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Combining Functional Equations and Computer Algebra Systems with Regard to XXI Century Mathematics Education

Andrés Iglesias
Department of Applied Mathematics and Comp. Sciences
University of Cantabria, Avda. de los Castros
s/n, E-39005, Santander, Spain
`iglesias@unican.es`
`http://personales.unican.es/iglesias`

Abstract

This paper explores the interplay between functional equations and computer algebra systems in order to derive consistent mathematical models of problems arising in different scientific and engineering fields. In particular, we claim that this combination of both techniques is particularly useful to assist our students to grasp the essential of mathematical modeling and selection of models for such problems as well as to solve them in an unified way. This scheme is illustrated by means of some examples of economical models. In this approach, functional equations are used to reach the mathematical expressions of the economical models for monopoly and duopoly, while a *Mathematica* package called `FSolve` is used to solve them symbolically.

1 Introduction

Last year, in the previous edition of this RIMS workshop, the author of this paper [9] emphasized that:

> *Today's students [...] are less skilled than their counterparts of the last previous decades in deduction, mathematical intuition and scientific reasoning and encounter more problems in solving questions with scientific content. Their background is also less solid in both science and arts. Furthermore, they also have less oral and written communication skills, with a much limited vocabulary and hence find some troubles for a full comprehension of concepts and ideas.*

Although this paragraph draws a quite pessimistic picture regarding the XXI century mathematical education, there are certainly not only shadows but also lights in the process. Coming back to previous year's words:

> *On the positive side, most current students come to college and university with greater computer proficiency and technology skills than their predecessors. Technology is natural to them as they got accustomed to use it from their childhood. [...] Much better, they are not only accustomed to technology but also they know how to use it efficiently. Therefore, proper use of computer tools and other technology turns out to be more than appropriate to promote their background to an upper level [7, 8].*

However, technology is just a tool, not the panacea or the real solution itself. A missing ingredient in the puzzle is our ability to teach students how to formalize problems and help them grasp the essentials of fundamental mental processes such as idealization of a problem and selection of appropriate models. In this paper we focus on these problems and try to give some useful hints to readers based on our experience dealing with a barely known field of mathematics: the functional equations. Our claim is that functional equations are one of the best mathematical tools available to achieve our educational goals. To this purpose, we shall describe some illustrative examples of applications of functional equations to several fields. The interested reader is referred to [5] for a comprehensive introduction to functional equations. See also [3] and [4] for further information on functional equations and their applications.

2 Functional Equations and Computer Algebra Systems

Modeling or idealization of a problem under consideration in Science and Engineering should be sufficiently simple, logically irrefutable, admitting a mathematical solution, and, at the same time, represent sufficiently well the actual problem. As in any other branch of knowledge, the selection of the idealized model should be achieved by detecting and representing the essential first-order factors, and discarding or neglecting the inessential second-order factors. Model building is usually based on an arbitrary or convenience selection of simple equations that seem to represent reality to a given quality level. However, on many occasions these models exhibit technical failures or inconsistencies which make them inadmissible. Functional equations are a tool that avoids arbitrariness and allows selection of models to be based on adequate constraints. In fact, one of the most appealing characteristics of functional equations is their capacity for model design.

Once a feasible and proper model is chosen, next step is to solve the problem and here is where computer algebra systems play a decisive role. Indeed, the combination of functional equations and computer algebra systems (CAS) provides a very convenient tool to solve many problems at full extent, starting with the modelization of the problem by using functional equations to solving the resulting equations by using a CAS.

This is the approach we follow in the present paper. All computer operations in this paper have been performed by using the *Mathematica* package `FSolve` described in [1, 2, 3]. The procedure is as follows: firstly, we make some assumptions about the functional structure of the functions describing the models. Such assumptions are given in terms of functional equations that account for the properties of the given problem. Then, the package cis applied to compute the solutions of these equations and check for inconsistencies. We start our discussion by loading the package:

```
In[1]:=<<FunctionalEquations'FSolve'
```

which includes the command

FSolve[*eqn*, {*functions*}, {*variables*}, *options*]

where *eqn* denotes the functional equation to be solved, {*functions*} is the list of unknown functions, {*variables*} is the list of variables and *options* allows the users to consider different domains for the variables and classes of feasible functions (see [3] for further details on this issue).

For instance, we can calculate the solution of the functional equation $f(x+y) = g(x)+h(y)$ where $x, y \in \mathbb{R}$ and f, g, h are continuous functions as:

```
In[2]:= FSolve[f[x+y]==g[x]+h[y],{f,g,h},{x,y},Domain->Real,
            Class->Continuous]
```

$Out[2] := \{f(x) \to C(1)\,x + C(2) + C(3), g(x) \to C(1)\,x + C(2), h(x) \to C(1)\,x + C(3)\}$

where $C(1)$, $C(2)$ and $C(3)$ are arbitrary constants. Note that the general solution can depend on one or more arbitrary constants and even on arbitrary functions (see *Out*[3] and *Out*[4] for two examples). Note also that a single equation can determine several unknown functions (such as f, g and h in this example). See [5] for a general introduction to the theory of functional equations and their applications.

3 Some examples of functional equations

This section describes some illustrative examples of how functional equations can be applied to solve some interesting problems arising in different fields.

3.1 First example: area of a rectangle

This example was firstly described by Legendre in 1791. Assume that the formula of the area of a rectangle is unknown but is given by $f(a,b)$, where b is the length of its base and a is its height. The area of such a rectangle remains the same if it is horizontally divided in two different subrectangles with the same base b and heights a_1 and a_2. According to our assumptions the areas of the subrectangles and the initial rectangle cannot be calculated, but they can be expressed in terms of our f function as $f(a_1,b)$, $f(a_2,b)$, and $f(a_1+a_2,b)$, respectively. Similarly, we can perform the division vertically and write the areas of the resulting rectangles as $f(a,b_1)$, $f(a,b_2)$, and $f(a,b_1+b_2)$, respectively. Stating that the areas of the initial rectangles must be equal to the sum of the areas of the subrectangles, we get the equations

$$\begin{aligned} f(a_1+a_2,b) &= f(a_1,b)+f(a_2,b) \\ f(a,b_1+b_2) &= f(a,b_1)+f(a,b_2). \end{aligned} \tag{1}$$

The solution of this functional equation is given by [5]:

$$f(a,b) = c_1(b)a = c_2(a)b,$$

where $c_1(b)$ and $c_2(a)$ are initially arbitrary functions, but due to the second identity, they must satisfy the condition

$$\frac{c_1(b)}{b} = \frac{c_2(a)}{a} = c,$$

which implies

$$f(a,b) = cab, \tag{2}$$

where c is an arbitrary positive constant. As a consequence, the area of a rectangle is the product of its base a, its height b and a constant c (the measurement unit).

3.2 Second example: simple interest

Let $f(x,t)$ be the future value of the capital x having been invested for a period of time of duration t. Then, if the assumtions of simple interest hold, the function $f(x,t)$ must satisfy the following conditions:

1.- At the end of the time period t, we receive the same interest if we deposit the amount $x+y$ in one account or if we deposit the amount x in one account, and the amount y in another account. Thus, we have:

$$f(x+y,t) = f(x,t) + f(y,t).$$

2.- At the end of the time period $t+u$, we receive the same interest if we deposit the amount x during a period of duration $t+u$ or if we deposit the amount x first during a period of duration t and later for a period of duration u. Thus, we have:

$$f(x,t+u) = f(x,t) + f(x,u).$$

That is, the following equations hold:

$$\left.\begin{array}{l} f(x+y,t) = f(x,t) + f(y,t) \\ f(x,t+u) = f(x,t) + f(x,u) \end{array}\right\}; \quad x,y,t,u \in \mathbb{R}_+ \tag{3}$$

The solution of the first equation is given by:

$$f(x,t) = c(t)x,$$

and substitution into the second leads to:

$$c(t+u)x = c(t)x + c(u)x \quad \Rightarrow \quad c(t+u) = c(t) + c(u) \quad \Rightarrow \quad c(t) = Kt,$$

and then, we finally obtain:

$$f(x,t) = kxt,$$

which is the well known formula of the simple interest.

It is important to note here that the above assumptions do not hold in reality, but they are the simple interest assumptions. It can be seen from the bank office that if we deposit a larger amount or we do it for a longer period the interest rate increases. We note that the bank policy has to be such that:

$$f(x+y,t) \geq f(x,t) + f(y,t).$$

Otherwise, the bank is inviting his clients to deposit their money in many accounts (a low amount in each account). In addition, we must have:

$$f(x,t+u) \geq f(x,t) + f(x,u).$$

Otherwise, the bank is inviting his clients to withdraw the money everyday and deposit it again in a new account. Consequently, *simple interest* is the optimal way of keeping account stability by giving the least possible interest.

A comparison of the system of equations of the rectangle area and of the simple interest examples shows that, apart from notation, the two systems of functional equations (1) and (3) are identical. This means that we have two physical problems: one geometric and one economic, leading to exactly the same mathematical model.

4 Application to Economical Models

Now we show some examples of application of our package `FSolve` to analyze some economical models for price and advertising policies (see [6] for more details). In particular, we focus on the problem of modeling the sales $S(p,v)$ of a single-product firm such that they depend on the price p of its product and on the advertising expenditure v. We will restrict our discussion here to the cases of monopoly and duopoly models.

4.1 The monopoly model

Let us assume a firm such that the sales S of a single product depend on the unitary price p and on the advertising expenditure v, that is, $S = S(p,v)$. The function S cannot be arbitrary, but it must satisfy the following properties:

(M1) The $S(p,v)$ function is continuous in both arguments.

(M2) $\forall\, v$, the $S(p,v)$ function, considered as a function of p only, must be convex from below and decreasing. This implies that, for the same advertising expenses, any increment in the unit price of the product leads to a reduction in sales, and that its derivative decreases with p.

(M3) $\forall\, p$, the $S(p,v)$ function, considered as a function of v only, must be concave from below and increasing. This implies that, for the same unit price, an increment in the advertising expenses leads to an increment in sales.

(M4) A multiplicative change in the advertising expenditure leads to an additive change in sales, that is,

$$S(p, v\,w) = S(p,v) + T(p,w), \tag{4}$$

where $p \geq 0$, $v \geq 0$, $w \geq 0$, $T(p,1) = 0$ and $T(p,w)$ is increasing with w.

(M5) The sales due to an increment q in price are equal to the previous sales times a real number, which depends on q and v, that is,

$$S(p+q,v) = S(p,v)\,R(q,v) \tag{5}$$

where $p \geq 0$, $p+q \geq 0$, $v \geq 0$, $R(0,v) = 1$ and $R(q,v)$ is decreasing in q.

Eq. (4) can be solved by using the package FSolve as follows:

```
In[3]:= em1=FSolve[S[p,v*w]==S[p,v]+T[p,w],{S,T},{p,v,w},
        Domain->RealPositiveZero,Class->Continuous]
```

$Out[3] := \{S(p,v) \to Arb1(p)\,Log(v) + Arb2(p), T(p,w) \to Arb1(p)\,Log(w)\}$

where $Arb1(p)$ and $Arb2(p)$ denote two arbitrary functions depending on the variable p. Similarly, we can solve eq. (5) as:

```
In[4]:= em2=FSolve[S[p+q,v]==S[p,v]*R[q,v],{S,R},{p,q,v},
        Domain->RealPositiveZero,Class->Continuous]
```

$Out[4] := \left\{S(p,v) \to Arb3(v)\,e^{p\,Arb4(v)}, R(q,v) \to e^{q\,Arb4(v)}\right\}$

where $Arb3(v)$ and $Arb4(v)$ denote two arbitrary functions depending on the variable v. Once we have solved functional equations (4) and (5) separately, the general solution of the system (4)-(5) is given by:

```
In[5]:= FSolve[Equal @@ (S[p,v] /.  First[#]& /@ {em1,em2}),
                  {Arb1,Arb2,Arb3,Arb4},{p,v}];
In[6]:= S[p,v] /.  %
```

$Out[6] := (C(1) + C(2)\,Log[v])\,Exp[-C(3)\,p]$

where $C(1)$, $C(2)$ and $C(3)$ are arbitrary constants. Note that in *Out[6]* we have no longer arbitrary functions, but arbitrary constants. This means that the parametric model is completely specified and that we can estimate its parameters $C(1)$, $C(2)$ and $C(3)$ using empirical data. The obtained solution shows a logarithmic increment of sales with advertising expenditures and an exponential decrease with price, in agreement with assumptions (M4) and (M5). One justification of this model of sales is the so-called Weber-Fechner law, that states that the stimuli of the intensity of perception is a linear function of the logarithm of the intensity of the stimulus. It can be argued, however, that the function R should depend on the price p, instead of v. Thus, we can replace (M5) by:

(M6) The sales due to an increment q in price are equal to the previous sales times a real number, which depends on q and p, that is,

$$S(p+q,v) = S(p,v)\,R(q,p) \tag{6}$$

where $p \geq 0$, $p+q \geq 0$, $v \geq 0$, $R(0,p) = 1$ and $R(q,p)$ is decreasing in q.

The general solution of (6) can be obtained by using the package FSolve as:

```
In[7]:= em3=FSolve[S[p+q,v]==S[p,v]*R[q,p],{S,R},{p,q,v},
        Domain->RealPositiveZero,Class->Continuous]
```

$$Out[7] := \left\{ S(p,v) \rightarrow Arb5(p)\, Arb6(v),\; R(q,p) \rightarrow \frac{Arb5(p+q)}{Arb5(p)} \right\}$$

Alternatively, we can assume a multiplicative, instead of an additive, change in the price p and we can question whether or not choosing between one of these assumptions influences the resulting model. In other words, we can assume:

(M7) The sales due to a multiplicative change (w times) in the price are equal to the previous sales times a real number, which depends on w and p, that is,

$$S(p\,w, v) = S(p,v)\, R(w,p) \tag{7}$$

where $p \geq 0$, $w \geq 0$, $v \geq 0$, $R(1,p) = 1$ and $R(w,p)$ is decreasing in w.

```
In[8]:= FSolve[S[p*w,v]==S[p,v]*R[w,p],{S,R},{p,v,w},
        Domain->RealPositiveZero,Class->Continuous]
```

$$Out[8] := \left\{ S(p,v) \rightarrow Arb5(p)\, Arb6(v),\; R(w,p) \rightarrow \frac{Arb5(p\,w)}{Arb5(p)} \right\}$$

Note that the S functions in *Out[7]* and *Out[8]* are identical. Thus, equations (6) and (7) are equivalent. Consequently, the above mentioned two assumptions (M6) and (M7) lead to the same model. Now, the solution of the system (4)-(6) can be obtained as:

```
In[9]:= FSolve[Equal @@ (S[p,v] /.  First[#]& /@ {em1,em3}),
                {Arb1,Arb2,Arb5,Arb6},{p,v}]
```

$$Out[9] := \begin{array}{l} \{Arb2(p) \rightarrow \frac{C(4)}{C(3)} Arb1(p), Arb5(p) \rightarrow \frac{-Arb1(p)}{C(3)}, \\ Arb6(p) \rightarrow -C(3)\, Log(p) - C(4)\} \end{array}$$

which leads to the model:

```
In[10]:= S[p,v] /.  %
```

$$Out[10] := Arb1(p)\, \left[Log(v) + \frac{C(4)}{C(3)} \right]$$

where the function $Arb1(p)$ and the constants $C(3)$ and $C(4)$ are arbitrary. For this solution to satisfy assumptions (M2) and (M3) above, $Arb1(p)$ must be convex from below and decreasing. Note that $Log(v) + \frac{C(4)}{C(3)}$ is increasing. We also remark that model in $Out[10]$ is more general than model in $Out[6]$. In fact, the resulting model is not completely specified because it depends on arbitrary functions. This means that new requirements might be established.

4.2 The duopoly model

Assume now that we have two different firms that compete in the market. Assume also that the sales S of the product by firm 1 depend on the unit prices p and q and on the advertising expenditures u and v, of the two firms, that is, $S = S(p,q,u,v)$. The function $S(p,q,u,v)$ must satisfy the following properties:

(D1) The $S(p,q,u,v)$ function is continuous in all arguments.

(D2) $S(p,q,u,v)$ is increasing in q and u.

(D3) $S(p,q,u,v)$ is decreasing in p and v.

(D4) A multiplicative change in the advertising expenditure of firm 1 leads to an additive change in sales, that is,

$$S(p,q,u\,w,v) = S(p,q,u,v) + T(p,q,w,v) \tag{8}$$

(D5) The sales due to an increment r in price of firm 1 are equal to the previous sales times a real number, which depends on r and p, that is,

$$S(p+r,q,u,v) = S(p,q,u,v)\,R(r,p,q,v) \tag{9}$$

where $p \geq 0$, $p+r \geq 0$, $v \geq 0$ and $R(0,p,q,v) = 1$.

The general solution of the system (8)-(9) is given by the following sequence of calculations: firstly, we compute the functions S, T and R of the previous equations, and then we apply the outputs to calculate the functional structure of function S.

```
In[11]:= ed1=FSolve[S[p,q,u*w,v]==S[p,q,u,v]+T[p,q,w,v],{S,T},
                {p,q,u,v,w},Domain->RealPositiveZero,
                Class->Continuous];
In[12]:= ed2=FSolve[S[p+r,q,u,v]==S[p,q,u,v]+R[r,p,q,v],{S,R},
                {p,q,r,u,v},Domain->RealPositiveZero,
                Class->Continuous];
In[13]:= FSolve[Equal @@ (S[p,q,u,v] /.  First[#]& /@ {ed1,ed2}),
                {Arb1,Arb2,Arb3,Arb4},{p,q,u,v}];
In[14]:= S[p,q,u,v] /.  %
```

$Out[14] := Arb1(p,q,v)\;[Log(u) + Arb2(q,v)]$

where $Arb1(p,q,v)$ and $Arb2(q,v)$ are arbitrary functions. In addition we can consider the following assumption:

(D6) The total sales of both firms is a constant K, that is,

$$S(p,q,u,v) + S(q,p,v,u) = K \tag{10}$$

which, using the previous output, leads to

```
In[15]:= FSolve[((S[p,q,u,v]+S[q,p,v,u]) /.  %)==K,{Arb1,Arb2},
         {p,q,u,v},Domain->RealPositiveZero,Class->Continuous]
         // FSimplify;
In[16]:= S[p,q,u,v] /.  %
```

$Out[16] := \dfrac{1}{Arb7(p) + Arb7(q)}\left[Log\left(\dfrac{u}{v}\right) + K\,Arb7(q)\right]$

where $Arb7(p)$ is an arbitrary but increasing function of p. The physical interpretation of this model is as follows: if the advertisement expenditures of both firms are coincident, the sales are proportional to the ratios $\dfrac{Arb7(q)}{Arb7(p)+Arb7(q)}$ and $\dfrac{Arb7(p)}{Arb7(p)+Arb7(q)}$ for firms 1 and 2, respectively. On the other hand, the advertisement expenditures influence sales directly proportional to the logarithm of the ratio $\dfrac{u}{v}$ and inversely proportional to $Arb7(p)+Arb7(q)$.

We can now consider two additional assumptions:

(D7) The sales $S(p+r, q+s, u, v)$ of firm 1 due to increments r and s in the prices of firms 1 and 2, respectively, are the initial sales $S(p, q, u, v)$ of firm 1 times two factors which consider the associated reduction and increments due to these two changes, that is,

$$S(p+r, q+s, u, v) = S(p, q, u, v)\, U(r, p, q)\, V(s, p, q) \tag{11}$$

(D8) The sales $S(p, q, u+r, v+s)$ of firm 1 due to increments r and s in the advertisement expenditures of firms 1 and 2, respectively, are the initial sales $S(p, q, u, v)$ of firm 1 times two factors which consider the associated increments and decrements due to these two changes, that is,

$$S(p, q, u+r, v+s) = S(p, q, u, v)\, U(r, u, v)\, V(s, u, v) \tag{12}$$

Combining now (D7) and (D8) and solving the system of equations (11)-(12), we get:

```
In[17]:= FSolve[S[p+r,q+s,u,v]==S[p,q,u,v]*U[r,p,q]*V[s,p,q],
              {S,U,V},{p,q,r,s,u,v},Domain->RealPositiveZero,
              Class->Continuous];
In[18]:= FSolve[S[p,q,u+r,v+s]==S[p,q,u,v]*U[r,u,v]*V[s,u,v],
              {S,U,V},{p,q,r,s,u,v},Domain->RealPositiveZero,
              Class->Continuous];
In[19]:= FSolve[Equal @@ (S[p,q,u,v] /. First[#]& /@ {%%,%}),
              {Arb1,Arb2,Arb3,Arb6,Arb7,Arb8},{p,q,u,v}];
In[20]:= S[p,q,u,v] /. %
```

Out[20] := $Arb1(p)\, Arb2(q)\, Arb6(u)\, Arb7(v)$

where the functions $Arb1(p)$ and $Arb7(v)$ are decreasing and the functions $Arb2(q)$ and $Arb6(u)$ are increasing, but otherwise arbitrary. The physical interpretation of this model is that all the factors (prices and advertisement expenditures) act independently and contribute to the total sales of firm 1 as a factor which is less than 1 and decreasing for p and v and greater than 1 and increasing for q and u.

5 Conclusions and Further Remarks

In this paper we focus on the interplay between functional equations and computer algebra systems as an effective way of getting a proper mathematical representation of a given problem in terms of functional equations and then solving them through the use of CAS. This approach is illustrated by means of some examples of economical models. Our experience is that the functional equations are an optimal technique to achieve these goals. They provide powerful and consistent methods to describe the common sense properties of the economical functions and, simultaneously, the mathematical tools for solving the resulting equations. The drawback of this approach is that most of this work must be performed by hand, as there is is only a few computer tools for solving functional equations. One remarkable exception is our *Mathematica* package, `FSolve`, which is intensively used in this paper in order to tackle this issue. Although functional equations are not commonly taught in standard mathematical courses (and this applies even for the degree in Mathematics) we still think they are a very valuable technique to develop the mathematical intuition of our students and consequently, we advice our readers to consider this approach very seriously.

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References

[1] Castillo, E., Gutierrez, J.M., Iglesias, A.: Solving a Functional Equation. *The Mathematica Journal*, **5**(1) (1995) 82-86.

[2] Castillo, E., Iglesias, A., Cobo, A.: A package for symbolic solutions of functional equations. In: Keranen, V., Mitic, P. (eds.) Proceedings of the First International Mathematica Symposium. *Computational Mechanics Publications*, Southampton (England), (1995) 85-92.

[3] Castillo, E., Iglesias, A.: A package for symbolic solution of real functional equations of real variables. *Aequationes Mathematicae*, **54** (1997) 181-198

[4] Castillo, E., Iglesias, A.: Some characterizations of families of surfaces using functional equations. *ACM Transactions on Graphics*, **16**(3) (1997) 296-318.

[5] Castillo, E., Iglesias, A., Ruiz-Cobo, R.: Functional Equations in Applied Sciences. *Elsevier Pub.*, Amsterdam (2005)

[6] Gálvez, A., Iglesias, A.: Symbolic Analysis of Economical Models with Mathematica. *Lectures Notes in Computer Science*, **3992** (2006) 414-421.

[7] Iglesias, A., Ipanaqué, R.: Using computer algebra systems to achieve Bologna's Declaration educational goals. A case study: symbolic proof of limits of functions. *International Journal of Computer Science and Software Technology*, **2**(1) (2009) 35-42.

[8] Iglesias, A.: Facing the challenges of the new European Space of Higher Education through effective use of computer algebra systems as an educational tool. *RIMS Kokyuroku Journal Series*, **1624** (2009) 114-128.

[9] Iglesias, A.: Computer Technologies for XXI Century Education: A New Way to Communicate and Learn at the University of Cantabria. *RIMS Kokyuroku Journal Series*, **1674** (2010) 114-128.