariant derivative along $\boldsymbol{v}$. Suppose that we have such a vector $\boldsymbol{e}^1$. Since we know that $\boldsymbol{v}$ is the unit normal of S^2, we have

$$\nabla_s \boldsymbol{e}^1 = \boldsymbol{e}^1_s - (\boldsymbol{e}^1_s \cdot \boldsymbol{v})\boldsymbol{v} = \boldsymbol{e}^1_s + (\boldsymbol{e}^1 \cdot \boldsymbol{v}_s)\boldsymbol{v}$$

where we have used $\boldsymbol{e}^1 \cdot \boldsymbol{v} = 0$. The above relation is a necessary condition that $\boldsymbol{e}^1$ should satisfy. Conversely, for any $t \geq 0$, we can define $\boldsymbol{e}^1(s,t)$ as the solution of the following linear ordinary differential equation in s

$$\begin{cases} \boldsymbol{e}^1_s + (\boldsymbol{e}^1 \cdot \boldsymbol{v}_s)\boldsymbol{v} = \boldsymbol{0}, & s > 0, \\ \boldsymbol{e}^1(0,t) = \boldsymbol{e}_1, \end{cases}$$

where $\boldsymbol{e}_i$ $(i = 1,2,3)$ will denote the standard orthonormal basis of $\mathbf{R}^3$. We see that

$$(\boldsymbol{e}^1 \cdot \boldsymbol{v})_s = \boldsymbol{e}^1_s \cdot \boldsymbol{v} + \boldsymbol{e}^1 \cdot \boldsymbol{v}_s = 0,$$

so $\boldsymbol{e}^1 \cdot \boldsymbol{v} = 0$. Also

$$(\boldsymbol{e}^1 \cdot \boldsymbol{e}^1)_s = -2(\boldsymbol{e}^1 \cdot \boldsymbol{v}_s)\boldsymbol{v} \cdot \boldsymbol{e}^1 = 0,$$

yielding $|\boldsymbol{e}^1| = 1$, and $\nabla_s \boldsymbol{e}^1 = \boldsymbol{0}$ from construction. So the solution is the desired vector. From this, we see that $\{\boldsymbol{v}, \boldsymbol{e}^1, \boldsymbol{v} \times \boldsymbol{e}^1\}$ is an orthonormal basis in $\mathbf{R}^3$ for every $s \geq 0$ and $t \geq 0$. Since $\boldsymbol{v} \cdot \boldsymbol{v}_s = 0$ from $|\boldsymbol{v}| = 1$, we can decompose $\boldsymbol{v}_s$ as

$$\boldsymbol{v}_s = q_1 \boldsymbol{e}^1 + q_2(\boldsymbol{v} \times \boldsymbol{e}^1). \tag{6.3}$$

For the same reason we also have

$$\boldsymbol{v}_t = p_1 \boldsymbol{e}^1 + p_2(\boldsymbol{v} \times \boldsymbol{e}^1).$$

The q_i and p_i are functions of s and t. From the way that we constructed $\boldsymbol{e}^1$, we have

$$\boldsymbol{e}^1_s = -(\boldsymbol{e}^1 \cdot \boldsymbol{v}_s)\boldsymbol{v} = -q_1 \boldsymbol{v}.$$

From $\boldsymbol{e}^1 \cdot \boldsymbol{v} = 0$ we have $\boldsymbol{e}^1_t \cdot \boldsymbol{v} = -\boldsymbol{e}^1 \cdot \boldsymbol{v}_t = -p_1$ so with $|\boldsymbol{e}^1| = 1$, we get

$$\boldsymbol{e}^1_t = -p_1 \boldsymbol{v} + \alpha(\boldsymbol{v} \times \boldsymbol{e}^1),$$

where α is an unknown function. By the equality $\boldsymbol{v}_{st} = \boldsymbol{v}_{ts}$ and comparing the components, we see that

$$\begin{cases} q_{1t} = p_{1s} + \alpha q_2, \\ q_{2t} = p_{2s} - \alpha q_1. \end{cases}$$

On the other hand, from $\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss}$ we get

$$p_1 = q_{2s}, \; p_2 = q_{1s}.$$

Finally from $\boldsymbol{e}^1_{ts} = \boldsymbol{e}^1_{st}$ we have

$$\alpha_s = p_1 q_2 - q_1 p_2 = -\frac{1}{2}\left\{(q_1)^2_s + (q_2)^2_s\right\}.$$

So $\alpha = -\frac{1}{2}\left\{(q_1)^2 + (q_2)^2\right\} + \alpha(0,t)$. Since $\boldsymbol{e}^1(0,t) = \boldsymbol{e}_1$, we see that $\boldsymbol{e}^1_t(0,t) = \boldsymbol{0}$, which makes $\alpha(0,t) = 0$. So if we set $q := q_1 - \mathrm{i}q_2$, q satisfies

$$\mathrm{i}q_t = q_{ss} + \frac{1}{2}|q|^2 q.$$

Since $\boldsymbol{v}(0,t) = \boldsymbol{e}_3$, differentiating by t yields $\boldsymbol{v}_t(0,t) = \boldsymbol{0}$. So the boundary condition for q becomes $q_s(0,t) = 0$. The initial datum $q_0 = q_{01} - \mathrm{i}q_{02}$ is defined by setting $t = 0$ in (6.3), i.e. from the coefficient of decomposition of $\boldsymbol{v}_{0s}$.

We must show that q_0 that we just defined satisfies the compatibility condition, which we will do by induction. For the convenience of expression, we set $\boldsymbol{e}^2 := \boldsymbol{v} \times \boldsymbol{e}^1$. By substituting (6.3) into the equation defining $\boldsymbol{e}^1$ we obtain

$$\boldsymbol{e}^1_{0s} = -q_{01}\boldsymbol{v}_0,$$

and from direct calculation we see that

$$\boldsymbol{e}^2_{0s} = -q_{20}\boldsymbol{v}_0.$$

The subscript 0 means that t is set to 0. By differentiating (6.3) with respect to s, we get

$$\boldsymbol{v}_{0ss} = q_{01s}\boldsymbol{e}^1_0 - (q_{01})^2\boldsymbol{v}_0 + q_{02s}\boldsymbol{e}^2_0 - (q_{02})^2\boldsymbol{v}_0.$$

From the compatibility condition for $\boldsymbol{v}_0$ we conclude that

$$\begin{aligned} \boldsymbol{0} = (\boldsymbol{v}_0 \times \boldsymbol{v}_{0ss})(0) &= q_{01s}(\boldsymbol{v}_0 \times \boldsymbol{e}^1_0) + q_{02s}(\boldsymbol{v}_0 \times \boldsymbol{e}^2_0)\big|_{s=0} \\ &= q_{01s}\boldsymbol{e}^2_0 - q_{02s}\boldsymbol{e}^1_0\big|_{s=0}. \end{aligned}$$

Since $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$ are perpendicular, $q_{01s}(0) = q_{02s}(0) = 0$ and the 0-th compatibility condition for q_0 is satisfied. Suppose that the compatibility condition holds up to

$n-1$ for some $n \geq 1$. By differentiating (6.3) $(2n+1)$ times with respect to s, we get

$$\partial_s^{2n+2}\boldsymbol{v}_0 = \sum_{k_1=0}^{2n+1} \binom{2n+1}{k_1} (\partial_s^{k_1} q_{01})(\partial_s^{2n+1-k_1}\boldsymbol{e}_0^1) + \sum_{k_1=0}^{2n+1} \binom{2n+1}{k_1} (\partial_s^{k_1} q_{02})(\partial_s^{2n+1-k_1}\boldsymbol{e}_0^2).$$

The terms where k_1 is odd are zero from the assumption of induction except for $k_1 = 2n+1$. When k_1 is even, $2n+1-k_1$ is an odd number greater than or equal to one. Setting $m_1 := 2n+1-k_1$ we have for $i = 1, 2$

$$\partial_s^{m_1}\boldsymbol{e}_0^i = -\sum_{k_2=0}^{m_1-1} \binom{m_1-1}{k_2} (\partial_s^{k_2} q_{0i})(\partial_s^{m_1-1-k_2}\boldsymbol{v}_0).$$

When k_2 is odd, it is less than or equal to $2n-1$, so those terms are zero from the assumption of induction, so only terms where k_2 is even remain. Then, $m_1 - 1 - k_2$ is an even number less than or equal to $2n$, so setting $k_1 = 2j_1$ and $k_2 = 2j_2$ we have

$$\begin{aligned}
\partial_s^{2n+2}\boldsymbol{v}_0(0) &= \partial_s^{2n+1} q_{01}\boldsymbol{e}_0^1 + \partial_s^{2n+1} q_{02}\boldsymbol{e}_0^2\Big|_{s=0} \\
&\quad + \sum_{j_1=0}^{n} \binom{2n+1}{2j_1} (\partial_s^{2j_1} q_{01}) \left\{ -\sum_{j_2=0}^{\frac{1}{2}(m_1-1)} \binom{m_1-1}{2j_2} (\partial_s^{2j_2} q_{01})(\partial^{m_1-1-2j_2}\boldsymbol{v}_0) \right\}\Bigg|_{s=0} \\
&\quad + \sum_{j_1=0}^{n} \binom{2n+1}{2j_1} (\partial_s^{2j_1} q_{02}) \left\{ -\sum_{j_2=0}^{\frac{1}{2}(m_1-1)} \binom{m_1-1}{2j_2} (\partial_s^{2j_2} q_{02})(\partial^{m_1-1-2j_2}\boldsymbol{v}_0) \right\}\Bigg|_{s=0}
\end{aligned}$$

Since the derivative on $\boldsymbol{v}_0$ in the summation is of even order, taking the exterior product with $\boldsymbol{v}_0$ will make all the terms zero from the compatibility condition on $\boldsymbol{v}_0$. So we finally arive at

$$\begin{aligned}
\mathbf{0} &= \boldsymbol{v} \times \partial_s^{2n+2}\boldsymbol{v}(0) \\
&= \partial_s^{2n+1} q_{01}\left(\boldsymbol{v}_0 \times \boldsymbol{e}_0^1\right) + \partial_s^{2n+1} q_{02}\left(\boldsymbol{v}_0 \times \boldsymbol{e}_0^2\right)\Big|_{s=0} \\
&= \partial_s^{2n+1} q_{01}\boldsymbol{e}_0^2 - \partial_s^{2n+1} q_{02}\boldsymbol{e}_0^1\Big|_{s=0}.
\end{aligned}$$

As before, since $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$ are perpendicular, $\partial_s^{2n+1} q_{01}(0) = \partial_s^{2n+1} q_{02}(0) = 0$ and this shows that q_0 satisfies compatibility condition.

6.2 Nonlinear Schrödinger Equation to the Vortex Filament Equation

Now given a smooth solution q of (6.2) with an initial datum q_0 satisfying the compatibility condition, we transform (6.2) to (6.1). First we define three vectors

$\boldsymbol{v}_0$, $\tilde{\boldsymbol{e}}^1$, and $\tilde{\boldsymbol{e}}^2$ by

$$\text{(6.4)}\qquad \begin{cases} \boldsymbol{v}_{0s} = q_{01}\tilde{\boldsymbol{e}}^1 + q_{02}\tilde{\boldsymbol{e}}^2, \\ \tilde{\boldsymbol{e}}^1_s = -q_{01}\boldsymbol{v}_0, \\ \tilde{\boldsymbol{e}}^2_s = -q_{02}\boldsymbol{v}_0, \\ (\boldsymbol{v}_0, \tilde{\boldsymbol{e}}^1, \tilde{\boldsymbol{e}}^2)(0) = (\boldsymbol{e}_3, \boldsymbol{e}_1, \boldsymbol{e}_2). \end{cases}$$

Here, these vectors are functions of s.

We must see if $\boldsymbol{v}_0$ satisfies the compatibility condition of (6.1), i.e. if $(\boldsymbol{v}_0 \times \partial_s^{2n}\boldsymbol{v}_0)(0) = \mathbf{0}$ for $n \in \mathbf{N}$. We note that by defining the matrix A as A$= (\boldsymbol{v}_0, \tilde{\boldsymbol{e}}^1, \tilde{\boldsymbol{e}}^2)$, we have

$$\mathrm{A}_s = \mathrm{A}\begin{pmatrix} 0 & q_{02s} & -q_{01s} \\ -q_{02s} & 0 & \frac{1}{2}|q_0|^2 \\ q_{01s} & -\frac{1}{2}|q_0|^2 & 0 \end{pmatrix} =: \mathrm{AP}.$$

We immediately see that since A and P are anti-symmetric,

$$\left(\mathrm{AA}^T\right)_s = \mathrm{APA}^T + \mathrm{A}(\mathrm{AP})^T = \mathrm{A}(\mathrm{P} + \mathrm{P}^T)\mathrm{A} = \mathrm{O},$$

where A^T is the transpose matrix of A. Thus we have $\mathrm{AA}^T(s) = \mathrm{AA}^T(0) = \mathrm{I}_3$. I_3 is the 3×3 unit matrix, and this shows that $\{\boldsymbol{v}_0, \tilde{\boldsymbol{e}}^1, \tilde{\boldsymbol{e}}^2\}_{s\geq 0}$ is an orthonormal basis of $\mathbf{R}^3$. $\tilde{\boldsymbol{e}}^2$ is actually $\boldsymbol{v}_0 \times \tilde{\boldsymbol{e}}^1$, but we use $\tilde{\boldsymbol{e}}^2$ for simplicity of notation. By differentiating the equation for $\boldsymbol{v}_0$ and using the other two equations we get

$$\boldsymbol{v}_{0ss} = q_{01s}\tilde{\boldsymbol{e}}^1 - (q_{01})^2\boldsymbol{v}_0 + q_{02s}\tilde{\boldsymbol{e}}^2 - (q_{02})^2\boldsymbol{v}_0.$$

Taking the exterior product of the above equation with $\boldsymbol{v}_0$ and setting $s = 0$, we see that $(\boldsymbol{v}_0 \times \boldsymbol{v}_{0ss})(0) = \mathbf{0}$. Thus the condition is true for $n = 1$. Suppose that it holds up to $n-1$. Differentiating the equation for $\boldsymbol{v}_0$ $(2n-1)$ times yields

$$\partial_s^{2n}\boldsymbol{v}_0 = \sum_{k_1=0}^{2n-1}\begin{pmatrix} 2n-1 \\ k_1 \end{pmatrix}(\partial_s^{k_1}q_{01})(\partial_s^{2n-1-k_1}\tilde{\boldsymbol{e}}^1) + \sum_{k_1=0}^{2n-1}\begin{pmatrix} 2n-1 \\ k_1 \end{pmatrix}(\partial_s^{k_1}q_{02})(\partial_s^{2n-1-k_1}\tilde{\boldsymbol{e}}^2).$$

At $s = 0$, the terms where k_1 is odd are zero from the compatibility condition for q_0. When k_1 is even, $2n-1-k_1$ is an odd number greater than or equal to one. Setting $m_1 := 2n-1-k_1$ we have for $i = 1, 2$

$$\partial_s^{m_1}\tilde{\boldsymbol{e}}^i = -\sum_{k_2=0}^{m_1-1}\begin{pmatrix} m_1-1 \\ k_2 \end{pmatrix}(\partial_s^{k_2}q_{0i})(\partial_s^{m_1-1-k_2}\boldsymbol{v}_0).$$

Again only terms where k_2 is even remain. Then, $m_1 - 1 - k_2$ is an even number less than or equal to $2(n-1)$, so setting $k_1 = 2j_1$ and $k_2 = 2j_2$ we have

$$\begin{aligned} \partial_s^{2n}\boldsymbol{v}_0(0) = &\sum_{j_1=0}^{n-1}\begin{pmatrix} 2n-1 \\ 2j_1 \end{pmatrix}(\partial_s^{2j_1}q_{01})\left\{-\sum_{j_2=0}^{\frac{1}{2}(m_1-1)}\begin{pmatrix} m_1-1 \\ 2j_2 \end{pmatrix}(\partial_s^{2j_2}q_{01})(\partial^{m_1-1-2j_2}\boldsymbol{v}_0)\right\}\Bigg|_{s=0} \\ &+\sum_{j_1=0}^{n-1}\begin{pmatrix} 2n-1 \\ 2j_1 \end{pmatrix}(\partial_s^{2j_1}q_{02})\left\{-\sum_{j_2=0}^{\frac{1}{2}(m_1-1)}\begin{pmatrix} m_1-1 \\ 2j_2 \end{pmatrix}(\partial_s^{2j_2}q_{02})(\partial^{m_1-1-2j_2}\boldsymbol{v}_0)\right\}\Bigg|_{s=0}. \end{aligned}$$

Since the derivative on $\boldsymbol{v}_0$ is of even order less than or equal to $2(n-1)$ on the right-hand side, taking the exterior product of this with $\boldsymbol{v}_0$ yields $(\boldsymbol{v}_0 \times \partial_s^{2n}\boldsymbol{v}_0)(0) = \mathbf{0}$ from the assumption of induction. So $\boldsymbol{v}_0$ constructed here satisfies the compatibility condition of (6.1).

Set $q(s,t) = q_1(s,t) - \mathrm{i}q_2(s,t)$. For any $s \geq 0$, we extend the three vectors in the t direction as the solution of

$$\begin{cases} \boldsymbol{v}_t = -q_{2s}\boldsymbol{e}^1 + q_{1s}\boldsymbol{e}^2, & t > 0, \\ \boldsymbol{e}^1_t = q_{2s}\boldsymbol{v} - \frac{1}{2}|q|^2\boldsymbol{e}^2, & t > 0, \\ \boldsymbol{e}^2_t = -q_{1s}\boldsymbol{v} + \frac{1}{2}|q|^2\boldsymbol{e}^1, & t > 0, \\ (\boldsymbol{v}, \boldsymbol{e}^1, \boldsymbol{e}^2)(s,0) = (\boldsymbol{v}_0(s), \tilde{\boldsymbol{e}}^1(s), \tilde{\boldsymbol{e}}^2(s)). \end{cases}$$

We express $\boldsymbol{v}, \boldsymbol{e}^1, \boldsymbol{e}^2$ as column vectors. Then we have

$$\begin{aligned} (\boldsymbol{v}, \boldsymbol{e}^1, \boldsymbol{e}^2)_t &= (-q_{2s}\boldsymbol{e}^1 + q_{1s}\boldsymbol{e}^2, q_{2s}\boldsymbol{v} - \frac{1}{2}|q|^2\boldsymbol{e}^2, -q_{1s}\boldsymbol{v} + \frac{1}{2}|q|^2\boldsymbol{e}^1) \\ &= (\boldsymbol{v}, \boldsymbol{e}^1, \boldsymbol{e}^2)\begin{pmatrix} 0 & q_{2s} & -q_{1s} \\ -q_{2s} & 0 & \frac{1}{2}|q|^2 \\ q_{1s} & -\frac{1}{2}|q|^2 & 0 \end{pmatrix}. \end{aligned}$$

As before, since the coefficient matrix is anti-symmetric, $\{\boldsymbol{v}, \boldsymbol{e}^1, \boldsymbol{e}^2\}$ forms an orthonormal basis and $\boldsymbol{e}^2 = \boldsymbol{v} \times \boldsymbol{e}^1$. From here we denote $\boldsymbol{e}^1$ as simply $\boldsymbol{e}$. Since $s \geq 0$ is arbitrary in the above argument, $0 = (\frac{1}{2}|\boldsymbol{v}|^2)_s = \boldsymbol{v} \cdot \boldsymbol{v}_s$. So $\boldsymbol{v}_s$ can be expressed as

$$\boldsymbol{v}_s = \tilde{q}_1\boldsymbol{e} + \tilde{q}_2(\boldsymbol{v} \times \boldsymbol{e}).$$

From $|\boldsymbol{e}| = 1$ and $\boldsymbol{e} \cdot \boldsymbol{v} = 0$ we see that

$$\begin{aligned} \boldsymbol{e}_s &= -\tilde{q}_1\boldsymbol{v} + \alpha(\boldsymbol{v} \times \boldsymbol{e}), \\ (\boldsymbol{v} \times \boldsymbol{e})_s &= -\tilde{q}_2\boldsymbol{v} - \alpha\boldsymbol{e}, \end{aligned}$$

where $\tilde{q}_i$ and α are unknown functions. From the way that we constructed $\tilde{\boldsymbol{e}}^1$, we see that at $t = 0$

$$\tilde{q}_1 = q_{01}, \ \tilde{q}_2 = q_{02}, \ \alpha = 0.$$

As before, from $\boldsymbol{v}_{st} = \boldsymbol{v}_{ts}$ and $\boldsymbol{e}_{st} = \boldsymbol{e}_{ts}$ we have

$$\begin{cases} \tilde{q}_{1t} = -q_{2ss} - \frac{1}{2}|q|^2\tilde{q}_2 - \alpha q_{1s}, \\ \tilde{q}_{2t} = q_{1ss} + \frac{1}{2}|q|^2\tilde{q}_1 - \alpha q_{2s}, \\ \alpha_t = \tilde{q}_1 q_{1s} + \tilde{q}_2 q_{2s} - (\frac{1}{2}|q|^2)_s, \\ (\tilde{q}_1, \tilde{q}_2, \alpha)(s,0) = (q_{01}(s), q_{02}(s), 0). \end{cases}$$

By setting $W_1 := \tilde{q}_1 - q_1$ and $W_2 := \tilde{q}_2 - q_2$ we have

$$\begin{cases} W_{1t} = -\frac{1}{2}|q|^2 W_2 - \alpha q_{1s}, \\ W_{2t} = \frac{1}{2}|q|^2 W_1 - \alpha q_{2s}, \\ \alpha_t = q_{1s}W_1 + q_{2s}W_2, \\ (W_1, W_2, \alpha)(s,0) = (0,0,0). \end{cases}$$

If we set $\boldsymbol{W} := (W_1, W_2, \alpha)^T$, we have

$$\boldsymbol{W}_t = \begin{pmatrix} 0 & -\frac{1}{2}|q|^2 & -q_{1s} \\ \frac{1}{2}|q|^2 & 0 & -q_{2s} \\ q_{1s} & q_{2s} & 0 \end{pmatrix} \boldsymbol{W}.$$

Since the coefficient matrix is anti-symmetric, we have $|\boldsymbol{W}(s,t)| = |\boldsymbol{W}(s,0)| = 0$, which is equivalent to $\tilde{q}_i = q_i$ for $i = 1,2$ and $\alpha = 0$. From direct calculation we have

$$\boldsymbol{v} \times \boldsymbol{v}_{ss} = (\boldsymbol{v} \times \boldsymbol{v}_s)_s = \{q_1(\boldsymbol{v} \times \boldsymbol{e}) - q_2\boldsymbol{e}\}_s = q_{1s}(\boldsymbol{v} \times \boldsymbol{e}) - q_{2s}\boldsymbol{e} = \boldsymbol{v}_t.$$

From the boundary condition imposed on q, we see that

$$\boldsymbol{v}_t(0,t) = -q_{2s}(0,t)\boldsymbol{e} + q_{1s}(0,t)(\boldsymbol{v} \times \boldsymbol{e}) = \boldsymbol{0}.$$

Integrating in t yields

$$\boldsymbol{v}(0,t) = \boldsymbol{v}_0(0) = \boldsymbol{e}_3.$$

So we see that the vector function $\boldsymbol{v}$ that we constructed is a solution of (6.1).

So we have proven that our initial-boundary value problem for the vortex filament equation is equivalent to an initial-boundary value problem for the nonlinear Schrödinger equation.

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