THE COHOMOLOGY RING OF THE GKM GRAPH
OF A FLAG MANIFOLD

1. Introduction

Let $T$ be a torus of dimension $n$ and $M$ a closed smooth $T$-manifold. The equivariant cohomology of $M$, denoted $H^*_{T}(M)$, contains a lot of geometrical information on $M$. Moreover it is often easier to compute $H^*_{T}(M)$ than $H^*(M)$ by virtue of the Localization Theorem which implies that the restriction map

(1.1) $i^*: H^*_{T}(M) \rightarrow H^*_{T}(M^T)$

to the $T$-fixed point set $M^T$ is often injective, in fact, this is the case when $H^{odd}(M) = 0$. When $M^T$ is isolated, $H^*_{T}(M^T) = \oplus_{p \in M^T} H^*_{T}(p)$ and hence $H^*_{T}(M^T)$ is a direct sum of copies of a polynomial ring in $n$ variables because $H^*_{T}(p) = H^*(BT)$.

Therefore we are in a nice situation when $H^{odd}(M) = 0$ and $M^T$ is isolated. Goresky-Kottwitz-MacPherson [2] (see also [3, Chapter 11]) found that under the further condition that the weights at a tangential $T$-module are pairwise linearly independent at each $p \in M^T$, the image of $i^*$ in (1.1) above is determined by the fixed point sets of codimension one subtori of $T$ when $Q$ is tensored in cohomology. Their result motivated Guillemin-Zara [4] to associate a labeled graph $\mathcal{G}_M$ with $M$ and define the "cohomology" ring $H^*(\mathcal{G}_M)$ of $\mathcal{G}_M$, which is a subring of $\oplus_{p \in M^T} H^*(BT)$. Then the result of Goresky-Kottwitz-MacPherson can be stated that $H^*_{T}(M) \otimes Q$ is isomorphic to $H^*(\mathcal{G}_M) \otimes Q$ as graded rings when $M$ satisfies the conditions mentioned above.

The result of Goresky-Kottwitz-MacPherson can be applied to many important $T$-manifolds $M$ such as flag manifolds and compact smooth toric varieties etc. When $M$ is such a nice manifold, $H^*_{T}(M)$ is often known to be isomorphic to $H^*(\mathcal{G}_M)$ without tensoring with $Q$ (see [1], [5], [6] for example). We determine the ring structure of $H^*(\mathcal{G}_M)$ or $H^*(\mathcal{G}_M) \otimes Z[\frac{1}{2}]$ when $M$ is a flag manifold of classical type directly without using the fact...
that $H^*_T(M)$ is isomorphic to $H^*(G_M)$ ([7]). In my talk, I introduced the result when $M$ is a flag manifold of type A. This is a joint work with Hiroaki Ishida and Mikiya Masuda and the details can be found in [7].

2. Labeled graph and its cohomology for type $A_{n-1}$

Let $\{t_i\}_{i=1}^n$ be a basis of $H^2(BT)$, so that $H^*(BT)$ can be identified with a polynomial ring $\mathbb{Z}[t_1, t_2, \ldots, t_n]$. We take an inner product on $H^2(BT)$ such that the basis $\{t_i\}$ is orthonormal. Then

\[(2.1) \quad \Phi(A_{n-1}) := \{\pm(t_i - t_j) | 1 \leq i < j \leq n\}\]

is a root system of type $A_{n-1}$.

**Definition.** The labeled graph associated with $\Phi(A_{n-1})$, denoted $\mathcal{A}_n$, is a graph with labeling $\ell$ defined as follows.

- The vertex set of $\mathcal{A}_n$ is the permutation group $S_n$ on $\{1, 2, \ldots, n\}$.
- Two vertices $w, w'$ in $\mathcal{A}_n$ are connected by an edge $e_{w,w'}$ if and only if there is a transposition $(i, j) \in S_n$ such that $w' = w(i, j)$, in other words,
  \[w'(i) = w(j), \quad w'(j) = w(i) \quad \text{and} \quad w'(r) = w(r) \quad \text{for} \quad r \neq i, j.\]
- The edge $e_{w,w'}$ is labeled by $\ell(e_{w,w'}) := t_{w(i)} - t_{w'(i)}$.

**Definition.** The cohomology ring of $\mathcal{A}_n$, denoted $H^*(\mathcal{A}_n)$, is defined to be the subring of $\text{Map}(V(\mathcal{A}_n), H^*(BT)) = \bigoplus_{v \in V(\mathcal{A}_n)} H^*(BT)$, where $V(\mathcal{A}_n)$ denotes the set of vertices of $\mathcal{A}_n$, i.e. $V(\mathcal{A}_n) = S_n$, satisfying the following condition:

\[f \in \text{Map}(V(\mathcal{A}_n), H^*(BT)) \text{ is an element of } H^*(\mathcal{A}_n) \text{ if and only if } f(v) - f(v') \text{ is divisible by } \ell(e) \text{ in } H^*(BT) \text{ whenever the vertices } v \text{ and } v' \text{ are connected by an edge } e \text{ in } \mathcal{A}_n.\]

For each $i = 1, \ldots, n$, we define elements $\tau_i, t_i$ of $\text{Map}(V(\mathcal{A}_n), H^*(BT))$ by

\[(2.2) \quad \tau_i(w) := t_{w(i)}, \quad t_i(w) := t_i \quad \text{for } w \in S_n.\]

In fact, both $\tau_i$ and $t_i$ are elements of $H^2(\mathcal{A}_n)$.

**Example.** The case $n = 3$. The root system $\Phi(A_2)$ is $\{\pm(t_i - t_j) | 1 \leq i < j \leq 3\}$. The labeled graph $\mathcal{A}_3$ and $\tau_i$ for $i = 1, 2, 3$ are as follows.
Theorem 2.1. Let $\mathcal{A}_n$ be the labeled graph associated with the root system $\Phi(A_{n-1})$ of type $A_{n-1}$ in (2.1). Then

$$\mathcal{H}^*(\mathcal{A}_n) = \mathbb{Z}[\tau_1, \cdots, \tau_n, t_1, \cdots, t_n]/(e_i(\tau) - e_i(t) | i = 1, \cdots, n),$$

where $e_i(\tau)$ (resp. $e_i(t)$) is the $i^{th}$ elementary symmetric polynomial in $\tau_1, \cdots, \tau_n$ (resp. $t_1, \cdots, t_n$).

To prove this theorem, we need the following two lemmas.

Lemma 2.2. $\mathcal{H}^*(\mathcal{A}_n)$ is generated by $\tau_1, \cdots, \tau_n, t_1, \cdots, t_n$ as a ring.

Proof. We shall prove the lemma by induction on $n$. When $n = 1$, $\mathcal{H}^*(\mathcal{A}_1)$ is generated by $t_1$ since $\mathcal{A}_1$ is a point; so the lemma holds.

Suppose that the lemma holds for $n - 1$. Then it suffices to show that any homogeneous element $f$ of $\mathcal{H}^*(\mathcal{A}_n)$, say of degree $2k$, can be expressed as a polynomial in $\tau_i$'s and $t_i$'s. For each $i = 1, \ldots, n$, we set

$$V_i := \{w \in S_n | w(i) = n\}$$

and consider the labeled full subgraph $\mathcal{L}_i$ of $\mathcal{A}_n$ with $V_i$ as the vertex set. Note that $\mathcal{L}_i$ can naturally be identified with $\mathcal{A}_{n-1}$ for any $i$.

Let

$$1 \leq q \leq \min\{k + 1, n\}$$

and assume that

$$f(\nu) = 0 \quad \text{for any } \nu \in V_i \text{ whenever } i < q.$$  

A vertex $w$ in $V_q$ is connected by an edge in $\mathcal{A}_n$ to a vertex $\nu$ in $V_i$ if and only if $\nu = w(i, q)$. In this case $f(w) - f(\nu)$ is divisible by $t_{w(i)} - t_{w(q)} = t_{w(i)} - t_n$ and $f(\nu) = 0$ whenever $i < q$ by (2.4), so $f(w)$ is divisible by $t_{w(i)} - t_n$ for $i < q$. Thus, for each $w \in V_q$, there is a homogeneous and of degree $2k$.$g^q(w) \in \mathbb{Z}[t_1, \cdots, t_n]$ such that

$$f(w) = (t_{w(1)} - t_n)(t_{w(2)} - t_n) \cdots (t_{w(q-1)} - t_n)g^q(w)$$

where $g^q(w)$ is homogeneous and of degree $2(k + 1 - q)$ because $f(w)$ is homogeneous and of degree $2k$.

One expresses

$$g^q(w) = \sum_{r=0}^{k+1-q} g^q_r(w)t_n^r$$

with homogenous polynomials $g^q_r(w)$ of degree $2(k+1-q-r)$ in $\mathbb{Z}[t_1, \cdots, t_{n-1}]$. Then there is a polynomial $G^q_r$ in $\tau_i$'s (except $\tau_q$) and $t_i$'s (except $t_n$) such that $G^q_r(w) = g^q_r(w)$ for any $w \in V_q$, because $g^q_r$ restricted to $\mathcal{L}_q$ is an element of $\mathcal{H}^*(\mathcal{L}_q) = \mathcal{H}^*(\mathcal{A}_{n-1})$. 

Since \( \tau_i(w) = t_{w(i)} \) and \( w(i) = n \) for \( w \in V_i \), we have

\[
(2.7) \quad \prod_{i=1}^{q-1} (\tau_i - t_n)(w) = 0 \quad \text{for any } w \in V_i \text{ whenever } i < q.
\]

Therefore, it follows from (2.5), (2.6), the Claim above and (2.7) that putting \( G^q = \sum_{r=0}^{k+1-q} G_{r}^{q} t_{n}^{r} \), we have

\[
(f - G^q \prod_{i=1}^{q-1} (\tau_i - t_n))(w) = f(w) - g^q(w) \prod_{i=1}^{q-1} (t_{w(i)} - t_n) = 0 \quad \text{for any } w \in V_i \text{ whenever } i \leq q.
\]

Therefore, subtracting the polynomial \( G^q \prod_{i=1}^{q-1} (\tau_i - t_n) \) from \( f \), we may assume that

\[
f(v) = 0 \quad \text{for any } v \in V_i \text{ whenever } i < q + 1.
\]

The above argument implies that \( f \) finally takes zero on all vertices of \( \mathcal{A}_n \) (which means \( f = 0 \)) by subtracting a polynomial in \( \tau_i \)'s and \( t_i \)'s, and this completes the induction step. \( \square \)

We abbreviate the polynomial ring \( \mathbb{Z}[\tau_1, \cdots, \tau_n, t_1, \cdots, t_n] \) as \( \mathbb{Z}[\tau, t] \). The canonical map \( \mathbb{Z}[\tau, t] \to H^*(\mathcal{A}_n) \) is a grade preserving homomorphism which is surjective by Lemma 2.2. Let \( e_i(\tau) \) (resp. \( e_i(t) \)) denote the \( i^{th} \) elementary symmetric polynomial in \( \tau_1, \cdots, \tau_n \) (resp. \( t_1, \cdots, t_n \)). It easily follows from (2.2) that \( e_i(\tau) = e_i(t) \) for \( i = 1, \cdots, n \). Therefore the canonical map above induces a grade preserving epimorphism

\[
(2.8)\quad \mathbb{Z}[\tau, t]/(e_1(\tau) - e_1(t), \cdots, e_n(\tau) - e_n(t)) \to H^*(\mathcal{A}_n).
\]

Remember that the Hilbert series of a graded ring \( A^* = \oplus_{j=0}^{\infty} A^j \), where \( A^j \) is the degree \( j \) part of \( A^* \) and of finite rank over \( \mathbb{Z} \), is a formal power series defined by

\[
F(A^*, s) := \sum_{j=0}^{\infty} (\text{rank}_\mathbb{Z} A^j) s^j.
\]

In order to prove that the epimorphism in (2.8) is an isomorphism, it suffices to verify the following lemma because the modules in (2.8) are both torsion free.

**Lemma 2.3.** The Hilbert series of the both sides at (2.8) coincide, in fact, they are given by \( \frac{1}{(1-s^2)^n} \prod_{i=1}^{n} (1 - s^{2i}) \).

**Proof.** (1) Calculation of LHS at (2.8). Let \( e_i := e_i(\tau) - e_i(t) \). It follows from the exact sequence

\[
0 \to (e_1, \cdots, e_n) \to \mathbb{Z}[\tau, t] \to \mathbb{Z}[\tau, t]/(e_1, \cdots, e_n) \to 0
\]
that we have

(2.9) \[ F(\mathbb{Z}[\tau, t]/(e_1, \ldots, e_n), s) = F(\mathbb{Z}[\tau, t], s) - F((e_1, \ldots, e_n), s). \]

Here, since \( \deg \tau_i = \deg t_i = 2 \), we have

(2.10) \[ F(\mathbb{Z}[\tau, t], s) = \frac{1}{(1-s^2)^{2n}} \]

as easily checked; so it suffices to calculate \( F((e_1, \ldots, e_n), s) \).

For \( I \subset [n] \) we set \( e_I := \prod_{i \in I} e_i \). Then it follows from the Inclusion-Exclusion principle that

(2.11) \[ F((e_1, \ldots, e_n), s) = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} F((e_I), s) \]

and since \( F((e_I), s) = s^{\deg e_I}/(1-s^2)^{2n} \) and \( \deg e_I = \sum_{i \in I} 2i \), it follows from (2.11) that

(2.12) \[ F((e_1, \ldots, e_n), s) = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} \frac{s^{\sum_{i \in I} 2i}}{(1-s^2)^{2n}}. \]

Therefore it follows from (2.9), (2.10) and (2.12) that

\[
F(\mathbb{Z}[\tau, t]/(e_1, \ldots, e_n), s) = \frac{1}{(1-s^2)^{2n}} - \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} \frac{s^{\sum_{i \in I} 2i}}{(1-s^2)^{2n}}
\]

(2.13) \[
= \frac{1}{(1-s^2)^{2n}} \sum_{I \subset [n]} \frac{(-1)^{|I|-1} \prod_{i=1}^{n} (1-s^{2i})}{(1-s^2)^{2n}}.
\]

(2) Calculation of RHS at (2.8). Let \( d_n(k) := \text{rank}_\mathbb{Z} \mathcal{H}^{2k}(\mathcal{A}_n) \). Then

(2.14) \[ F(\mathcal{H}^*(\mathcal{A}_n), s) = \sum_{k=0}^{\infty} d_n(k)s^{2k}. \]

Recall the argument in the proof of Lemma 2.2. Since \( g^q \) in (2.6) belongs to \( \mathcal{H}^{2(k+1-q-r)}(\mathcal{L}_q) = \mathcal{H}^{2(k+1-q-r)}(\mathcal{A}_{n-1}) \) as shown in the Claim there, the rank of the module consisting of those \( g^q \) in (2.5) and (2.6) is given by

\[
\sum_{r=0}^{k+1-q} d_{n-1}(k+1-q-r) = \sum_{r=0}^{k+1-q} d_{n-1}(r).
\]

Therefore, noting (2.3), we see that the argument in the proof of Lemma 2.2 implies

\[
d_n(k) = \sum_{q=1}^{\min(k+1,n)} \sum_{r=0}^{k+1-q} d_{n-1}(r),
\]
in other words, if we set $d_{n-1}(j) = 0$ for $j < 0$, then

\begin{equation}
\begin{aligned}
d_n(k) = \begin{cases} 
\sum_{i=1}^{n} id_{n-1}(k + 1 - i) & \text{if } k \leq n - 1, \\
\sum_{i=1}^{n} id_{n-1}(k + 1 - i) + n \sum_{j=n+1}^{k+1} d_{n-1}(k + 1 - i) & \text{if } k \geq n.
\end{cases}
\end{aligned}
\end{equation}

We shall abbreviate $F(\mathcal{H}^*(\mathcal{A}_n), s)$ as $F_n(s)$. Then, plugging (2.15) in (2.14), we obtain

\begin{align*}
F_n(s) &= \sum_{k=0}^{\infty} \left( d_{n-1}(k) + \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} id_{n-1}(k+1-i) + \cdots + nd_{n-1}(k+1-n) \right) s^{2k} \\
&= F_{n-1}(s) + 2s^2F_{n-1}(s) + \cdots + ns^{2n}F_{n-1}(s) \\
&= F_{n-1}(s)\left( 1 + 2s^2 + \cdots + ns^{2n-2} \right) + \frac{1 - s^{2n}}{1 - s^2} F_{n-1}(s) \\
&= \frac{1 - s^{2n}}{1 - s^2} F_{n-1}(s).
\end{align*}

On the other hand, $F_1(s) = 1/(1 - s^2)$ since $\mathcal{H}^*(\mathcal{A}_1) = \mathbb{Z}[t_1]$. It follows that

\begin{align*}
F_n(s) &= \frac{1}{(1 - s^2)^n} \prod_{i=1}^{n} (1 - s^{2i}).
\end{align*}

This together with (2.13) proves the lemma. \qed

REFERENCES


