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Odds theorem in Markov-dependent trials with multiple selection chances

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Abstract

We study the optimal multiple stopping problem which maximizes the probability of selecting the last success with multiple selecting chances in nonstationary Markov-dependent trials. We provide the sufficient condition for the optimal multiple stopping rule to be of Sum-the-Odds form[1].

1 Introduction

For a positive integer N, let $X_1, X_2, \ldots, X_N$ denote 0/1 random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. These 0/1 random variables appears according to nonstationary Markov chain with the transition probability such that

$$\mathbf{P}_k = \begin{pmatrix} 1-\alpha_k & \alpha_k \\ \beta_k & 1-\beta_k \end{pmatrix}, \tag{1.1}$$

where $\alpha_k := \mathrm{P}(X_{k+1} = 1 | X_k = 0)$ and $\beta_k := \mathrm{P}(X_{k+1} = 0 | X_k = 1)$. We observe these X_k's sequentially and claim that the ith trial is a success if $X_k = 1$. The problem lies in finding a rule $\tau \in \mathcal{T}$ to maximize the probability of selecting the last success, where $\mathcal{T}$ is the class of all selection rules such that $\{\tau = j\} \in \sigma(X_1, X_2, \cdots, X_j)$; that is, the decision of whether to select the jth success depends on the information up to j. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ and for $j = k+1, \cdots, N$

$$p_{jk} := \mathrm{P}(X_j = 1 | X_k = 1, X_{k+1} = \cdots = X_{j-1} = 0, X_{j+1} = \cdots = X_N = 0). \tag{1.2}$$

Let $q_{jk} := 1 - p_{jk}$ for $k+1 \leq j$ and $k, j \in \mathcal{N}$. In addition, let r_{jk}, $k, j \in \mathcal{N}$, denote the odds of the jth trial; that is, $r_{jk} = p_{jk}/q_{jk}$, where we set $r_{jk} = +\infty$ if $p_{jk} = 1$.

When exactly one selection chance was allowed in an independent trials, that is, for the case of $p_{jk} = p_j = \mathrm{P}(X_j = 1)$ for all $k \in \mathcal{N}$ and $r_j = p_j/(1-p_j)$, Bruss [4] solved the stopping problem with elegant simplicity as follows; The optimal selection rule $\tau_*^{(1)}$ selects the first success

[1]This paper is an abbreviated version of Ano, Kakie and Miyoshi [1]

after the sum of the future odds becomes less than one; that is, $\tau_*^{(1)} = \min\{i \geq i_*^{(1)} : X_k = 1\}$ with $i_*^{(1)} = \min\{i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1\}$, where $\min \emptyset = +\infty$ and $\sum_{j=a}^{b} \cdot = 0$ when $b < a$ conventionally. Furthermore, the maximum probability of "win" (selecting the last success) is given by $\mathrm{P}^{(1)}(\text{win}) = \mathrm{P}_N^{(1)}(p_1, \ldots, p_N) = \prod_{k=i_*^{(1)}}^{N} q_k \sum_{k=i_*^{(1)}}^{N} r_k$.

This result, referred to as the sum-the-odds theorem or odds theorem in short, is attractive because it can be applied to many basic optimal stopping problems such as betting, the classical secretary problem (CSP) and the group-interview secretary problem proposed by Hsiau and Yang [12]. Bruss [4] also proved that $\mathrm{P}^{(1)}(\text{win})$ is bounded below by $R^{(1)} e^{-R^{(1)}}$ with $R^{(1)} = \sum_{j=i_*^{(1)}}^{N} r_j$. Remarkably, in [5], he found that it is bounded below by e^{-1} when $\sum_{j=1}^{N} r_j \geq 1$. These results generalize the known lower bounds for the CSP, where each p_k has the specific value of $p_k = 1/i$ for $i \in \mathcal{N}$ (e.g., Hill and Krengel [11]).

After Bruss [4], which includes the problem with random number of observations, the odds theorem has been extended in several directions. Bruss and Paindaveine [6] extended it to the problem of selecting the last ℓ (> 1) successes. Hsiau and Yang [13] considered the problem in Markov-dependent trials with the case of $\alpha_k + \beta_k \leq 1$. Recently, Ferguson [9] extended the odds theorem in several ways, where infinite number of trials are allowed, the payoff for not selecting till the end is different from the payoff for selecting a success that is not the last, and the trials are generally dependent. Furthermore, he applied his extension to the stopping game of Sakaguchi [14]. Recently, Ano, Kakinuma and Miyoshi [2] succeeded to extend the multiple selection problem in independent trials. They provides the optimal stopping rule as odds form. In this paper, we consider yet another extension of the result by Bruss [4]; that is, we are interested in the problem with multiple selection chances in Markov-dependent trials.

2 Single selection

Let $V_k^{(1)}$ be the maximum probability of win (that is, selecting the last success) when we observe $X_k = 1$ and select the success;

$$V_k^{(1)} = \mathrm{P}(X_{k+1} = X_{k+2} = \cdots = X_N = 0 | X_k = 1), \ k \in \mathcal{N}. \tag{2.1}$$

Let $M_k^{(1)}$ be the maximum probability of win when we observe $X_k = 1$ and behave optimally. By the principle of optimality, we have the optimal equation (dynamic programming equation) as follows; for $k = 1, 2, \cdots, N-1$

$$M_k^{(1)} = \max\left\{ V_k^{(1)}, \sum_{j=k+1}^{N} \mathbf{P}_{kj} M_j^{(1)} \right\}, \tag{2.2}$$

where $\mathbf{P}_{kj}$ denotes the probability that the first success after $X_k = j$ appears at j; for $j > k$,

$$\mathbf{P}_{kj} := \mathrm{P}(X_{k+1} = \cdots = X_{j-1} = 0, X_j = 1 | X_k = 1). \tag{2.3}$$

When $X_N = 1$, we definitely win and thus we have $M_N^{(1)} = V_N^{(1)} = 1$ as the boundary condition.

Next theorem gives the optimal single selecting rule. It is worthy to note that our result contains the result of Hsiau and Young [13] that studied only for the case of $\alpha_k + \beta_k \geq 1$. Further, it may be elegant that under some condition the optimal selection rule in Markov-dependent trials case can be described as the same simple Sum-the-Odds form as of the optimal selection rule in independent trials case of Burss [4].

When $\beta_k = 1$ and $\alpha_k = 0$ for all $k \in \mathcal{N}$, 0 and 1 appears by turns. So that it is clearly optimal to stop the success on the stage $N-1$ or N. To ensure $V_k^{(1)} \neq 0$ for all $k \in \mathcal{N}$, we assume that $0 < \alpha_k, \beta_k < 1$ for all $k \in \mathcal{N}$. Let $H_k^{(1)} := 1 - \sum_{j=k+1}^{N} r_{jk}$.

Theorem 2.1 *Assume that $0 < \alpha_k, \beta_k < 1$ for each $k \in \mathcal{N}$. If $H_k^{(1)}$ is nondecreasing in k and changes sign from nonpositive to positive at most once, then the optimal single selecting rule is given by*

$$\tau_1^* = \inf\left\{ k \in \mathcal{N} : X_k = 1 \text{ and } \sum_{j=k+1}^{N} r_{jk} < 1 \right\}. \tag{2.4}$$

The sufficient condition of Theorem 2.1 is equivalent to that $k \mapsto \sum_{j=k+1}^{N} r_{jk}$ is nonincreasing and goes to below 1. In independent trials, $p_{jk} = \mathrm{P}(X_j = 1)$. Putting $p_j = p_{jk}$ and $r_j = r_{jk}$, $\sum_{j=k+1}^{N} r_j$ is obviously nonincreasing in k. In this case, τ_1^* coincides with Bruss' optimal rule. Note that $H_k^{(1)}$ is nondecreasing in k if and only if

$$\frac{(1-\beta_{k+1})(\beta_{k+2})}{\beta_{k+1}(1-\alpha_{k+2})} \leq \frac{(1-\beta_{k+1})\beta_{k+1}}{\beta_k(1-\alpha_{k+1})} + \frac{\alpha_{k+1}\beta_{k+2}}{(1-\alpha_{k+1})(1-\alpha_{k+2})}. \tag{2.5}$$

Proof. Let $B^{(1)}$ be the monotone selecting region that the probability of win by selecting the current success is greater than the probability of win by passing the current success and selecting the next first appearing success. We have for $V_k^{(1)} \neq 0$, $B^{(1)} = \{k \in \mathcal{N} : G_k^{(1)} \geq 0\}$ where

$$G_k^{(1)} := V_k^{(1)} \left\{ 1 - \sum_{j=k+1}^{N} \frac{\mathbf{P}_{kj} V_j^{(1)}}{V_k^{(1)}} \right\}. \tag{2.6}$$

By Markov property, $V_j^{(1)} = \mathrm{P}(X_{j+1} = \cdots = X_N = 0 | X_j = 1) = \mathrm{P}(X_{j+1} = \cdots = X_N = 0 | X_j = 1, X_k = 1, X_{k+1} = \cdots = X_{j-1} = 0)$. Therefore,

$$\frac{\mathbf{P}_{kj} V_j^{(1)}}{V_k^{(1)}} = \frac{\mathbf{P}_{kj} \mathrm{P}(X_{j+1} = \cdots = X_N = 0 | X_j = 1, X_k = 1, X_{k+1} = \cdots = X_{j-1} = 0)}{\mathrm{P}(X_{j+1} = \cdots = X_N = 0 | X_j = 1)}. \tag{2.7}$$

From the definition of the conditional probability, it follows that

$$\begin{aligned} &\text{Numerator of RHS in (2.7)} \\ &\quad = \frac{\mathrm{P}(X_k = 1, X_{k+1} = \cdots = X_{j-1} = 0, X_j = 1)}{\mathrm{P}(X_k = 1)} \\ &\qquad \times \frac{\mathrm{P}(X_k = 1, X_{k+1} = \cdots = X_{j-1} = 0, X_j = 1, X_{j+1} = \cdots = X_N = 0)}{\mathrm{P}(X_k = 1, X_{k+1} = \cdots = X_{j-1} = 0, X_j = 1)} \\ &\quad = \mathrm{P}(X_{k+1} = \cdots = X_{j-1} = 0, X_j = 1, X_{j+1} = \cdots = X_N = 0 | X_k = 1) \end{aligned} \tag{2.8}$$

From $0 < \alpha_k, \beta_k < 1$ for each k, it follows that $\mathrm{P}(X_{k+1} = \cdots = X_{j-1} = 0, X_{j+1} = \cdots = X_N = 0 | X_k = 1) = \beta_k(1-\alpha_{k+1}) \cdots (1-\alpha_{j-2})\alpha_{j-1}\beta_j(1-\alpha_{j+1}) \cdots (1-\alpha_{N-1}) \neq 0$. Hence from (2.7) and (2.8), we obtain

$$\frac{\mathbf{P}_{kj}V_j^{(1)}}{V_k^{(1)}} = \frac{\mathrm{P}(X_{k+1}=\cdots=X_{j-1}=0, X_j=1, X_{j+1}=\cdots=X_N=0|X_k=1)}{\mathrm{P}(X_{k+1}=\cdots=X_N=0|X_k=1)}$$
$$= \frac{\mathrm{P}(X_{k+1}=\cdots=X_{j-1}=0, X_j=1, X_{j+1}=\cdots=X_N=0|X_k=1)}{\mathrm{P}(X_{k+1}=\cdots=X_{j-1}=0, X_{j+1}=\cdots=X_N=0|X_k=1)}$$
$$\div \frac{\mathrm{P}(X_{k+1}=\cdots=X_N=0|X_k=1)}{\mathrm{P}(X_{k+1}=\cdots=X_{j-1}=0, X_{j+1}=\cdots=X_N=0|X_k=1)} \tag{2.9}$$

From the definition of the conditional probability, it follows that

Numerator of RHS in (2.9)

$$= \frac{\mathrm{P}(X_k=1, X_{k+1}=\cdots=X_{j-1}=0, X_j=1, X_{j+1}=\cdots=X_N=0)/\mathrm{P}(X_k=1)}{\mathrm{P}(X_k=1, X_{k+1}=\cdots=X_{j-1}=0, X_{j+1}=\cdots=X_N=0)/\mathrm{P}(X_k=1)}$$
$$= \mathrm{P}(X_j=1|X_k=1, X_{k+1}=\cdots=X_{j-1}=0, X_{j+1}=\cdots=X_N=0)$$
$$= p_{jk}. \tag{2.10}$$

Denominator of RHS in (2.9)

$$= \frac{\mathrm{P}(X_k=1, X_{k+1}=\cdots=X_N=0)/\mathrm{P}(X_k=1)}{\mathrm{P}(X_k=1, X_{k+1}=\cdots=X_{j-1}=0, X_{j+1}=\cdots=X_N=0)/\mathrm{P}(X_k=1)}$$
$$= \mathrm{P}(X_j=0|X_k=1, X_{k+1}=\cdots=X_{j-1}=0, X_{j+1}=\cdots=X_N=0)$$
$$= 1-p_{jk}. \tag{2.11}$$

Through (2.6)-(2.11), we obtain $\sum_{j=k+1}^N(\mathbf{P}_{kj}V_j^{(1)}/V_k^{(1)}) = 1-\sum_{j=k+1}^N\{p_{jk}/(1-p_{jk})\} = 1-\sum_{j=k+1}^N r_{jk}$. Finally we have

$$G_k^{(1)} = V_k^{(1)}\left\{1-\sum_{j=k+1}^N r_{jk}\right\}, \quad B^{(1)} = \left\{k\in\mathcal{N} : X_k=1 \text{ and } \sum_{j=k+1}^N r_{jk}<1\right\}.$$

When $\sum_{j=k+1}^N r_{jk}$ is nonincreasing in k, $B^{(1)}$ is "closed" in the sense of the monotone problem in Chow et al [7]; that is, $k\in B^{(1)}$ implies that $j\in B^{(1)}$ for all $j=k,k+1,\ldots,N$. Hence, the optimal rule for the single selection problem is given by (2.4). The proof completes. □

3 Multiple selection

Suppose that we are given m ($\in\mathcal{N}$) selection chances in the problem described in the preceding section. Let $V_k^{(m)}$, $k\in\mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_k=1$ and select this success when we have at most m selection chances left. Let $W_k^{(m)}$, $i\in\mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_k=1$ and ignore this success when we have at most m selection chances left. Furthermore, let $M_k^{(m)}$, $k\in\mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_k=1$ and decide whether to select when we have at most m selection chances left. The optimality equation is then given by

$$M_k^{(m)} = \max\{V_k^{(m)}, W_k^{(m)}\}, \quad i\in\mathcal{N}. \tag{3.1}$$

Clearly, if $m > N-k$ (the remaining selection chances are more than the remaining observations) and we observe $X_k = 1$, then the decision to select results in win with probability 1, so that $M_k^{(m)} = V_k^{(m)} = 1$ for $k > N-m$. In particular, we have $M_N^{(m)} = V_N^{(m)} = 1$ and $W_N^{(m)} = 0$ for any $m \in \mathcal{N}$.

We observe that $V_k^{(m)}$ is represented as the sum of two conditional probabilities; the first is that no success appears in $k+1, \dots, N$ provided that $X_k = 1$ and the second is that we finally win in starting at $k+1$ with $m-1$ selection chances provided that $X_k = 1$. Since the latter conditional probability is equal to $W_k^{(m-1)}$, we have

$$V_k^{(m)} = \mathrm{P}(X_{k+1} = X_{k+2} = \cdots = X_N = 0 \mid X_k = 1) + W_k^{(m-1)}, \quad k \in \mathcal{N}, \tag{3.2}$$

where we set $W_k^{(0)} := 0$ for $i \in \mathcal{N}$ conventionally. On the other hand, $W_k^{(m)}$ is given as the conditional probability based on which we make the optimal decision at the first success after i and finally win provided that $X_k = 1$, so that,

$$W_k^{(m)} = \sum_{j=k+1}^{N} \mathbf{P}_{kj} M_j^{(m)}, \quad k \in \mathcal{N}, \tag{3.3}$$

where $\mathbf{P}_{kj}$ is the probability that the first success appears at the state j after $X_k = 1$, which is given by (2.3).

We can now state the optimal rules for the multiple selection problem. For each $k \in \mathcal{N}$, we recursively define $H_k^{(m)}$, $m \in \mathcal{N}$, by

$$H_k^{(1)} := 1 - \sum_{j=k+1}^{N} r_{jk}, \tag{3.4}$$

$$H_k^{(m)} := H_k^{(1)} + \sum_{j=(k+1)\vee k_*^{(m-1)}}^{N} r_{jk} H_j^{(m-1)}, \tag{3.5}$$

$$k_*^{(m)} := \min\{k \in \mathcal{N} : H_k^{(m)} > 0\}, \tag{3.6}$$

where $a \vee b = \max\{a, b\}$ for a, $b \in \mathbf{R}$. In (3.5), if there exists a $j \in \{k+1, \dots, N\}$ such that $p_{jk} = 1$ (that is, $r_{jk} = +\infty$), then we set $H_k^{(m)} := -\infty$.

Theorem 3.1 *Suppose that we have at most m ($\in \mathcal{N}$) selection chances. Assume that $0 < \alpha_k, \beta_k < 1$ for each $k \in \mathcal{N}$. If $H_k^{(1)}$ is nonincreasing in k and changes sign from nonpositive to positive at most once, then, the optimal selection rule $\tau_*^{(m)}$ is given by*

$$\tau_*^{(m)} = \min\{k \geq k_*^{(m)} : X_k = 1\}, \tag{3.7}$$

where $\min \emptyset = +\infty$. *Furthermore, we have*

$$1 \leq k_*^{(m)} \leq k_*^{(m-1)} \leq \cdots \leq k_*^{(1)} \leq N. \tag{3.8}$$

Proof. The monotone selection region for the problem with m ($\in \mathcal{N}$) selection chances is defined by $B^{(m)} := \{k \in \mathcal{N} : G_k^{(m)} > 0\}$, where

$$G_k^{(m)} := V_k^{(m)} - \sum_{j=k+1}^{N} \mathbf{P}_{kj} V_j^{(m)}. \quad k \in \mathcal{N}. \tag{3.9}$$

Suppose that for the problems with $1, 2, \cdots, m-1$ selection chances, Theorem 3.1 holds. Then for each $\ell = 1, 2, \cdots, m-1$,

$$M_k^{(\ell)} = \begin{cases} V_k^{(\ell)}, & H_k^{(\ell)} \geq 0 \\ W_k^{(\ell)}, & H_k^{(\ell)} < 0. \end{cases} \tag{3.10}$$

By induction on ℓ, we shall show that for each $\ell = 1, 2, 3, \ldots, m$

$$H_k^{(\ell)} = \frac{G_k^{(\ell)}}{V_k^{(1)}}, \quad k = 1, 2, \cdots, N. \tag{3.11}$$

In Section 2, we have already seen $H_k^{(1)} = G_k^{(1)}/V_k^{(1)}$. As an induction hypothesis, assume that $H_k^{(m-1)} = G_k^{(m-1)}/V_k^{(1)}$. It suffices to show that $H_k^{(m)} = G_k^{(m)}/V_k^{(1)}$. Now we have

$$\begin{aligned} G_k^{(m)} &= V_k^{(m)} - \sum_{j=k+1}^{N} \mathbf{P}_{kj} V_j^{(m)} \\ &= \left\{V_k^{(1)} + W_k^{(m-1)}\right\} - \sum_{j=k+1}^{N} \mathbf{P}_{kj} \left\{V_j^{(1)} + W_j^{(m-1)}\right\} \\ &= G_k^{(1)} + \left\{W_k^{(m-1)} - \sum_{j=k+1}^{N} \mathbf{P}_{kj} W_j^{(m-1)}\right\} \\ &= G_k^{(1)} + \sum_{j=k+1}^{N} \mathbf{P}_{kj} \left\{M_j^{(m-1)} - W_j^{(m-1)}\right\}. \end{aligned}$$

From (3.10), it follows

$$\begin{aligned} M_j^{(m-1)} - W_j^{(m-1)} &= \begin{cases} V_j^{(m-1)} - W_j^{(m-1)}, & H_j^{(m-1)} \geq 0 \\ 0, & H_j^{(m-1)} < 0 \end{cases} \\ &= \begin{cases} V_j^{(m-1)} - \sum_{h=j+1}^{N} \mathbf{P}_{kj} M_h^{(m-1)}, & H_j^{(m-1)} \geq 0 \\ 0, & H_j^{(m-1)} < 0 \end{cases} \end{aligned}$$

Since if $H_j^{(m-1)} \geq 0$, then $M_h^{(m-1)} = V_h^{(m-1)}$ for $h \geq j$, we have

$$\begin{aligned} M_j^{(m-1)} - W_j^{(m-1)} &= \begin{cases} V_j^{(m-1)} - \sum_{h=j+1}^{N} \mathbf{P}_{kj} V_h^{(m-1)}, & H_j^{(m-1)} \geq 0 \\ 0, & H_j^{(m-1)} < 0 \end{cases} \\ &= \left(V_j^{(m-1)} - \sum_{h=j+1}^{N} \mathbf{P}_{kj} V_h^{(m-1)}\right) I\{H_k^{(m-1)} \geq 0\} \\ &= G_j^{(m-1)} I\{H_j^{(m-1)} \geq 0\}. \end{aligned}$$

Thus

$$G_k^{(m)} = G_k^{(1)} + \sum_{j=k+1}^{N} \left\{\mathbf{P}_{kj} G_j^{(m-1)} I\{H_j^{(m-1)} \geq 0\}\right\}.$$

From the induction hypothesis, $G_j^{(m-1)} = V_j^{(1)} H_j^{(m-1)}$. Therefore

$$\begin{aligned}\frac{G_k^{(m)}}{V_k^{(1)}} &= \frac{G_k^{(1)}}{V_k^{(1)}} + \sum_{j=k+1}^{N} \frac{\mathbf{P}_{kj} V_j^{(1)}}{V_k^{(1)}} H_j^{(m-1)} I\{j \geq i_*^{(m-1)}\} \\ &= H_k^{(1)} + \sum_{j=\max\{k+1, i_*^{(m-1)}\}}^{N} \frac{p_{jk}}{1-p_{jk}} H_j^{(m-1)} \\ &= H_k^{(1)} + \sum_{j=\max\{k+1, i_*^{(m-1)}\}}^{N} r_{jk} H_j^{(m-1)} \\ &= H_k^{(m)}.\end{aligned}$$

Consequently, $B^{(m)} = \{k \in \mathcal{N} : G_k^{(m)} > 0\} = \{k \in \mathcal{N} : H_k^{(m)} > 0\}$. We can show that $B^{(m)}$ is closed under the condition of $H_k^{(1)}$. For more details, see Ano, Kakie and Miyoshi [1]. □

Further interesting question is what the values of the limiting maximum probability of selecting the last success with multiple selection chances as $N \to \infty$ are for each $m = 1, 2, \cdots$. These values are still unknown.

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