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Bayesian multiple stopping problem on geometric random walk[1]

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Abstract

This paper studies the optimal multiple stopping times for the Bayesian multiple stopping problem on geometric random walk, where the upward probability, p, is assumed to be unknown. We assume that the prior distribution of p is Beta. Under some conditions, we show that optimal stopping times are of threshold type, and it is optimal to exercise if the number of upward below some threshold value.

1 Introduction

Image that an investor wants to invest and maximize the expected cumulative discounted return from a certain project. He is allowed to invest a project at most m different times. We try to impose investor's subjective view for future performance. The performance (discounted cash flow) of the investment to the project varies according to the geometric random walk, and the upward probability, p, of the random walk is unknown. Assume that the prior distribution of p is Beta. What are the optimal multiple stopping (exercise) times? How to solve them? These are main interesting subjects of this paper. Let $\{S_n\}_{n=0}^{N}$ be a geometric random walk;

$$S_n = S_0 s_n, s_n = s^{X_1+\cdots+X_n}, n \in \mathbb{N}, \tag{1.1}$$

where $S_0 = s, P(X_n = 1) = p = 1 - P(X_n = -1), 0 < p < 1$. Upward probability p is unknown. Assume that prior distribution of p is $Beta(\alpha, \beta)$. For example, if investor believes that the upward probability is between 0.5 and 0.8 with his subjective probability 0.9, then he can select the parameters (α, β) satisfying $P(0.5 \leq p \leq 0.8) = 0.90$. Note that these values, (α, β), are not unique pair. Let N is the last time. Under some conditions, we show that optimal stopping times are of threshold type, and it is optimal to exercise if the number of upward below some threshold value that is unique root of a certain equation.

2 Formulation

We suppose *random sampling* from the distribution with unknown parameter θ, that is, for given θ, $f_n(x_1, \cdots, x_n|\theta) = f(x_1|\theta)\cdots f(x_n|\theta)$. It is nice to see that if a sufficient statistic exists, then the prior and posterior distributions are in the same distribution family (cf. DeGroot (1971)). The definition of the sufficient statistics is as follows; for any prior density $g(\theta)$, the

[1]This paper is an abbreviated version of Ano [1].

posterior density can be expressed by $g(\theta|\vec{x}_n) = g(\theta|Y_n(\vec{x}_n))$. We denote a sufficient statistics for $\{f(x|\theta), \theta \in \mathbb{R}\}$ by $Y_n(\vec{x}_n)$. Our problem is

$$V_n^{[m]}(\vec{x}_n|\theta_n) := \sup_{n \leq \tau_m < \cdots < \tau_1 \leq N} \mathbb{E}_{\vec{x}_n|\theta_n}\left[\sum_{k=1}^{m} a^{\tau_k} G(X_{\tau_k})\right], \tag{2.1}$$

where in this paper we specify the reward function as $G(x) = (x - K)^+$ motivated by American put option. Let $U_n^{[m]}(\vec{x}_n|\theta_n)$:= stopping reward, that is, conditional maximum expected reward when investor observed $X_1 = x_1, \cdots, X_n = x$, he can exercise at most m times, and he exercises at time n, then

$$U_n^{[m]}(\vec{x}_n|\theta_n) = a^n G(x_n) + \mathbb{E}_{\vec{x}_n|\theta_n}[V_{n+1}^{[m-1]}(\vec{X}_{n+1}|\theta_{n+1})]. \tag{2.2}$$

Optimality equations for our finite horizon Bayesian optimal multiple stopping problem are given by for each k,

$$V_n^{[k]}(\vec{x}_n|\theta_n) = \max\{U_n^{[k]}(\vec{x}_n|\theta_n), \mathbb{E}_{\vec{x}_n|\theta_n}[V_{n+1}^{[k]}(\vec{X}_{n+1}|\theta_{n+1})]\}, \ 0 \leq n \leq N-1, \tag{2.3}$$

where $V_N^{[k]}(\vec{x}_N|\theta_N) = U_N^{[k]}(\vec{x}_N|\theta_N) = a^N G(x_N)$ for each k. Generally, it is not easy to solve these optimality equations because these equations include the history of the observations. However, we can obtain the reduced optimality equations generated from the sufficient statistics sequence $\{Y_n\}_{n\in\mathbb{N}}$ as follows; for $Y_n = y$, $Y_0 \equiv 0$,

$$V_n^{[k]}(y|\theta_n) = \max\{U_n^{[k]}(y|\theta_n), \mathbb{E}_{y|\theta_n}[V_{n+1}^{[k]}(Y_{n+1}|\theta_{n+1})]\}, \tag{2.4}$$

$$U_n^{[k]}(y|\theta_n) = a^n G(y) + \mathbb{E}_{y|\theta_n}[V_{n+1}^{[k-1]}(Y_{n+1}|\theta_{n+1})]), \ n = 0, 1, \cdots, N-1, \tag{2.5}$$

where $V_N^{[k]}(y|\theta_N) = U_N^{[k]}(y|\theta_N) = a^N G(y)$ for each $k = 1, 2, \cdots, m$. So we have the optimal stopping region: for each k

$$B^{[k]} = \bigcup_{n=1}^{N} B_n^{[k]}, \ B_n^{[k]} = \{y : U_n^{[k]}(y|\theta_n) = V_n^{[k]}(y|\theta_n)\} \tag{2.6}$$

But (2.6) gives us no any useful information, since $V_n^{[k]}(y|\theta_n)$ is implicit.

We specify the probability density function $g(p)$ of p Beta, that is,

$$g(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{Be(\alpha, \beta)} I_{\{0<p<1\}}, \tag{2.7}$$

where $Be(\alpha, \beta) := \int_0^1 p^{\alpha-1}(1-p)^{\beta-1}dp, (0 < p < 1)$ is Beta function. After observing $\vec{x}_n = (x_1, \cdots, x_n)$, the posterior distribution, for which we denote the density by $g(\vec{x}_n)$, is again $Beta(\alpha_{n+1}, \beta_{n+1})$ distribution with parameters $\alpha_{n+1} := \alpha + y_n, \beta_{n+1} := \beta + n - y_n$, where $y_n := \sum_{i=1}^{n} I_{\{x_i=1\}}$is the number of upward among $\{X_1, X_2, \cdots, X_n\}$. Indeed, y_n is the sufficient statistics satisfying $g(p|\vec{x}_n) = g(p|y_n)$. Since the number of downward until time n is $n - y_n$, it follows that $S_n = S_0 s^{2y_n - n}$.

3 How to solve

3.1 Single stopping

Define a new operator $\mathcal{L}^{[1]}(y|\theta_n)$ given $Y_n = y$ by

$$\mathcal{L}_n^{[1]}(y|\theta_n) := \mathbb{E}_{y|\theta_n}[a^{n+a}G(Y_{n+a})] - a^n G(y). \tag{3.1}$$

This may be regarded as a discrete version of an infinitesimal generator (cf. Abdel-Hameed [4]).

Theorem 3.1 *For the Bayesian single stopping problem, the optimal stopping region is given by* $B^{[1]} = \bigcup_{n=1}^{N} B_n^{[1]}$, *where for* $Y_n = y$

$$B_n^{[1]} = \left\{ y : \mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{\tau-1} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right] \le 0 \right\} \tag{3.2}$$

and $B_N^{[1]} = \{y : G(y) \ge 0\}$. *The optimal stopping time is* $\tau_1^* = \inf\{n \in \{0, 1, \cdots, N\} : Y_n \in B_n^{[1]}\}$. *The maximum expected reward is* $\mathbb{E}_{Y_0|\theta_0}[a^{\tau_1^*}G(Y_{\tau_1^*})]$.

Proof. $B_n^{[1]}$ follows immediately from a discrete version of Dynkin formula (cf. Abdel-Hameed [4]); for any finite (a.s) stopping time τ,

$$\mathbb{E}_{y|\theta_n}[a^\tau G(Y_\tau)] = a^n G(y) + \mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{\tau-1} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right], Y_n = y. \tag{3.3}$$

□

Lemma 3.1 $B_n^{[1]} \subseteq B_{n+1}^{[1]}, n = 1, 2, \cdots, N.$

Proof. Since $B_n^{[1]} = \{y : a^n G(y) \ge \mathbb{E}_{y|\theta_n}[V_{n+1}(Y|\theta_n)]\}$ and $B_{n+1}^{[1]} = \{y : a^{n+1}G(y) \ge \mathbb{E}_{y|\theta_{n+1}}[V_{n+2}(Y|\theta_{n+1})]\}$, it suffices to show that

$$\mathbb{E}_{y|\theta_n}[V_{n+2}(Y|\theta_n)] \le a\mathbb{E}_{y|\theta_{n+1}}[V_{n+1}(Y|\theta_{n+1})], n = 1, 2, \cdots, N-1. \tag{3.4}$$

By backward induction on n, we can prove this inequality. □

Consider the following conditions. For each $n = 1, 2, \cdots, N-1$, $Y_n = y$ and any integer $j \in \mathbb{N}$

(A1): $y \mapsto \mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right]$ changes sign at most once from positive to nonpositive.

(A2): $y \mapsto \mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right]$ changes sign at most once from negative to nonnegative.

These conditions ensure that there is a unique root, $y_n^{[1]*}$, of the equation, $\mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{\tau-1} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right]$ = 0. Under (A1), when $\mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right] > 0$ (≥ 0) for all y, we set $y_n^{[1]*} \equiv \infty$ ($-\infty$, respectively). Under (A2), when $\mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right] < 0$ (≤ 0) for all y, we set $y_n^{[1]*} \equiv \infty$ ($-\infty$, *resp.*).

Corollary 3.1

(i) If (A1) holds, then $B_n^{[1]} = \{y : y \geq y_n^{[1]*}\}$, $\{y_n^{[1]*}\}_{n=1}^N$ *is nonincreasing sequence and* $\tau_1^* = \inf\{n \in \{0, 1, \cdots, N\} : Y_n \geq y_n^{[1]*}\}$.

(ii) If (A2) holds, then $B_n^{[1]} = \{y : y \leq y_n^{[1]*}\}$, $\{y_n^{[1]*}\}_{n=1}^N$ *is nondecreasing sequence and* $\tau_1^* = \inf\{n \in \{0, 1, \cdots, N\} : Y_n \leq y_n^{[1]*}\}$.

Proof. For the proof of (i), use Theorem 3.1, Lemma 3.1 and

$$B_n^{[1]} \subseteq B_{n+1}^{[1]} \Longleftrightarrow \{y : y \geq y_n^{[1]*}\} \subseteq \{y : y \geq y_{n+1}^{[1]*}\}.$$

□

3.2 Multiple stopping

Define an operator $\mathcal{L}_n^{[k]}(y)$ given $Y_n = y$ as follows;for each $\ell = 1, 2 \cdots, N-n$ and $k = 1, 2, \cdots, m$,

$$\mathcal{L}_n^{[k]}(y|\theta_n) = \mathbb{E}_{y|\theta_n}[U_{n+1}^{[k]}(Y_{n+1})] - U_n^{[k]}(y). \tag{3.5}$$

Theorem 3.2 *For the Bayesian multiple stopping problem, the optimal stopping region is given by* $B^{[k]} = \bigcup_{n=1}^N B_n^{[k]}$ *for each* $k = 1, 2, \cdots, m$, *as for* $Y_n = y$

$$B_n^{[k]} = \left\{ y : \mathbb{E}_{y|\theta_n} \left[\sum_{\ell=n}^{\tau-1} \mathcal{L}_\ell^{[k]}(Y_\ell|\theta_n) \right] \leq 0 \right\} \tag{3.6}$$

and $B_N^{[k]} = \{y : G(y) \geq 0\}$. *The optimal stopping time is* $\tau_k^* = \inf\{n \in \{0, 1, \cdots, N\} : Y_n \in B_n^{[k]}\}$. *The maximum expected reward is* $\mathbb{E}_{Y_0|\theta_0}[\sum_{k=1}^m a^{\tau_k^*} G(Y_{\tau_k^*})]$.

Proof. It is as same as the proof of Theorem 3.1. □

Corollary 3.2 *For each* $k = 1, 2, \cdots, m-1$ *and* $n = 1, 2, \cdots, N$, $B_n^{[k]} \subseteq B_n^{[k+1]}$.

Proof. Since $B_n^{[k]} = \{y : \alpha^n G(y) \geq \mathbb{E}_{y|\theta_n}[V_{n+1}^{[k]}(Y) - V_{n+1}^{[k-1]}(Y)]\}$, it suffices to prove that for all y

$$\mathbb{E}_{y|\theta_n}[V_{n+1}^{[k]}(Y) - V_{n+1}^{[k-1]}(Y)] \geq \mathbb{E}_{y|\theta_n}[V_{n+1}^{[k+1]}(Y) - V_{n+1}^{[k]}(Y)]. \tag{3.7}$$

We can show this inequality by induction on k. □

Corollary 3.3 *For each* $k = 1, 2, \cdots, m$ *and* $n = 1, 2, \cdots, N-1$, $B_n^{[k]} \subseteq B_{n+1}^{[k]}$.

Proof. Since $B_n^{[k]} = \{y : \alpha^n G(y) \geq \mathbb{E}_{y|\theta_n}[V_{n+1}^{[k]} - V_{n+1}^{[k-1]}]\}$ and $B_{n+1}^{[k]} = \{y : \alpha^{n+1} G(y) \geq \mathbb{E}_{y|\theta_{n+1}}[V_{n+2}^{[k]} - V_{n+2}^{[k-1]}]\}$, it suffices to prove that

$$\mathbb{E}_{y|\theta_{n+1}}[V_{n+2}^{[k]} - V_{n+2}^{[k-1]}] \leq \alpha \mathbb{E}_{y|\theta_n}[V_{n+1}^{[k]} - V_{n+1}^{[k-1]}]. \tag{3.8}$$

By induction on n, we can prove this. □

Consider the following conditions. For each $k = 1, 2, \cdots, m$ and $n = 1, 2, \cdots, N-1$, $Y_n = y$ and any integer $j \in \mathbb{N}$

(B1): $y \mapsto \mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[k]}(Y_\ell|\theta_n)\right]$ changes sign at most once from positive to nonpositive.

(B2): $y \mapsto \mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[k]}(Y_\ell|\theta_n)\right]$ changes sign at most once from negative to nonnegative.

These conditions ensure that there is a unique root, $y_n^{[1]*}$, of the equation, $\mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{\tau-1} \mathcal{L}_\ell^{[k]}(Y_\ell|\theta_n)\right] = 0$. Under (B1), when $\mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[k]}(Y_\ell|\theta_n)\right] > 0$ (≥ 0) for all y, we set $y_n^{[k]*} \equiv \infty$ ($-\infty$, respectively). Under (B2), when $\mathbb{E}_{y|\theta_n}\left[\sum_{\ell=n}^{j} \mathcal{L}_\ell^{[1]}(Y_\ell|\theta_n)\right] < 0$ (≤ 0) for all y, we set $y_n^{[k]*} \equiv \infty$ ($-\infty$, *resp.*).

Corollary 3.4

(i) If (B1) holds for each k, then $B^{[k]} = \bigcup_{n=1}^{N} B_n^{[k]}$, $B_n^{(k)} = \{y : y \geq y_n^{[k]*}\}$, $\tau_k^* = \inf\{n \in \{0, 1, \cdots, N\} : Y_n \geq y_n^{[k]*}\}$ *where* $y_n^{[k]*}$ *is nonincreasing in n and k, and the maximum expected reward is* $\mathbb{E}_{y_0|\theta_0}\left[\sum_{k=1}^{m} \alpha^{\tau_k *} G(Y_{\tau_k^*})\right]$.

(ii) If (B2) holds for each k, then $B^{[k]} = \bigcup_{n=1}^{N} B_n^{[k]}$, $B_n^{(k)} = \{y : y \leq y_n^{[k]*}\}$, $\tau_k^* = \inf\{n \in \{0, 1, \cdots, N\} : Y_n \leq y_n^{[k]*}\}$ *where* $y_n^{[k]*}$ *is nondecreasing in n and k, and the maximum expected reward is* $\mathbb{E}_{y_0|\theta_0}\left[\sum_{k=1}^{m} \alpha^{\tau_k *} G(Y_{\tau_k^*})\right]$.

For each $k = 1, 2, \cdots, m$ and $\ell = 2, 3, \cdots, N - n$,

$$\begin{aligned} B_{n+1,n+\ell}^{[k]} &:= \{Y_{n+1} \notin B_{n+1}^{[k]}, \cdots, Y_{n+\ell-1} \notin B_{n+\ell-1}^{[k]}, Y_{n+\ell} \in B_{n+\ell}^{[k]}\}, \\ B_{n+1,n+1}^{[k]} &:= \{Y_{n+1} \in B_{n+1}^{[k]}\} \end{aligned}$$

Lemma 3.2 *For each* $k = 1, 2, \cdots, m$,

$$\mathcal{L}_n^{[k]}(y|\theta_n) = \mathcal{L}_n^{[1]}(y|\theta_n) + \mathbb{E}_{y|\theta_n}\left[\sum_{\ell=1}^{N-n-1}\sum_{j=0}^{\ell-1} \mathcal{L}_{n+1+j}^{[k-1]}(Y_{n+1+j}|\theta_n) I_{B_{n+1,n+1}^{[k-1]}} I_{B_{n+1,n+1+\ell}^{[k-1]}}\right].$$

Proof. See Ano [1]. □

Corollary 3.5 *If (A1) holds, then (B1) satisfies. If (A2) holds then (B2) satisfies.*

Proof. They follow from Lemma 3.2. □

4 Multiple stopping problem on geometric random walk

Let us calculate $\mathcal{L}_n^{[1]}(y|\alpha, \beta)$ for our Bayesian multiple stopping problem on geometric random walk with unknown upward probability.

$$\begin{aligned} \mathbb{E}_{y|\alpha.\beta}[a^{n+1}G(Y_{n+1})] &= a^{n+1}\left\{\int_{-\infty}^{\infty} G(y + x_{n+1}) f(x_{n+1}|y) dx_{n+1}\right\} \\ &= a^{n+1}\left\{\int_{-\infty}^{\infty} G(y + x_{n+1}) \int_0^1 f(x_{n+1}|p) g(p|y) dp dx_{n+1}\right\} \\ &= a^{n+1}\left\{\int_0^1 \int_{-\infty}^{\infty} G(y + x_{n+1}) f(x_{n+1}|p) g(p|y) dx_{n+1} dp\right\} \end{aligned}$$

From $f(x|p) = pI_{\{x=1\}} + qI_{\{x=0\}}$, it follows that for $G(y) = (s^{2y-n+1} - K)^+$,

$$\begin{aligned}
\mathbb{E}_{y|\alpha,\beta}[a^{n+1}G(Y_{n+1})] &= a^{n+1}\left\{\int_0^1 p\left(s^{2y-n+2} - K\right)^+ g(p|y)dp + \int_0^1 q(s^{2y-n} - K)^+ g(p|y)dp\right\} \\
&= a^{n+1}\left\{\int_0^1 p(s^{2y-n+2} - K)^+ \frac{p^{\alpha+y-1}q^{\beta+n-y-1}}{Be(\alpha+y,\beta+n-y)}dp\right. \\
&\qquad \left. + \int_0^1 q(s^{2y-n} - K)^+ \frac{p^{\alpha+y-1}q^{\beta+n-y-1}}{Be(\alpha+y,\beta+n-y)}dp\right\} \\
&= a^{n+1}\left\{\frac{Be(\alpha+y+1,\beta+n-y)}{Be(\alpha+y,\beta+n-y)}(s^{2y-n+2} - K)^+\right. \\
&\qquad \left. + \frac{Be(\alpha+y,\beta+n-y+1)}{Be(\alpha+y,\beta+n-y)}(s^{2y-n} - K)^+\right\} \\
&= a^{n+1}\sum_{k=0}^{1}\binom{1}{k}\frac{Be(\alpha+y+k,\beta+n-y+1-k)}{Be(\alpha+y,\beta+n-y)}(s^{2y-n+2k} - K)^+.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}_n^{[1]}(y|\alpha,\beta) &= a^{n+1}\sum_{k=0}^{1}\binom{1}{k}\frac{Be(\alpha+y+k,\beta+n-y+1-k)}{Be(\alpha+y,\beta+n-y)}(s^{2y-n+2k} - K)^+ \\
&\qquad - a^n(s^{2y-n+1} - K)^+. \qquad (4.1)
\end{aligned}$$

Therefore by Corollaries 3.4 and 3.5, we have

Theorem 4.1 *Suppose that* $G(y) = (s^{2y-n+1} - K)^+$.

(i) If (A1) holds, then for each $k = 1, 2, \cdots, m$ (i) $B^{[k]} = \bigcup_{n=1}^N B_n^{[k]}$, $B_n^{(k)} = \{y : y \geq y_n^{[k]*}\}$. (ii) $\tau_k^* = \inf\left\{n \in [0,N] : S_n \geq s^{2y_n^{[k]*}-n+1}\right\} = \inf\left\{n \in [0,N] : Y_n \geq y_n^{[k]*}\right\}$, *where* $y_n^{[k]*}$ *is nonincreasing in* n *and* k. (iii) *Maximum expected reward is* $\mathbb{E}_{S_0|\alpha,\beta}[\sum_{k=1}^m a^{\tau_k}(S_{\tau_k} - K)^+]$.

(ii) If (A2) holds, then for each $k = 1, 2, \cdots, m$ (i) $B^{[k]} = \bigcup_{n=1}^N B_n^{[k]}$, $B_n^{(k)} = \{y : y \leq y_n^{[k]*}\}$. (ii) $\tau_k^* = \inf\left\{n \in [0,N] : S_n \leq s^{2y_n^{[k]*}-n+1}\right\} = \inf\left\{n \in [0,N] : Y_n \leq y_n^{[k]*}\right\}$, *where* $y_n^{[k]*}$ *is nondecreasing in* n *and* k. (iii) *Maximum expected reward is* $\mathbb{E}_{S_0|\alpha,\beta}[\sum_{k=1}^m a^{\tau_k}(S_{\tau_k} - K)^+]$.

In the same way, we have

Theorem 4.2 *Suppose that* $G(y) = (K - s^{2y-n+1})^+$.

(i) If (A1) holds, then for each $k = 1, 2, \cdots, m$ (i) $B^{[k]} = \bigcup_{n=1}^N B_n^{[k]}$, $B_n^{(k)} = \{y : y \geq y_n^{[k]*}\}$. (ii) $\tau_k^* = \inf\left\{n \in [0,N] : S_n \geq s^{2y_n^{[k]*}-n+1}\right\} = \inf\left\{n \in [0,N] : Y_n \geq y_n^{[k]*}\right\}$, *where* $y_n^{[k]*}$ *is nonincreasing in* n *and* k. (iii) *Maximum expected reward is* $\mathbb{E}_{S_0|\alpha,\beta}[\sum_{k=1}^m a^{\tau_k}(K - S_{\tau_k})^+]$.

(ii) If (A2) holds, then for each $k = 1, 2, \cdots, m$ (i) $B^{[k]} = \bigcup_{n=1}^N B_n^{[k]}$, $B_n^{(k)} = \{y : y \leq y_n^{[k]*}\}$. (ii) $\tau_k^* = \inf\left\{n \in [0,N] : S_n \leq s^{2y_n^{[k]*}-n+1}\right\} = \inf\left\{n \in [0,N] : Y_n \leq y_n^{[k]*}\right\}$, *where* $y_n^{[k]*}$ *is nondecreasing in* n *and* k. (iii) *Maximum expected reward is* $\mathbb{E}_{S_0|\alpha,\beta}[\sum_{k=1}^m a^{\tau_k}(K - S_{\tau_k})^+]$.

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