PART I

§1, Background....

Manin (Denninger, Kurokawa, Kapranov-Smironov...) suggested \( \exists \) a curve \( C = \overline{\text{Spec} \mathbb{Z}} \) "defined over" \( \mathbb{F}_1 \), whose "zeta function" \( \zeta_C(s) \) is the complete Riemann zeta function \( \zeta_{\mathbb{Q}}(s) \):

\[
\zeta_C(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) =: \zeta_{\mathbb{Q}}(s)
\]

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p: \text{primes}} \frac{1}{1-p^{-s}} \quad (\Re(s) > 1)
\]

\[
\Gamma(s):= \int_{0}^{\infty} x^{s-1} e^{-x} dx \quad (\Re(s) > 0)
\]

Furthermore, they suggested the Riemann hypothesis may be solved in a fashion similar to the Weil conjecture for smooth schemes defined over a finite field \( \mathbb{F}_q \) (\( q \to 1 \)).

Kato, Kurokawa-Ochiai-Wakayama, Deitmar, Toen-Vaquie, Haran, Durov, Soulé, Connes-Conani... proposed some similar notions of \( \mathbb{F}_1 \)-schemes.

(commutative rings \( \to \) commutative monoid with 0)

Deitmar-Kurokawa-Koyama, Kurokawa-Ochiai, Soule, Connes-Consani proposed different kinds of zeta functions of \( \mathbb{F}_1 \)-schemes.
\[ |X(\mathbb{F}_{q^{n}})| \to |Y(\mathbb{Z}/n)| \]
\[ |(\text{Spec } R)(\mathbb{F}_{q^{n}})| = |\text{Hom}_{\text{rings}}(R, \mathbb{F}_{q^{n}})| \]
\[ |(\text{Spec } A)(\mathbb{Z}/n)| = |\text{Hom}_{\text{groups}}(A, \mathbb{Z}/n)| \]

§2. The plan of the paper

PART I: (Soule, Connes-Consani) Rough idea of the $F_{1}$-scheme and the zeta function for some class of $F_{1}$-scheme.

PART II: (Connes-Consani) A similarity between "the counting functions" of the "hypothetical $C = \text{Spec } \mathbb{Z}$" , and an irreducible smooth projective algebraic curve defined over a finite field.

PART III: (Connes-Consani, Deitmar-Kurokawa-Koyama, M) $F_{1}$-zeta functions of Deitmar-Kurokawa-Koyama and Kurokawa-Ochiai, some invariants for finite abelian groups, and an expression of the Soule-Connes-Consani zeta function for general, not necessarily torsion free, Noetherian $F_{1}$-schemes.

§3. A rough idea of the $F_{1}$-scheme

There is a very general theory of $F_{1}$-scheme, e.g.

[CC] Alain Connes and Caterina Consani, "Schemes over $F_{1}$ and zeta functions", ArXiv0903.2024

which employs the functor-of-points philosophy for the category $\text{Ring} \cup_{\text{adjoint}} \text{Monoid}_{0}$. 

\[
\text{Monoid}_{0} \rightleftarrows \text{Ring} \\
\text{Hom}_{\text{Ring}}(Z[M], R) \cong \text{Hom}_{\text{Monoid}_{0}}(M, R) \\
M \mapsto Z[M] \quad (0_{M} \mapsto 0_{Z[M]}) \\
R \leftrightarrow R
\]
\[
\text{Ob} (\text{Ring} \cup_{\text{adjoint}} \text{Monoid}_{0}) = \text{Ob} (\text{Ring}) \coprod \text{Ob} (\text{Monoid}_{0})
\]

\[
\text{Hom}_{\text{Ring} \cup_{\text{adjoint}} \text{Monoid}_{0}} (X, Y) = \begin{cases} 
\text{Hom}_{\text{Ring}} (X, Y) & \text{if } X, Y \in \text{Ring} \\
\text{Hom}_{\text{Monoid}_{0}} (X, Y) & \text{if } X, Y \in \text{Monoid}_{0} \\
\text{Hom}_{\text{Ring}} (\mathbb{Z}[X], Y) & \text{if } X \in \text{Ring}, Y \in \text{Monoid}_{0} \\
\cong \text{Hom}_{\text{Monoid}_{0}} (X, Y) & \text{if } X \in \text{Monoid}_{0}, Y \in \text{Ring}
\end{cases}
\]

A $F_{1}$-functor is by definition a functor

\[
\text{Ring} \cup_{\text{adjoint}} \text{Monoid}_{0} \to \text{Set},
\]

which is equivalent to the following data:

- $X : \text{Monoid}_{0} \to \text{Set}$
- $X_{Z} : \text{Ring} \to \text{Set}$
- $e : X \to X_{Z} \circ \beta$, where $\beta : \text{Monoid}_{0} \to \text{Ring}$, $M \mapsto \mathbb{Z}[M]$ ($0_{M} \mapsto 0_{\mathbb{Z}[M]}$)

\[
e \iff e : X \circ \beta^{*} \to X_{Z}, \text{ where } \beta^{*} : \text{Ring} \to \text{Monoid}_{0}, R \mapsto R
\]

Connes-Consani defined a $F_{1}$-scheme $\mathcal{X}$ to be a $F_{1}$-functor

\[
\text{Ring} \cup_{\text{adjoint}} \text{Monoid}_{0} \to \text{Set}
\]

s.t.

- $X_{Z}$, its restriction to $\text{Ring}$, is a $\mathbb{Z}$-scheme.
- $X$, its restriction to $\text{Monoid}_{0}$, is a $\mathfrak{M}_{0}$-scheme.
- the natural transformation $e : X \circ \beta^{*} \to X_{Z}$, associated to a field, is a bijection of sets. In particular,

\[
\begin{array}{c}
\text{Hom}_{-\text{sch}} (\text{Spec} F_{q}, X_{Z}) \\
\downarrow e
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
\text{Hom}_{-\text{sch}} (\text{Spec} F_{1^{(q-1)}}, \underline{X})
\end{array}
\]

\[
\begin{array}{c}
\text{Hom}_{-\text{sch}} (\text{Spec} F_{q^{n}}, X_{Z}) \\
\downarrow
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
\text{Hom}_{-\text{sch}} (\text{Spec} F_{1^{n}}, \underline{X})
\end{array}
\]

Here, \( \lim_{q \to 1} F_{q^{n}} \simeq F_{1^{n}} := F_{1} [\mathbb{Z}/n \mathbb{Z}] := \mathbb{Z}/n \mathbb{Z} \cup \{0\} \)

For Noetherian $F_{1}$-scheme $\mathcal{X}$ (both $X_{Z}$ and $X$ admit a finite open cover by Noetherian affine representables in each category),

(1) there are just finitely many “points” in $X$.
(2) at each such a point $x \in X$, the “residue field” $\kappa(x) = F_{1} [\mathcal{O}_{x}^{\times}]$ is a finitely generated abelian group $\mathcal{O}_{x}^{\times} = \mathbb{Z}^{n(x)} \times \prod_{j} \mathbb{Z}/m_{j}(x) \mathbb{Z}$
\begin{enumerate}
\item \( X(F_{1^{n}}) = \text{Hom}_{\mathcal{M}_{0}-\text{sch}}(\text{Spec}F_{1^{n}}, X) = \bigsqcup_{x \in X} \text{Hom}_{\mathfrak{U}b}(O_{x}^{\times}, \mathbb{Z}/n\mathbb{Z}) \),
\end{enumerate}

where

\[ \left( \lim_{q \to 1} F_{q^{n}} \simeq \right) F_{1^{n}} := F_{1}[\mathbb{Z}/n\mathbb{Z}] := \mathbb{Z}/n\mathbb{Z} \cup \{0\} \]

In general, a \( \mathcal{M}_{0} \)-scheme \( X \) is caleed \textit{torsion free}, if \( O_{x}^{\times} \) is a torsion free abelian group for any \( x \in X \).

\section{4}

The zeta function for some class of \( F_{1} \)-scheme by Soule, Connes-Consani

(Deitmar, Connes-Consani) For a Noetherian \( F_{1} \)-scheme \( X \) with \( X \) torsion free, \( \exists N(u+1) \in \mathbb{Z}_{\geq 0}[u] \) s.t.

\[ |X(F_{1^{n}})| = N(n+1), \quad \forall n \in \mathbb{N} \]

In particular,

\[ |X_{\mathbb{Z}}(F_{q})| = |X(F_{1^{n}(q-1)})| = N(q), \quad \forall q, \text{ a prime power} \]

So, set \( N(u+1) := \sum_{x \in X} u^{n(x)} \in \mathbb{Z}_{\geq 0}[u]. \)

So, we are naively lead to define the zeta function of \( X \) as the Hasse zeta function of \( X_{\mathbb{Z}} \), as our first attempt:

\[ \zeta(s, X_{\mathbb{Z}}) := \prod_{p \text{ prime}} \zeta(s, X_{\mathbb{Z}}/F_{p}), \]

where \( \zeta(s, X_{\mathbb{Z}}/F_{p}) \) is the congruence zeta function

\[ \zeta(s, X_{\mathbb{Z}}/F_{p}) = \exp \left( \sum_{m=1}^{\infty} \frac{|X_{\mathbb{Z}}(F_{p^{m}})|}{m} p^{-ms} \right) \]

\textbf{Bad New. (Soule, Deitmar, Kurokawa)} When \( N(v) = N(u+1) = \sum_{x \in X} u^{n(x)} = \sum_{x \in X} u^{n(x)} = \sum_{x \in X} (v-1)^{n(x)} = \sum_{k=0}^{d} a_{k} v^{k} \), \( (a_{k} \in \mathbb{Z}) \),

\[ \zeta(s, X_{\mathbb{Z}}) = \prod_{k=0}^{d} \zeta(s-k)^{a_{k}}, \quad \zeta(s, X_{\mathbb{Z}}/F_{p}) = \prod_{k=0}^{d} (1 - p^{k-s})^{-a_{k}} \]

(Too complicated and redundant for such simple (comparing with \( C = \overline{\text{Spec} \mathbb{Z}} \)) \( X ! \))
Good News. (Soule, predicted by Manin, corrected by Kurokawa) For a Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$ with $\mathcal{X}$ torsion free (so, $\exists N(v) = \sum_{k=0}^{d} a_k v^k \in \mathbb{Z}[v]$ s.t. $|\mathcal{X}(\mathbb{F}_1^n)| = N(n+1)$, $\forall n \in \mathbb{N}$),

$$\zeta_\mathcal{X}(s) := \lim_{p \to 1^-} \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p)(p-1)^{N(1)} = \prod_{k=0}^{d} (s - k)^{-a_k}$$

(Kurokawa) In an ideal case, for the $l$-th Betti number $b_l$ of $X_{\mathbb{Z}}/\mathbb{F}_p$,

$$\prod_{k=0}^{d} (1 - p^k p^{-s})^{-a_k} = \zeta(s, X_{\mathbb{Z}}/\mathbb{F}_p) = \prod_{l=0}^{m} \left( \prod_{j=1}^{b_l} (1 - \alpha_{l,j} p^{-s}) \right)^{(-1)^{l+1}}$$

$$= \prod_{l=0}^{m} \left( 1 - p^{l/2} p^{-s} \right)^{(-1)^{l} b_l} \quad \Rightarrow \quad b_l = \begin{cases} a_{l/2} & l : \text{even} \\ 0 & l : \text{odd} \end{cases}$$

Thus, $N(v) = \sum_{k=0}^{d} a_k \in \mathbb{Z}_{\geq 0}[v]$,

$$N(1) = \sum_{k=0}^{d} a_k = \sum_{l=0}^{m} (-1)^{l} b_l$$

the Euler characteristic of $X_{\mathbb{Z}}/\mathbb{F}_p$.

Example (Toric variety)

fan picture: lattice $\underline{N}$: a group $N \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$.

convex cone $\sigma$ in $\underline{N}_\mathbb{R}$: a convex subset $\sigma \subseteq \underline{N}_\mathbb{R} := N \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}_{\geq 0} \sigma = \sigma$.

A convex cone $\sigma$ is called:
- polyhedral: if it is finitely generated,
- rational: if the generators lie in the lattice $N$,
- proper: if it does not contain a non-zero sub vector space of $N_{\mathbb{R}}$.

fan $\triangle$ in $\underline{N}$: a finite collection $\triangle$ of proper convex rational polyhedral cones $\sigma$ in the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ s.t.

- every face of a cone in $\triangle$ is in $\triangle$,
- the intersection of two cones in $\triangle$ is a face of each.

(Here zero is considered a face of every cone.)

monoid picture: dual lattice $\underline{M}$: $M := \mbox{Hom}(N, \mathbb{Z})$

dual cone $\check{\sigma}$ in $M_{\mathbb{R}} := \mbox{Hom}(N_{\mathbb{R}}, \mathbb{R})$: $\check{\sigma} := \{ \alpha \in M_{\mathbb{R}} \mid \alpha(\sigma) \geq 0 \}$

monoid $A_{\sigma}$: $A_{\sigma} := \check{\sigma} \cap M$; face inclusion $\tau \subseteq \sigma$ $\Rightarrow$ $A_{\tau} \supseteq A_{\sigma}$

affine open $U_{\sigma}$: $U_{\sigma} := \mbox{Spec}(\mathbb{C}[A_{\sigma}]) = \mbox{Spec}(\mathbb{C}[\check{\sigma} \cap M])$

toriv variety $X_{\triangle}$: $X_{\triangle}$ is obtained by glueing $U_{\sigma} = \mbox{Spec}(\mathbb{C}[A_{\sigma}])$ along $U_{\tau} \to U_{\sigma}$ for each face inclusion $\tau \subseteq \sigma$.
This construction allows us to define a $\mathbb{F}_1$-scheme $\mathcal{X}_\Delta$.

(Deitmar) Given a fan $\triangle \subseteq N \cong \mathbb{Z}^n$, for $j = 0, 1, 2, \ldots, n$, let

\[ f_j \text{ be the number of cones in } \triangle \text{ of dimension } j, \text{ and set } c_j := \sum_{k=j}^{n} f_{n-k} (-1)^{k+j} \binom{k}{j} \]

Then,

\[ \zeta_{\mathcal{X}_\Delta}(s) = \prod_{j=0}^{n} (s - j)^{-c_j} \]

\[ \therefore N(u + 1) = \sum_{x \in \mathcal{X}_\Delta} u^{n(x)} = \sum_{k=0}^{n} f_k u^{n-k} = \sum_{k=0}^{n} f_{n-k} u^k \]

\[ \Rightarrow N(u) = \sum_{k=0}^{n} f_{n-k} (u - 1)^k = \sum_{k=0}^{n} f_{n-k} \sum_{j=0}^{k} \binom{k}{j} x^j (-1)^{k-j} \]

\[ = \sum_{j=0}^{n} x^j \sum_{k=j}^{n} f_{n-k} \binom{k}{j} (-1)^{k-j} \quad \square \]

**Question** Can we define $\zeta_{\mathcal{X}}(s)$ for more general $\mathbb{F}_1$-scheme $\mathcal{X}$?

**Good News.** Connes-Consani proposed two solutions.

**Solution 1:** This proceeds as follows:

- Extend "canonically" $N(n + 1) := \lvert \mathcal{X}(\mathbb{F}_1^n) \rvert$, $(n \in \mathbb{N})$ to

\[ N : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \text{s.t. } \exists C > 0, \exists k \in \mathbb{N}, \text{ s.t. } \lvert N(u) \rvert \leq Cu^k \]

- As far as zero points and poles concerns, can characterize $\zeta_{N}(s)$ (which is supposed to be $\zeta_{\mathcal{X}}(s)$) by

\[ \frac{\partial_s \zeta_{N}(s)}{\zeta_{N}(s)} = - \int_1^\infty N(u) u^{-s} d^*u, \quad d^*u = du/u \]
\[
\zeta_N(s) := \lim_{q \to 1} \exp \left( \sum_{r \geq 1} N(q^r) \frac{(q^{-s})^r}{r} \right) (q - 1)^\chi, \quad \chi = N(1)
\]

\[
\Rightarrow \quad \frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = \lim_{q \to 1} \partial_s \left( \sum_{r \geq 1} N(q^r) \frac{(q^{-s})^r}{r} \right) = \lim_{q \to 1} \sum_{r \geq 1} N(q^r) \frac{(q^{-r})^s}{r} \log(q^{-r}) = - \lim_{q \to 1} \sum_{r \geq 1} N(q^r)(q^r)^{-s} \log q = - \lim_{q \to 1} \sum_{r \geq 1} N(q^r)(q^r)^{-s} \left( \log(q^r) - \log(q^{r-1}) \right) = - \int_1^\infty N(u) u^{-s} d\log u = - \int_1^\infty N(u) u^{-s} du/u
\]

**Solution 2:** Rather than extending to \( N : \mathbb{R}_{\geq 0} \to \mathbb{R} \), consider \( \zeta_N^{\text{disc}}(s) \), whose zero points and poles are characterized by

\[
\frac{\partial_s \zeta_N^{\text{disc}}(s)}{\zeta_N^{\text{disc}}(s)} = - \sum_{n \geq 1} N(n) n^{-s-1}
\]

**Good News. (Connes-Consani)** For any Noetherian \( F_1 \)-scheme \( \mathcal{X} \),

\[ \exists h(z), \text{ an entire function, s.t.} \quad \zeta_N^{\text{disc}}(s) = \zeta_N(s) \exp(h(z)) \]

Therefore, \( \zeta_N^{\text{disc}}(s) \) and \( \zeta_N(s) \) have the same zero points and poles including multiplicities.

\( \zeta_N^{\text{disc}}(s) \) may be defined for more general, not necessarily Noetherian, \( F_1 \)-schemes...
PART II

§5, Hypothetical computation of $N(n+1) = |(\text{Spec } \mathbb{Z}) (\mathbb{F}_1^n)|$
(Connes-Consani)

Using results of Ingham, Connes-Consani observed:

- Regard $w(u) = \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1}$ as a distribution on $[1, \infty)$.

Then,

(1) $N(u) := u - \frac{d}{du} w(u) + 1 = u - \frac{d}{du} \left( \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} \right) + 1$,

where the derivative is in the sense of distributions, enjoys

$$\frac{-\partial_s \zeta_Q(s)}{\zeta_Q(s)} = \int_1^\infty N(u) u^{-s} d^*u$$

- The evaluation $\omega(1) = \lim_{s \to 1} w(s) = \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) \frac{1}{\rho+1} = \frac{1}{2} + \frac{1}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}$, plays an essential role in establishing (2).

Connes-Consani further pointed out the following analogue:

- $X$, an irreducible, smooth projective algebraic curve over $\mathbb{F}_p$,

- If $X$ comes from $\mathbb{F}_1$ by “scalar extension” , $N(q) = |X(\mathbb{F}_q)| = q - \sum_\alpha \alpha^r + 1$, $q = p^r$,

where $\alpha$'s are the complex roots of the characteristic polynomial of the Frobenius on $H^1_{et}(X \otimes \mathbb{F}_p, \mathbb{Q}_\ell)$ ($\ell \neq p$)

- Expressing these roots in the form $\alpha = p^\rho$, for $\rho \in \mathbb{Z}'$

the set of zeros of the Hasse-Weil zeta function of $X$,

(3) $N(q) = |X(\mathbb{F}_q)| = q - \sum_{\rho \in \mathbb{Z}'} \text{order}(\rho) q^\rho + 1$.

- Now, compare (3) with the formal differentiation of (1):

$$N(u) \sim u - \sum_{\rho \in \mathbb{Z}} \text{order}(\rho) u^\rho + 1$$
PART III

§6, Invariants $\mu_r(A)$ for an abelian group $A$

For a finite abelian group $A$, define the $r$-th $\mu$-invariant $\mu_r(A)$ ($r \in \mathbb{N}$) by

$$\mu_r(A) := \frac{1}{|A|^r} \sum_{k_1, \ldots, k_r=1}^{|A|} |\text{Hom}_{\mathfrak{Ab}}(A, \mathbb{Z}/(k_1k_2 \cdots k_r)\mathbb{Z})|.$$ 

$\mu_r(A)$ is essentially the average of the random variable

$$\tilde{X}_r(A) : \tilde{\Omega} := \mathbb{N}^r \rightarrow \mathbb{N} \quad (k_1, k_2, \ldots, k_r) \mapsto |\text{Hom}_{\mathfrak{Ab}}(A, \mathbb{Z}/(k_1k_2 \cdots k_r)\mathbb{Z})|,$$

when the infinite set $\tilde{\Omega} = \mathbb{N}^r$ is given the homogeneous measure.

$$\Rightarrow \quad \mu_r(A) = E\left[\tilde{X}_r(A)\right], \quad E\left[\tilde{X}_r(A)^w\right] = \mu_r(A^w) \quad (w \in \mathbb{N})$$

The invariants $\mu_r(A)$ were first considered by Deitmar-Kurokawa-Koyama and Kurokawa-Ochiai, through their study of, what they call, multiplicative Igusa-type zeta functions of $\mathbb{F}_1$-scheme, which we review by comparing with the Connes-Consani modified zeta function.

(i) The modified zeta function $\zeta_X^{disc}(s)$ for a Noetherian $\mathbb{F}_1$-scheme $X$, defined and studied by Connes-Consani [CC] is characterized by the following property:

$$\begin{cases} -\frac{\zeta_X^{disc}(s)'}{\zeta_X^{disc}(s)} & \equiv \sum_{x \in X} \sum_{m \geq 1} |\text{Hom}_{\mathfrak{Ab}}(O_x^*, \mathbb{Z}/m\mathbb{Z})| (m+1)^{-s-1} \pmod{\text{constant N(1)}} \\ \zeta_X(s) & = e^{h(z)} \zeta_X^{disc}(s) \quad (\zeta_X(s) : \text{Soule zeta function, } h(z) : \text{entire}) \end{cases}$$

(ii) The multivariable ($r$ variable) Igusa type zeta function $Z_X^{Igusa}(s_1, \ldots, s_r)$ for a Noetherian $\mathbb{F}_1$-scheme $X$ ([DKK] for $r = 1$ and [KO] for general $r \in \mathbb{N}$) is given by

$$Z_X^{Igusa}(s_1, \ldots, s_r) := \sum_{x \in X} \sum_{m_1, \ldots, m_r \geq 1} |\text{Hom}_{\mathfrak{Ab}}(O_x^*, \mathbb{Z}/m_1 \cdots m_r\mathbb{Z})| m_1^{-s_1} \cdots m_r^{-s_r}$$
Analyzing analytic properties of

$$Z_{\text{Spec} F_{1}[A]}^{\text{Igusa}}(s_{1}, \ldots, s_{r}) = \sum_{m_{1}, \ldots, m_{r} \geq 1}^{\infty} \left| \text{Hom}_{\mathfrak{Ab}}(A, \mathbb{Z}/m_{1} \cdots m_{r}\mathbb{Z}) \right|m_{1}^{-s_{1}} \cdots m_{r}^{-s_{r}},$$

some very mysterious looking identity of elementary number theory, which expresses $\mu_{r}(A)$ in two different ways, was obtained in the following two cases:

[DKK] $r = 1$ and arbitrary finite abelian group $A$.
[KO] Cyclic groups $A = \mathbb{Z}/n\mathbb{Z}$ and arbitrary $r \in \mathbb{N}$.

I reported a purely elementary proof of some slight generalization of these identities at the Vanderbilt conference in May, 2009:

[M1] Norihiko Minami, 
"On the random variable $\mathbb{N}^{r} \ni (k_{1}, k_{2}, \ldots, k_{r}) \mapsto \gcd(n, k_{1}k_{2}\ldots k_{r}) \in \mathbb{N}, \"arXiv:0907.0916.
[M2] Norihiko Minami, "On the random variable $\mathbb{N} \ni l \mapsto \gcd(l, n_{1})\gcd(l, n_{2})\ldots\gcd(l, n_{k}) \in \mathbb{N},\"arXiv:0907.0918.
Theorem of [DKK] type. For a finite abelian group $A = \prod_{j=1}^{k} (\mathbb{Z}/n_{j}\mathbb{Z})$,

$$
\mu_{1}(A) = \mu_{1} \left( \prod_{j=1}^{k} (\mathbb{Z}/n_{j}\mathbb{Z}) \right)
$$

$$
:= \frac{1}{\text{lcm}(n_{1}, n_{2}, \ldots, n_{k})} \sum_{l=1}^{\text{lcm}(n_{1}, n_{2}, \ldots, n_{k})} \gcd(l, n_{1}) \gcd(l, n_{2}) \cdots \gcd(l, n_{k})
$$

$$
= \prod_{p \mid \text{lcm}(n_{1}, n_{2}, \ldots, n_{k})} \left[ p^{\nu_{p,0} + \cdots + \nu_{p,k-1}} \sum_{j=0}^{\nu_{p,j}} p^{(k-j)\nu_{p,j}-l} \right]
$$

Here, for each prime $p \mid n$,

$$
\{\nu_{p,1}, \nu_{p,2}, \ldots, \nu_{p,k-1}, \nu_{p,k}\} = \{\text{ord}_{p}(n_{1}), \text{ord}_{p}(n_{2}), \ldots, \text{ord}_{p}(n_{k-1}), \text{ord}_{p}(n_{k})\}
$$

$$
\nu_{p,0} := 0 \leq \nu_{p,1} \leq \nu_{p,2} \leq \ldots \leq \nu_{p,k-1} \leq \nu_{p,k}
$$

Set $nH_{r} := \binom{n+r-1}{r}$. Then, we have:

Theorem of [KO] type. For $n, r \in \mathbb{N}$, $w \in \mathbb{C}$,

$$
\frac{1}{n^{r}} \sum_{k_{1}, \ldots, k_{r}=1}^{n} \gcd(n, k_{1} \cdots k_{r})^{w}
$$

$$
= \begin{cases}
\prod_{p \mid n} \left[ \left( \frac{1-p^{-1}}{1-p^{w-1}} \right) + p^{\text{ord}_{p}(n)(w-1)} \sum_{l=0}^{\text{ord}_{p}(n)-1} \text{ord}_{p}(n)H_{l} \left\{ (1-p^{-1})^{l} - \left( \frac{1-p^{-1}}{1-p^{w-1}} \right)^{r} (1-p^{w-1})^{l} \right\} \right] & \text{if } w \neq 1 \\
\prod_{p \mid n} \left[ \sum_{l=0}^{r} \text{ord}_{p}(n)H_{l} \left( 1 - \frac{1}{p} \right)^{l} \right] & \text{if } w = 1
\end{cases}
$$

$$
= \begin{cases}
\prod_{p \mid n} \left[ \left( \frac{1-p^{-1}}{1-p^{w-1}} \right) + p^{\text{ord}_{p}(n)(w-1)} (1-p^{-1})^{r} \sum_{l=0}^{\text{ord}_{p}(n)-1} \text{ord}_{p}(n)H_{l} \left\{ (1-p^{-1})^{l-r} - (1-p^{w-1})^{l-r} \right\} \right] & \text{if } w \neq 1 \\
\prod_{p \mid n} \left[ \sum_{l=0}^{r} \text{ord}_{p}(n)H_{l} \left( 1 - \frac{1}{p} \right)^{l} \right] & \text{if } w = 1
\end{cases}
$$

Corollary [KO]. For $n, r \in \mathbb{N}$,

$$
\mu_{r}(\mathbb{Z}/n\mathbb{Z}) = \prod_{p \mid n} \left[ \sum_{l=0}^{r} \text{ord}_{p}(n)H_{l} \left( 1 - \frac{1}{p} \right)^{l} \right]
$$
§7 Motivation of the rest of talk

When we play with $\mu_r(A)$, the following questions seem to be very natural:

- Is there any more conceptual interpretation or description of $\mu_r(A)$?
- Is $\mu_r(A)$, whose origin is the Igusa-type zeta functions for $F_1$-schemes of Kurokawa and his collaborators, useful to study $F_1$-scheme?
- Is there any relationship between the zeta functions of Soullé, Connes-Consani, and the Igusa-type zeta functions, which was the origin of $\mu_r(A)$?

§8 $\mu_1(A)$ and the zeta functions of Soullé, Connes-Consani.

The logarithmic derivative of the deformed modified zeta function of Soulé type $\zeta_X^{disc}(s; w)$:

$$\frac{\partial_s \zeta_X^{disc}(s; w)}{\zeta_X^{disc}(s; w)} \equiv - \sum_{x \in \mathcal{X}} \sum_{m \geq 1} |\text{Hom}_{\mathfrak{A}}(\mathcal{O}^x, \mathbb{Z}/m\mathbb{Z})|^w (m+1)^{-s-1} \pmod{\text{constant}}$$

is a meromorphic function of $s$ with all of its poles simple.

This gives us the following expression of the deformed modified zeta function of Soulé type:

[M3] Norihiko Minami,

"Meromorphicity of some deformed multivariable zeta functions for $F_1$-schemes," arXiv:0910.3879
\[
\zeta_{\mathcal{X}}^{\text{disc}}(s;w) = e^{h(s;w)} \prod_{x \in \underline{X}} \left( \prod_{j=0}^{n(x)w} (s-j)^{(-n(x)^w)(-1)^{n(x)w-j}} \right)^{\frac{\sum_{k=1}^{l(x)} \left| \text{Hom}_{\mathfrak{U}b}(A_{x},\mathbb{Z}/k\mathbb{Z}) \right|^w}{l(x)}}
\]

where, for each \(x \in X\), \(\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x\), \(l(x) = \text{lcm}\{m_j(x)\}\), and \(h(s;w)\) is some entire function of \(s\) depending upon \(w \in \mathbb{N}\).

Restricting to the case \(w = 1\) further, we obtain the following:

For a Noetherian \(F_1\)-scheme \(\mathcal{X}\), there are some entire functions \(h_1(s), h_2(s)\) s.t.

\[
\zeta_{\mathcal{X}}(s) = e^{h_1(s)} \zeta_{\mathcal{X}}^{\text{disc}}(s) = e^{h_2(s)} \prod_{x \in \underline{X}} \left( \prod_{j=0}^{n(x)} (s-j)^{(-n(x)^j)(-1)^{n(x)^j-j}} \right)^{\mu_1(A_x)}
\]

where, for each \(x \in X\), \(\mathcal{O}_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x\),

Message:

- \(\mu_1\) measures "local contribution of ramification"!
- locally, torsion does not creat any new singularity.
An outline of the proof of the $\zeta_X^{\text{disc}}(s; w)$ formula.

\[ - \sum_{x \in X} \sum_{m=1}^{\infty} \left| \text{Hom}_{\mathfrak{U}b} \left( \mathcal{O}_x^*, \mathbb{Z}/m\mathbb{Z} \right) \right|^w (m+1)^{-s-1} \]

\[ = - \sum_{x \in X} \sum_{j=0}^{n(x)w} \binom{n(x)w}{j} (-1)^{n(x)w-j} \frac{l(x)}{l(x)} \sum_{k=1}^{l(x)} \left| \text{Hom}_{\mathfrak{U}b} \left( A_x, \mathbb{Z}/k\mathbb{Z} \right) \right|^w \zeta(s+1-j, \frac{k+1}{l(x)}) , \]

where the Hurwitz zeta function

\[ \zeta(s, q) := \sum_{n \geq 0} (n+q)^{-s} (\Re(s) > 1, \Re(q) > 0) \]

only has a pole of residue 1 at $s = 1$. Thus, the singularities of (4) are poles at $s = j \in \bigcup_{x \in X} \{0, \cdots, n(x)\}$ with residue

\[ \sum_{x \in X} \sum_{j=0}^{n(x)w} \binom{n(x)w}{j} (-1)^{n(x)w-j} \frac{\sum_{k=1}^{l(x)} \left| \text{Hom}_{\mathfrak{U}b} \left( A_x, \mathbb{Z}/k\mathbb{Z} \right) \right|^w}{l(x)} \]

Now the claim follows immediately. \qed

§9. The conceptual interpretation of $\mu_1(A)$.

For any finite abelian group $A$,

\[ \mu_1(A) := \frac{1}{|A|} \sum_{k=1}^{|A|} \left| \text{Hom}_{\mathfrak{U}b} \left( A, \mathbb{Z}/k\mathbb{Z} \right) \right| = \sum_{a \in A} \frac{1}{|a|} \]
If we interpret that \( \frac{1}{|a|} = \frac{1}{\infty} = 0 \) for an element \( a \) of infinite order, we may generalize the definition of \( \mu_1(A) \) to finitely generalized abelian groups, as well as to finite (not necessary commutative) groups.

\[
\begin{align*}
\text{Proof of } \mu_1(A) & = \sum_{a \in A} \frac{1}{|a|}. \\
\frac{1}{|A|} \sum_{l=1}^{|A|} \left| \text{Hom}_{ab}(A, \mathbb{Z}/l\mathbb{Z}) \right| & = \frac{1}{|A|} \sum_{l=1}^{|A|} \left| \text{Hom}_{ab}(\mathbb{Z}/l\mathbb{Z}, A) \right| \\
& = \frac{1}{|A|} \sum_{C \subset A} \sum_{l=1}^{|A|} \left| \text{Epi}_{ab}(\mathbb{Z}/l\mathbb{Z}, C) \right| = \frac{1}{|A|} \sum_{C \subset A} \sum_{l=1}^{|A|} \left| \text{Epi}_{ab}(\mathbb{Z}/l\mathbb{Z}, C) \right| \\
& = \frac{1}{|A|} \sum_{C \subset A} \sum_{l=1}^{|A|} \left| \text{Mono}_{ab}(C, \mathbb{Z}/l\mathbb{Z}) \right| = \frac{1}{|A|} \sum_{C \subset A} \frac{|A|}{|C|} \phi(|C|) \\
& = \sum_{C \subset A} \frac{\phi(|C|)}{|C|} = \sum_{h \in \text{Hom}(\mathbb{Z}, A)} \frac{1}{|h(1)|} = \sum_{a \in A} \frac{1}{|a|}
\end{align*}
\]

\( \square \)

§10, \( \mu_r(A) \) for general \( r \in \mathbb{N} \).

For any abelian group \( A \) and \( r \in \mathbb{N} \), we have

\[
\mu_r(A) = \sum_{a \in A} \frac{\text{KO}_{r-1}(|a|)}{|a|} = \prod_{p || A} \left[ \sum_{l=0}^{r-1} \text{ord}_p(|a|) H_l \left( 1 - \frac{1}{p} \right)^l \right]
\]

where \( \text{KO} \) stands for Kurokawa-Ochiai [KO]:

\[
\text{KO}_r(n) := \begin{cases} 
1 & (r = 0) \\
\mu_r(\mathbb{Z}/n\mathbb{Z}) = \prod_{p || n} \left[ \sum_{l=0}^{r-1} \text{ord}_p(n) H_l \left( 1 - \frac{1}{p} \right)^l \right] & (r \geq 1)
\end{cases}
\]
§11, Connes-Consani modified Soulé type zeta function, again

To recap, let us combine the two theorem:

For a Noetherian $F_1$-scheme $\mathcal{X}$, there are some entire functions $h_1(s), h_2(s)$ s.t.

$$\zeta_{\mathcal{X}}(s) = e^{h_1(s)} \zeta^\text{disc}_{\mathcal{X}}(s)$$

$$= e^{h_2(s)} \prod_{x \in X} \left( \sum_{s \in A_x} \frac{1}{|s|} \right)$$

where, for each $x \in X$, $O_x^* = \mathbb{Z}^{n(x)} \times \prod_j \mathbb{Z}/m_j(x)\mathbb{Z} =: \mathbb{Z}^{n(x)} \times A_x$.

Once again, the above result is in the following:

[M3] Norihiko Minami,

"Meromorphicity of some deformed multivariable zeta functions for $F_1$-schemes," arXiv:0910.3879

I would like to end this paper with the following question to transformation group theorists:

*Is there any application of the invariants $\mu_r(A)$ to the transformation group theory?*