数理解析研究所講究録
第 1736 巻 2011 年 200-215

Financing and Investment under Different Debt Structures

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1 Introduction

In general, there exist two types of debt issued by firms: market debt and bank debt. When a firm is unable to service contractual debt payments, that is, default, the firm usually asks creditors to accept debt payment concessions. The bank may grant concessions through bilateral renegotiation if the firm is suffered by temporary financial distress, not economically inefficient. However, since the creditors of market debt are dispersive, it is hard to reach an agreement on debt payment concessions through renegotiation. In that case, the firm has to go into bankruptcy directly after default. The creditors seize the alienable physical assets of the firm after paying the bankruptcy cost. Modelling such corporate features in default is critical in debt valuation literature and also has large impact on firms' financing and investment decisions.

Except for Hackbarth, Hennessy, and Leland (2007), most existing models assume that firms issue a single class of debt: nonrenegotiable debt (see Leland, 1994; Goldstein, Ju, and Leland, 2001; Sundaresan and Wang, 2007a) or renegotiable debt (see Mella-Barral and Perraudin, 1997; Fan and Sundaresan, 2000; Sundaresan and Wang, 2007b).[1] Table 1 summarizes the features of related structural trade-off models. In the table, a symbol "Y" implies that the model incorporates a feature that increases realism. The first row represents each model (e.g., "L" stands for Leland, 1994; "GJL" stands for Goldstein, Ju, and Leland, 2001; etc.).

Table 1: Features of related structural trade-off models.

	L	GJL	SW(a)	MP	FS	SW(b)	HHL
nonnegotiable debt	Y	Y	Y	N	N	N	Y
negotiable debt	N	N	N	Y	Y	Y	Y
bargaining power	N	N	N	N	Y	Y	N
endogenous capital structure	Y	Y	Y	N	Y	Y	Y
endogenous investment	N	N	Y	N	N	Y	N

As Hart and Moore (1995) argue, models with single class of debt cannot explain the existence of different debt structures observed in practice, especially the mixed debt structure. Concerning bank debt and market debt, existing literature find that the percentage of market debt in

[1] While Sundaresan and Wang (2007b) consider debt renegotiation, Sundaresan and Wang (2007a) focus on the debt overhang problem and ignore debt renegotiation.

total debt is increasing in firm size and age (see Houston and James, 1996; Johnson, 1997; Krishnaswami *et al.*, 1999; and Denis and Mihov, 2003). Blackwell and Kidwell (1988) document that while small firms issue privately placed debt almost exclusively, large firms are more likely to issue market debt.

In this paper, we examine the financing and investment decisions under different debt structures: exclusive market debt (corresponding to Sundaresan and Wang, 2007a), exclusive bank debt (corresponding to Sundaresan and Wang, 2007b), and the mixture of market debt and bank debt. The major difference between market debt and bank debt stems from the possibility of renegotiation once the firm falls into financial distress. While the market debt cannot be renegotiated because of the dispersion of debtholders, the bank may grant state-contingent debt payment concessions in costless bilateral renegotiation.

The main questions are as follows: How do different debt structures affect firm's financing and investment decisions? What is the optimal debt structure? Especially, if the third one is optimal, then what is the optimal mixture of market debt and bank debt? Do the results depend on firm characteristics?

The most related literature of this chapter is Hackbarth *et al.* (2007) and Sundaresan and Wang (2007b). Hackbarth *et al.* (2007) examine the optimal mixture and priority structure of bank debt and market debt, without considering the investment decision. Sundaresan and Wang (2007b) investigate investment under uncertainty with strategic debt service. They provide a framework to analyze both the financing and investment decisions. However, the debt structure is limited to a single class.

The contribution of this paper is that we integrate the two strands of literature: investment and debt structure. We adopt a setting that resembles Hackbarth *et al.* (2007) and extend their model in the following dimensions by applying the framework of Sundaresan and Wang (2007b). First, we incorporate investment decision into the model. In Hackbarth *et al.* (2007), debt is issued at the same exogenous investment timing under different debt structures. However, since the financing and investment decisions interact with each other, the optimal timing of investment varies under different debt structures and so do the timing of debt issuance. Thus, it is necessary to incorporate the investment decision to consider the optimal debt structure. Second, we accommodate varying bargaining powers to the equityholders and the bank during debt renegotiation in financial distress. This is more flexible in that it comprises the two extreme case (either the equityholders or the bank can make take-it-or-leave-it offers in debt service) analyzed in Hackbarth *et al.* (2007). Third, we consider a reasonable restoration of contractual debt payment and the associated tax benefits when the EBIT improves.[2] In Hackbarth *et al.* (2007), once the debt renegotiation begins, the debt payment concessions continue forever, regardless of whether the EBIT has recovered or not. Fourth, we consider that both the bank debtholders and market debtholders receive a part of the remaining firm value upon bankruptcy. Thus, we are able to obtain an interior solution of the mixed debt structure, which means that

[2] In contrast with our model, Hackbarth *et al.* (2007) assign bargaining powers to the bank and market debtholders to examine deviations from absolute priority upon bankruptcy. In their paper, if the bank has full bargaining power, the senior position of the bank is inviolable.

the optimal bank debt and market debt interact with each other. This result is more realistic than Hackbarth *et al.* (2007), who substantially assume that the market debtholders receive nothing upon bankruptcy in the main part of their model. In such an extreme case, they cna only obtain a corner solution of the mixed debt structure, which means that the optimal bank debt and market debt are determined separately and the optimal debt structure is to issue the bank debt until its capacity and thereafter issue a positive amount of market debt.

2 Model setup

The model is set in a continuous-time risk-neutral framework. We suppose that the firm owns a privileged right to undertake a project with an irreversible investment cost I. The potential EBIT generated by the project is given by the following geometric Brownian motion process:

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}z(t), \tag{2.1}$$

where μ and $\sigma > 0$ are constants and $(z(t))_{t\geq 0}$ denotes a standard Brownian motion under risk-neutral measure $\mathbb{P}$. The initial value $X(0)$ is sufficiently low; i.e., the potential EBIT has not yet been favorable enough to undertake the project. Let $r > 0$ denote the discount rate. As in most real option analysis, we assume $r > \mu$ for convergence.

When the EBIT process $X(t)$ reaches the investment threshold x^i (the superscript "i" stands for investment), the firm decides to exercise the investment option by paying the fixed investment cost I, which can be financed by equity and debt. For simplicity, we assume that the issued debt has infinite maturity. The contractual continuous coupon of the perpetual debt is c, which is tax deductible. Let the corporate tax rate be τ. After engaging in the investment project, at each instant, the firm receives the EBIT $X(t)$ and must pay coupon c to debtholders. When the EBIT $X(t)$ is sufficiently low to hit the default threshold x^d (the superscript "d" stands for default), the firm fails to pay the contractual coupon. That is, default occurs.

2.1 All-equity financing

First, we consider an all-equity financed firm. Based on our setup, the after-tax all-equity financed firm value after investment is given by

$$\Pi(x) = \mathbb{E}\Big[\int_t^\infty (1-\tau)\mathrm{e}^{-r(s-t)}X(s)\mathrm{d}s\Big|X(t) = x\Big] = \frac{1-\tau}{r-\mu}x, \tag{2.2}$$

where $\mathbb{E}[\cdot|X(t) = x]$ denotes the expectation operator under the risk-neutral measure $\mathbb{P}$, given that $X(t) = x$.

Let the *ex ante* firm value (firm value before investment, option value of investment) be $V_{\mathrm{U}}^{o}(x)$ (the superscript "o" and subscript "U" stand for option value and unlevered firm, respectively).

By using the standard real options approach, we obtain the *ex ante* firm value as:

$$V_{\mathrm{U}}^{o}(x) = \left[\Pi(x_{\mathrm{U}}^{i}) - I\right]\left(\frac{x}{x_{\mathrm{U}}^{i}}\right)^{\beta}, \quad x \leq x_{\mathrm{U}}^{i}, \tag{2.3}$$

where β is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2 y(y-1)/2 + \mu y - r = 0. \tag{2.4}$$

That is,

$$\beta = \frac{1}{\sigma^2}\left[-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}\right] > 1. \tag{2.5}$$

Then, we determine the optimal investment threshold x_{U}^i by maximizing the *ex ante* firm value in Eq.(2.3). The results under all-equity financing are summarized in the following proposition.

Proposition 2.1 (All-equity financing)
The optimal investment threshold is given by

$$x_{\mathrm{U}}^i = \frac{\beta}{\beta - 1}\frac{I}{\Pi(1)}. \tag{2.6}$$

The ex ante firm value is

$$V_{\mathrm{U}}^o(x) = \left(\frac{x}{x_{\mathrm{U}}^i}\right)^\beta \frac{I}{\beta - 1}, \quad x \le x_{\mathrm{U}}^i. \tag{2.7}$$

3 Equity and exclusive debt financing

From now on, we consider that the firm is partially financed with exclusive debt (market debt or bank debt), which is issued upon investment. The contractual continuous coupon of the perpetual market debt and bank debt are c_{M} (the subscript "M" stands for market debt) and c_{B} (the subscript "B" stands for bank debt), respectively. As mentioned in Section 1, throughout this paper, we assume that the firm behaves in equityholders' interests. Since equity is issued in all the cases, we hereafter write "debt financing" instead of "equity and debt financing" for abbreviation.

3.1 Exclusive market debt financing

In this subsection, we examine the case of exclusive market debt financing. We assume that the market debt cannot be renegotiated when the firm fails to pay the contractual coupon, because of the dispersion of debtholders. Then, the firm has to declare bankruptcy once falling into financial distress.

Following Leland (1994), we consider a stock-based definition of bankruptcy whereby equityholders default on their debt obligations the first time equity value is equal to zero. Let x_{M}^b denote the bankruptcy threshold (the superscript "b" stands for bankruptcy). We assume that the bankruptcy value is $(1-\alpha)\Pi(x_{\mathrm{M}}^b)$, i.e., a fraction $(1-\alpha)$ of the unlevered after-tax firm value $\Pi(x_{\mathrm{M}}^b)$ upon bankruptcy. The parameter $\alpha \in (0,1)$ measures the losses in firm value incurred by default costs.

Now, we consider the financing and investment decisions. The investment decision is characterized by an endogenously determined investment threshold x_{M}^{i}. The capital structure decision involves the choice of the coupon level of debt and an endogenous bankruptcy threshold. The coupon level $c_{\mathrm{M}}(x_{\mathrm{M}}^{i})$, which is characterized by a trade-off between the tax benefits and default costs of debt financing, is determined simultaneously with the investment decision. In contrast, the bankruptcy threshold $x_{\mathrm{M}}^{b}(c_{\mathrm{M}})$, which depends on the coupon level, is determined after investment option is exercised. Note that the three endogenous variables in our model (i.e., x_{M}^{i}, $c_{\mathrm{M}}(x_{\mathrm{M}}^{i})$, and $x_{\mathrm{M}}^{b}(c_{\mathrm{M}})$) form a nested structure, and therefore enables us to examine the interaction between financing and investment decisions.

In the following, we first derive the bankruptcy threshold from the values after investment. Then, we determine the coupon level, which depends on investment threshold. Finally, we derive the optimal investment threshold from the values before investment.

3.1.1 Bankruptcy decision

Let the equity value after investment be $E_{\mathrm{M}}(x)$. It must satisfy the following ODE:

$$rE_{\mathrm{M}}(x) = \mu x E_{\mathrm{M}}'(x) + \frac{1}{2}\sigma^2 x^2 E_{\mathrm{M}}''(x) + (1-\tau)x - (1-\tau)c_{\mathrm{M}}. \tag{3.1}$$

In addition, $E_{\mathrm{M}}(x)$ must satisfy the following boundary conditions:

$$\lim_{x\to\infty} \frac{E_{\mathrm{M}}(x)}{x} < \infty, \quad E_{\mathrm{M}}(x_{\mathrm{M}}^{b}) = 0, \quad E_{\mathrm{M}}'(x_{\mathrm{M}}^{b}) = 0. \tag{3.2}$$

Solving the ODE (3.1) with the boundary conditions above, we obtain

$$E_{\mathrm{M}}(x) = \Pi(x) - (1-\tau)\frac{c_{\mathrm{M}}}{r} - \left[\Pi(x_{\mathrm{M}}^{b}) - (1-\tau)\frac{c_{\mathrm{M}}}{r}\right]\left(\frac{x}{x_{\mathrm{M}}^{b}}\right)^{\gamma}, \quad x \geq x_{\mathrm{M}}^{b}, \tag{3.3}$$

where γ is the negative root of the quadratic equation (2.4). That is,

$$\gamma = -\frac{1}{\sigma^2}\left[\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}\right] < 0. \tag{3.4}$$

The optimal bankruptcy threshold is given by

$$x_{\mathrm{M}}^{b} = \frac{\gamma}{\gamma - 1}\frac{r-\mu}{r}c_{\mathrm{M}}. \tag{3.5}$$

Similarly, the debt value after investment can be derived as:

$$D_{\mathrm{M}}(x) = \frac{c_{\mathrm{M}}}{r} - \left[\frac{c_{\mathrm{M}}}{r} - (1-\alpha)\Pi(x_{\mathrm{M}}^{b})\right]\left(\frac{x}{x_{\mathrm{M}}^{b}}\right)^{\gamma}, \quad x \geq x_{\mathrm{M}}^{b}. \tag{3.6}$$

The firm value after investment is the sum of equity value and debt value.

$$V_{\mathrm{M}}(x) = \Pi(x) + \frac{\tau c_{\mathrm{M}}}{r}\left[1 - \left(\frac{x}{x_{\mathrm{M}}^{b}}\right)^{\gamma}\right] - \alpha\Pi(x_{\mathrm{M}}^{b})\left(\frac{x}{x_{\mathrm{M}}^{b}}\right)^{\gamma}, \quad x \geq x_{\mathrm{M}}^{b}. \tag{3.7}$$

This expression is intuitive. It says that the firm value consists of three terms: the unlevered firm value, plus the present value of tax benefits, and minus the present value of default costs.

3.1.2 Coupon level and investment decisions

Before turning to the analysis of coupon level and investment decisions, it is important to make a clear distinction between the *ex ante* equity value and *ex post* equity value. While the *ex post* equity value is given by the present value of the cash flow accruing to equityholders after debt has been issued (see Eq.(3.3)), the *ex ante* equity value is given by the sum of the *ex post* equity value and debt value (see Eq.(3.7)) upon investment. As a result, although equityholders choose the bankruptcy threshold to maximize the *ex post* equity value, they choose the coupon level to maximize the *ex ante* equity value, internalizing both the tax benefits and default costs of debt financing.

By maximizing the firm value $V_{\mathrm{M}}(x_{\mathrm{M}}^i; c_{\mathrm{M}})$ upon investment with c_{M}, we have

$$c_{\mathrm{M}}(x_{\mathrm{M}}^i) = \frac{\gamma - 1}{\gamma} \frac{r}{r - \mu} \frac{x_{\mathrm{M}}^i}{h_{\mathrm{M}}}, \tag{3.8}$$

where

$$h_{\mathrm{M}} = \left[1 - \gamma(1 - \alpha + \frac{\alpha}{\tau})\right]^{-\frac{1}{\gamma}} > 1. \tag{3.9}$$

Note that the coupon c_{M} is a linear function of x_{M}^i, which is endogenously determined later. Combining Eq.(3.8) with Eq.(3.5), we find that $x_{\mathrm{M}}^i / x_{\mathrm{M}}^b = h_{\mathrm{M}} > 1$. In other words, the ratio of investment threshold to bankruptcy threshold is constant.

Substituting c_{M} and x_{M}^b into Eq.(3.7) with $x = x_{\mathrm{M}}^i$, the firm value upon investment can be rewritten as:

$$V_{\mathrm{M}}(x_{\mathrm{M}}^i) = \psi_{\mathrm{M}} \Pi(x_{\mathrm{M}}^i), \tag{3.10}$$

where

$$\psi_{\mathrm{M}} = 1 + \frac{\tau}{1 - \tau} \frac{1}{h_{\mathrm{M}}} > 1. \tag{3.11}$$

Having derived the firm value, we next choose the optimal investment threshold x_{M}^i to maximize the *ex ante* firm value V_{M}^o. Since the investment cost financed by equity is $I - D_{\mathrm{M}}(x_{\mathrm{M}}^i)$, the value-matching condition at the investment threshold is

$$V_{\mathrm{M}}^o(x_{\mathrm{M}}^i) = E_{\mathrm{M}}(x_{\mathrm{M}}^i) - \left[I - D_{\mathrm{M}}(x_{\mathrm{M}}^i)\right] = V_{\mathrm{M}}(x_{\mathrm{M}}^i) - I. \tag{3.12}$$

The associated smooth-pasting condition is

$$V_{\mathrm{M}}^{o\prime}(x_{\mathrm{M}}^i) = V_{\mathrm{M}}'(x_{\mathrm{M}}^i). \tag{3.13}$$

Therefore, the *ex ante* firm value is given by

$$V_{\mathrm{M}}^o(x) = \left[V_{\mathrm{M}}(x_{\mathrm{M}}^i) - I\right] \left(\frac{x}{x_{\mathrm{M}}^i}\right)^{\beta}, \quad x \leq x_{\mathrm{M}}^i. \tag{3.14}$$

The results under exclusive market debt financing are summarized in the following proposition.

Proposition 3.1 (Exclusive market debt financing)
The optimal solution set of investment threshold, bankruptcy threshold, and coupon level is[3]

$$\left(x^i_{\mathrm{M}}, x^b_{\mathrm{M}}, c_{\mathrm{M}}\right) = \left(\frac{x^i_{\mathrm{U}}}{\psi_{\mathrm{M}}}, \frac{x^i_{\mathrm{M}}}{h_{\mathrm{M}}}, \frac{\zeta_{\mathrm{M}}}{\psi_{\mathrm{M}} h_{\mathrm{M}}} I\right), \tag{3.15}$$

where h_{M} and ψ_{M} are defined in Eq.(3.9) *and Eq.*(3.11), *and*

$$\zeta_{\mathrm{M}} = \frac{\gamma - 1}{\gamma} \frac{\beta}{\beta - 1} \frac{r}{1 - \tau} > 0. \tag{3.16}$$

The ex ante firm value is

$$V^o_{\mathrm{M}}(x) = \psi^\beta_{\mathrm{M}} V^o_{\mathrm{U}}(x) = \left(\frac{\psi_{\mathrm{M}} x}{x^i_{\mathrm{U}}}\right)^\beta \frac{I}{\beta - 1}, \quad x \leq x^i_{\mathrm{U}}. \tag{3.17}$$

The leverage upon investment is

$$L_{\mathrm{M}}(x^i_{\mathrm{M}}) = \frac{D_{\mathrm{M}}(x^i_{\mathrm{M}})}{V_{\mathrm{M}}(x^i_{\mathrm{M}})} = \frac{\gamma - 1}{\gamma} \frac{1 - \xi_{\mathrm{M}}}{\psi_{\mathrm{M}} h_{\mathrm{M}}} \frac{1}{1 - \tau}, \tag{3.18}$$

where

$$\xi_{\mathrm{M}} = \left[1 - (1 - \alpha)(1 - \tau)\frac{\gamma}{\gamma - 1}\right] h^\gamma_{\mathrm{M}} \in (0, 1). \tag{3.19}$$

3.2 Exclusive bank financing

In this subsection, we examine the case of exclusive bank debt financing. Since bankruptcy is typically costly, there exist ample opportunities for equityholders and debtholders to have debt renegotiation instead of bankruptcy once the firm falls into financial distress. We assume that the bank may grant state-contingent coupon concessions in costless bilateral renegotiation. Empirical evidences, including Gilson *et al.* (1990), report that banks tend to be more understanding toward firms in financial distress compared to other debtholders.

As before, we solve the decision making problems using backward induction. First, we investigate the renegotiation process by applying Nash bargaining between equityholders and debtholders. Both the reduced level of debt service and the renegotiation threshold are derived. Then, we examine the coupon level and investment decisions.

3.2.1 Renegotiation decision

The first step is to derive the reduced level of debt service and the renegotiation threshold from values after investment. We suppose that debt renegotiation begins once the EBIT process hits an endogenously determined threshold x^s_{B} (the superscript "s" and subscript "B" stand for strategic debt service and bank debt, respectively). During the renegotiation region ($0 \leq x \leq x^s_{\mathrm{B}}$), the contractual coupon c_{B} is reduced to $s_{\mathrm{B}}(x)$, which is derived later. The equityholders continue to operate the firm. We assume that the tax benefits of debt are suspended. That is,

[3]This representation of the optimal solution set follows Shibata and Nishihara (2010), in which agency conflicts are examined.

the debt in the renegotiation region is treated as equity. As soon as the EBIT goes back to the normal region ($x \geq x_B^s$), the contractual coupon c_B and the tax benefits of debt are restored.

Based on the setup above, the firm value satisfies the following ODEs:

$$
\begin{aligned}
rV_B^n(x) &= (1-\tau)x + \tau c_B + \mu x V_B^{n\prime}(x) + \frac{1}{2}\sigma^2 x^2 V_B^{n\prime\prime}(x), \quad && x \geq x_B^s, \\
rV_B^s(x) &= (1-\tau)x + \mu x V_B^{s\prime}(x) + \frac{1}{2}\sigma^2 x^2 V_B^{s\prime\prime}(x), && 0 \leq x \leq x_B^s,
\end{aligned}
\tag{3.20}
$$

where the subscripts "n" and "s" denote the normal region and the renegotiation region with strategic debt service, respectively. The boundary conditions are as follows:

$$
\lim_{x\to\infty} \frac{V_B^n(x)}{x} < \infty, \quad V_B^s(0) = 0, \quad V_B^n(x_B^s) = V_B^s(x_B^s), \quad V_B^{n\prime}(x_B^s) = V_B^{s\prime}(x_B^s). \tag{3.21}
$$

The last condition implies that the first-order derivative of the firm value should be continuous, since renegotiation is reversible (see Dumas (1991)).

Solving the ODEs (3.20) with the boundary conditions above, we obtain the firm value as follows:

$$
\begin{aligned}
V_B^n(x) &= \Pi(x) + \tau\frac{c_B}{r}\left[1 - \frac{\beta}{\beta-\gamma}\right]\left(\frac{x}{x_B^s}\right)^{\gamma}, \quad && x \geq x_B^s, \\
V_B^s(x) &= \Pi(x) + \tau\frac{c_B}{r}\frac{\gamma}{\gamma-\beta}\left(\frac{x}{x_B^s}\right)^{\beta}, && 0 \leq x \leq x_B^s.
\end{aligned}
\tag{3.22}
$$

Now, we describe the renegotiation process. Following Fan and Sundaresan (2000), let $\eta \in [0,1]$ denote the equityholders' bargaining power, and then $1-\eta$ is the debtholders' bargaining power. Let θ be the fraction of $V_B^s(x)$ that equityholders receive from renegotiation. The values upon bankruptcy for equityholders and debtholders are reference points of the bargaining process. The incremental value for the equityholders to participate in debt renegotiation is $\theta V_B^s(x)$, because the equity value is zero upon bankruptcy. On the other hand, the incremental value for the debtholders is $(1-\theta)V_B^s(x) - (1-\alpha)\Pi(x)$, because the reservation value of debtholders is $(1-\alpha)\Pi(x)$ upon bankruptcy. Thus, the Nash bargaining solution is characterized by maximizing

$$
[\theta V_B^s(x)]^{\eta}\left[(1-\theta)V_B^s(x) - (1-\alpha)\Pi(x)\right]^{1-\eta}. \tag{3.23}
$$

After simple calculations, we obtain

$$
\theta = \eta - \eta\frac{(1-\alpha)\Pi(x)}{V_B^s(x)}. \tag{3.24}
$$

Therefore, the equity value and debt value in the renegotiation region ($x \leq x_B^s$) are

$$
\begin{aligned}
E_B^s(x) &= \eta\left[V_B^s(x) - (1-\alpha)\Pi(x)\right] = \eta\left[\alpha\Pi(x) - \tau\frac{c_B}{r}\frac{\gamma}{\beta-\gamma}\left(\frac{x}{x_B^s}\right)^{\beta}\right], \\
D_B^s(x) &= (1-\eta)V_B^s(x) + \eta(1-\alpha)\Pi(x) = (1-\eta\alpha)\Pi(x) + (1-\eta)\tau\frac{c_B}{r}\frac{\gamma}{\gamma-\beta}\left(\frac{x}{x_B^s}\right)^{\beta}.
\end{aligned}
\tag{3.25}
$$

Substituting $E^s_{\mathrm{B}}(x)$ into the ODE:

$$rE^s_{\mathrm{B}}(x) = (1-\tau)x - s(x) + \mu x E^{s\prime}_{\mathrm{B}}(x) + \frac{\sigma^2}{2}x^2 E^{s\prime\prime}_{\mathrm{B}}(x), \quad x \le x^s_{\mathrm{B}}, \tag{3.26}$$

we obtain the reduced level of debt service as:

$$s_{\mathrm{B}}(x) = (1-\eta\alpha)(1-\tau)x, \quad x \le x^s_{\mathrm{B}}. \tag{3.27}$$

We can easily confirm that $s_{\mathrm{B}}(x)$ is lower than the contractual coupon c_{B}. The larger the bargaining power that equityholders have, the greater the concessions that equityholders can receive during debt renegotiation.

Next, we derive the values in the normal region and determine the debt renegotiation threshold. The following value-matching and smooth-pasting conditions describe the equityholders' optimal debt renegotiation decision by choosing the renegotiation threshold x^s_{B}:

$$E^n_{\mathrm{B}}(x^s_{\mathrm{B}}) = E^s_{\mathrm{B}}(x^s_{\mathrm{B}}), \quad E^{n\prime}_{\mathrm{B}}(x^s_{\mathrm{B}}) = E^{s\prime}_{\mathrm{B}}(x^s_{\mathrm{B}}). \tag{3.28}$$

Therefore, we obtain the equity value in the normal region ($x \ge x^s_{\mathrm{B}}$) as:

$$E^n_{\mathrm{B}}(x) = \Pi(x) - \frac{(1-\tau)c_{\mathrm{B}}}{r} - \left[(1-\eta\alpha)\Pi(x^s_{\mathrm{B}}) - \frac{c_{\mathrm{B}}}{r}\left(1-\tau-\tau\frac{\eta\gamma}{\beta-\gamma}\right)\right]\left(\frac{x}{x^s_{\mathrm{B}}}\right)^{\gamma},$$

and the debt renegotiation threshold as:

$$x^s_{\mathrm{B}}(c_{\mathrm{B}}) = \frac{\gamma}{\gamma-1}\frac{1-\tau(1-\eta)}{(1-\eta\alpha)\Pi(1)}\frac{c_{\mathrm{B}}}{r}. \tag{3.29}$$

For the same level of exogenously given coupon, the renegotiation threshold x^s_{B} is higher than the bankruptcy threshold x^b_{M} in Eq.(3.5). As the bankruptcy cost α increases, the difference between the two thresholds widens. If the equityholders' bargaining power η is zero, then the renegotiation threshold x^s_{B} in Eq.(3.29) is equal to the bankruptcy threshold x^b_{M} in Eq.(3.5), provided the same coupon level.

Similarly, the debt value in the normal region ($x \ge x^s_{\mathrm{B}}$) can be derived as:

$$D^n_{\mathrm{B}}(x) = \frac{c_{\mathrm{B}}}{r} - \frac{c_{\mathrm{B}}}{r}\left[\frac{1}{1-\gamma} + \frac{\gamma}{\gamma-1}\frac{\tau(\beta-1)(1-\eta)}{\beta-\gamma}\right]\left(\frac{x}{x^s_{\mathrm{B}}}\right)^{\gamma}. \tag{3.30}$$

3.2.2 Coupon level and investment decisions

As before, the optimal coupon level is chosen to maximize the firm value in the normal region upon investment. By maximizing $V^s_{\mathrm{B}}(x)$ in Eq.(3.22) at $x = x^i_{\mathrm{B}}$, we have

$$c_{\mathrm{B}}(x^i_{\mathrm{B}}) = \frac{\gamma-1}{\gamma}\frac{(1-\eta\alpha)\Pi(1)}{1-\tau(1-\eta)}\frac{r}{h_{\mathrm{B}}}x^i_{\mathrm{B}}, \tag{3.31}$$

where

$$h_{\mathrm{B}} = \left[\frac{\beta}{\beta-\gamma}(1-\gamma)\right]^{-\frac{1}{\gamma}} > 1. \tag{3.32}$$

Note that the coupon c_B is a linear function of x^i_B, which is endogenously determined later. Combining Eq.(3.31) with Eq.(3.29), we find that $x^i_B/x^s_B = h_B > 1$. In other words, the ratio of the investment threshold to the renegotiation threshold is constant. Simple calculations give that $h_B < h_M$. That is, the ratio of the investment threshold to the renegotiation threshold is lower than the ratio of the investment threshold to the bankruptcy threshold. Moreover, compared to the coupon level of market debt c_M in Eq.(3.8), the coupon level of bank debt c_B in Eq.(3.31) depends on the tax rate and bargaining power.

Substituting c_B and x^s_B into Eq.(3.22) with $x = x^i_B$, we obtain the firm value upon investment as:

$$V^n_B(x^i_B) = \psi_B \Pi(x^i_B), \tag{3.33}$$

where

$$\psi_B = 1 + \frac{\tau(1-\eta\alpha)}{1-\tau(1-\eta)} \frac{1}{h_B} > 1. \tag{3.34}$$

Then, we determine the optimal investment threshold x^i_B to maximize the *ex ante* firm value:

$$V^o_B(x) = \left[V^n_B(x^i_B) - I\right] \left(\frac{x}{x^i_B}\right)^{\beta}, \quad x \leq x^i_B. \tag{3.35}$$

The results under exclusive bank debt financing are summarized in the following proposition.

Proposition 3.2 (Exclusive bank debt financing)
The optimal solution set of investment threshold, renegotiation threshold, coupon level is

$$(x^i_B, x^s_B, c_B) = \left(\frac{x^i_U}{\psi_B}, \frac{x^i_B}{h_B}, \frac{\zeta_B}{\psi_B h_B} I\right), \tag{3.36}$$

where h_B and ψ_B are defined in Eq.(3.32) and Eq.(3.34), and

$$\zeta_B = \frac{\gamma-1}{\gamma} \frac{\beta}{\beta-1} \frac{r(1-\eta\alpha)}{1-\tau(1-\eta)} > 0. \tag{3.37}$$

The ex ante firm value is

$$V^o_B(x) = \psi^{\beta}_B V^o_U(x) = \left(\frac{\psi_B x}{x^i_U}\right)^{\beta} \frac{I}{\beta-1}, \quad x \leq x^i_B. \tag{3.38}$$

The leverage upon investment is

$$L_B(x^i_B) = \frac{D_B(x^i_B)}{V_B(x^i_B)} = \frac{\gamma-1}{\gamma} \frac{1-\xi_B}{\psi_B h_B} \frac{1-\eta\alpha}{1-\tau(1-\eta)}, \tag{3.39}$$

where

$$\xi_B = \left[1 - \gamma\tau(1-\eta)\frac{\beta-1}{\beta-\gamma}\right] \frac{h^{\gamma}_B}{1-\gamma} \in (0,1). \tag{3.40}$$

3.3 Comparison between exclusive debt financing and all-equity financing

In this subsection, we compare the results under exclusive debt financing with those under the benchmark (all-equity financing). First, we compare the investment thresholds. Since $x^i_{\mathrm{M}} = x^i_{\mathrm{U}}/\psi_{\mathrm{M}}$, $x^i_{\mathrm{B}} = x^i_{\mathrm{U}}/\psi_{\mathrm{B}}$, and $\psi_{\mathrm{M}} > 1$, $\psi_{\mathrm{B}} > 1$, we obtain the following corollary:

Corollary 3.1 (Investment threshold)
The investment thresholds satisfy the following inequalities:

$$x^i_{\mathrm{M}} < x^i_{\mathrm{U}}, \quad x^i_{\mathrm{B}} < x^i_{\mathrm{U}}. \tag{3.41}$$

The economic interpretation of Corollary 3.1 is that, investment advances with debt financing.

Second, we examine the payoff upon investment. Since $V_{\mathrm{M}}(x^i_{\mathrm{M}}) = \psi_{\mathrm{M}}\Pi(x^i_{\mathrm{M}})$, $V_{\mathrm{B}}(x^i_{\mathrm{B}}) = \psi_{\mathrm{B}}\Pi(x^i_{\mathrm{B}})$, we find that the firm values upon investment are all proportional to the investment threshold. Let $\psi_j x^i_j$, $(j \in \{\mathrm{M}, \mathrm{B}, *\})$ denote the *ex ante* firm value (gross payoff upon investment) in general. The equityholders choose x^i_j to maximize the *ex ante* firm value, which is given by the product of the net payoff upon investment and the investment probability, i.e., $(x/x^i_j)^\beta(\epsilon x^i_j - I)$. Consequently, $\psi_j x^i_j = \beta I/(\beta-1)$.

Corollary 3.2 (Firm value upon investment)
The firm values upon investment are identical:

$$\Pi(x^i_{\mathrm{U}}) = V_{\mathrm{M}}(x^i_{\mathrm{M}}) = V_{\mathrm{B}}(x^i_{\mathrm{B}}) = V_*(x^i_*) = \frac{\beta}{\beta-1}I. \tag{3.42}$$

The economic implication of Corollary 3.2 is that, as long as the firm value upon investment (gross payoff to equityholders) is proportional to investment threshold, the net payoffs upon investment are identical and independent of financing structures.

Combining the results in Corollary 3.1 and Corollary 3.2, we immediately obtain the following corollary.

Corollary 3.3 (Option value of investment)
The option values of investment satisfy the following inequalities:

$$V^o_{\mathrm{M}}(x) > V^o_{\mathrm{U}}(x), \quad V^o_{\mathrm{B}}(x) > V^o_{\mathrm{U}}(x). \tag{3.43}$$

Because the *ex ante* firm value is determined by the ordering of $(1/x^i)^\beta$, the ordering of *ex ante* firm values is the opposite of the ordering of investment thresholds.

Since the comparison between the exclusive market debt financing and exclusive bank debt financing depends on different parameter values, there is no unique dominance between the two exclusive debt financing. However, when the bank has full bargaining power ($\eta = 0$), we have the following proposition.

Proposition 3.3 (Weak firm)
For weak firms where the bank has full bargaining power ($\eta = 0$), bank debt dominates market debt in that $x^i_{\mathrm{B}} < x^i_{\mathrm{M}}$, $V^o_{\mathrm{B}}(x) > V^o_{\mathrm{M}}(x)$. In other words, exclusive bank debt financing is the optimal debt structure.

This result can be confirmed as follows. According to Eq.(3.34), $\psi_{\mathrm{B}} = 1+\tau/[(1-\tau)h_{\mathrm{B}}] > 1+\tau/[(1-\tau)h_{\mathrm{M}}] = \psi_{\mathrm{M}}$, because $h_{\mathrm{B}} < h_{\mathrm{M}}$. The economic interpretation is that, in the case of a weak firm, where the bank has full bargaining power, the renegotiation threshold x^s_{B} in Eq.(3.29) is equal to the bankruptcy threshold x^b_{M} in Eq.(3.5). Since the bank debt dominates market debt from the point view of avoiding costly bankruptcy, exclusive bank debt financing is the optimal debt structure.

When the equityholders have bargaining power ($\eta > 0$), we cannot determine which exclusive debt financing is better. Thus, we need to discuss the optimal mixed debt structure in the next section.

4 Equity and mixed debt financing

In this section, we examine the case of mixed debt financing. The procedures to solve the problem are similar with those in Section 3.2. As before, we solve the decision making problems using backward induction.

4.1 Bankruptcy and renegotiation decisions

The firm value after investment satisfies the same ODE as in (3.20), except that the normal region and the renegotiation region are $x \geq x^s_*$ and $x^b_* \leq x \leq x^s_*$, where the subscript "$*$" corresponds to the expressions with mixed debt financing. The boundary conditions are similar with thoes in (3.21), except that the second one changes to $V^s_*(x^b_*) = (1-\alpha)\Pi(x^b_*)$. Solving the ODEs with the boundary conditions, we obtain the firm value as follows:

$$
\begin{aligned}
V^n_*(x) &= \Pi(x) + \frac{\tau}{r}(c_{\mathrm{B}*}+c_{\mathrm{M}*})\left[1 - \frac{\beta}{\beta-\gamma}\left(\frac{x}{x^s_*}\right)^{\gamma} + \frac{\gamma}{\beta-\gamma}\left(\frac{x^b_*}{x^s_*}\right)^{\beta}\left(\frac{x}{x^b_*}\right)^{\gamma}\right] - \alpha\Pi(x^b_*)\left(\frac{x}{x^b_*}\right)^{\gamma}, \quad x \geq x^s_* \\
V^s_*(x) &= \Pi(x) + \frac{\gamma}{\beta-\gamma}\frac{\tau}{r}(c_{\mathrm{B}*}+c_{\mathrm{M}*})\left[\left(\frac{x^b_*}{x^s_*}\right)^{\beta}\left(\frac{x}{x^b_*}\right)^{\gamma} - \left(\frac{x}{x^s_*}\right)^{\beta}\right] - \alpha\Pi(x^b_*)\left(\frac{x}{x^b_*}\right)^{\gamma}, \quad x^b_* \leq x \leq x^s_*.
\end{aligned} \tag{4.1}
$$

We assume that the bank debt and the market debt have equal priority at the bankruptcy threshold.[4] Then, the market debt value is

$$
D_{\mathrm{M}*}(x) = \frac{c_{\mathrm{M}*}}{r} - \left[\frac{c_{\mathrm{M}*}}{r} - \frac{c_{\mathrm{M}*}}{c_{\mathrm{B}*}+c_{\mathrm{M}*}}(1-\alpha)\Pi(x^b_*)\right]\left(\frac{x}{x^b_*}\right)^{\gamma}, \quad x \geq x^b_*. \tag{4.2}
$$

Now, we describe the renegotiation process. The incremental value for the equityholders to participate in debt renegotiation is $\theta_*\left(V^s_*(x) - D_{\mathrm{M}*}(x)\right)$, because the equityholders should pay market debt coupon even in the renegotiation region. On the other hand, the incremental value for the debtholders is $(1-\theta_*)\left(V^s_*(x) - D_{\mathrm{M}*}(x)\right) - (1-\alpha)\Pi(x)c_{\mathrm{B}*}/(c_{\mathrm{B}*}+c_{\mathrm{M}*})$, because the bank debtholders receive $c_{\mathrm{B}*}/(c_{\mathrm{B}*}+c_{\mathrm{M}*})$ part of the remaining firm value upon bankruptcy. Thus,

[4] A number of papers, including Weiss (1990) and Goldstein *et al.* (2001), report that the priority of claims is frequently violated in bankruptcy. It is typical that all unsecured debt receives the same recovery rate, regardless of the issuance date.

the Nash bargaining solution is characterized by maximizing

$$[\theta_* (V_*^s(x) - D_{M*}(x))]^\eta \left[(1-\theta_*)(V_*^s(x) - D_{M*}(x)) - \frac{c_{B*}}{c_{B*} + c_{M*}}(1-\alpha)\Pi(x)\right]^{1-\eta}. \quad (4.3)$$

After simple calculations, we obtain

$$\theta_* = \eta - \eta \frac{(1-\alpha)\Pi(x)}{V_*^s(x) - D_{M*}(x)}. \quad (4.4)$$

Therefore, we can obtain the equity value $E_*^s(x)$ and bank debt value $D_*^s(x)$ in the renegotiation region $(x_*^b \leq x \leq x_*^s)$.The reduced level of debt service is

$$s_*(x) = \left[1 - \eta + \eta \frac{(1-\alpha)c_{B*}}{c_{B*} + c_{M*}}\right] x + (1-\eta)c_{M*}, \quad x_*^b x \leq x_*^s. \quad (4.5)$$

By maximizing the equity value in the renegotiation region $E_*^s(x; x_*^b)$ with x_*^b, we find that the bankruptcy threshold is determined by the following equation:

$$\tau(c_{B*} + c_{M*}) \left(\frac{x_*^b}{x_*^s}\right)^\beta - \frac{1-\gamma}{\gamma} r\Pi(x_*^s) \left[\alpha + \frac{(1-\alpha)c_{M*}}{c_{B*} + c_{M*}}\right] - c_{M*} = 0 \quad (4.6)$$

Also, with similar boundary conditions in Eq.(3.28), we obtain the equity value and bank debt value in the normal region $(x \geq x_*^s)$.The renegotiation threshold is determined by the following equation:

$$\frac{\eta\tau(c_{B*} + c_{M*})}{\beta - \gamma} \left[\beta \left(\frac{x_*^b}{x_*^s}\right)^{\beta-\gamma} - \gamma\right] + \frac{1-\gamma}{\gamma} r\Pi(x_*^s) \left[1 - \eta + \frac{\eta(1-\alpha)c_{B*}}{c_{B*} + c_{M*}}\right] + (1-\tau)(c_{B*} + c_{M*}) - \eta c_{M*} = 0. \quad (4.7)$$

4.2 Coupon level and investment decisions

The optimal coupons of bank debt and market debt are obtained by maximizing $V_*^n(x_*^i; c_{B*}, c_{M*})$ with c_{B*} and c_{M*}), respectively. Since the expressions are a little long, we omit the two equations od optimization here. The investment threshold is obtained by maximizing

$$V_*^o(x) = [V_*^n(x_*^i) - I] \left(\frac{x}{x_*^i}\right)^\beta. \quad (4.8)$$

After simple calculations, we find that the investment threshold is determined by the following equation:

$$(\beta - 1)\Pi(x_*^i) + \frac{\tau}{r}(c_{B*} + c_{M*}) \left[\beta \left[1 - \left(\frac{x_*^i}{x_*^s}\right)^\gamma\right] + \gamma \left(\frac{x_*^b}{x_*^s}\right)^\beta \left(\frac{x_*^i}{x_*^b}\right)^\gamma\right] - \alpha(\beta - \gamma)\Pi(x_*^b) \left(\frac{x_*^i}{x_*^b}\right)^\gamma = \beta I. \quad (4.9)$$

Since the equations above are all nonlinear in the thresholds, analytical solutions in closed forms are impossible. In the next section, we calibrate the model to analyze the characteristics of the solutions and provide several empirical predictions.

5 Comparison among debt structures

The basic parameters are set as follows: $\mu = 0.01$, $\sigma = 0.25$, $r = 0.06$, $\tau = 0.4$, $\alpha = 0.4$, $\eta = 1$, $I = 10$, $x = 1$. The growth rate $\mu = 0.01$ and volatility $\sigma = 0.25$ of the EBIT are selected to match the data of an average Standard and Poor's (S&P) 500 firms (see Strebulaev (2007)). The discount rate $r = 0.06$ is taken from the yield curve on Treasury bonds. The tax rate $\tau = 0.4$ follows the estimation by Kemsley and Nissim (2002). The parameter of proportional bankruptcy cost $\alpha = 0.4$ is chosen to be consistent with Gilson (1997), who report that default costs are equal to 0.365 and 0.455 for the median firm in his samples.

Figure 1 plots the investment threshold and the *ex ante* firm value with positive equityholders' bargaining power. We find that the investment threshold is the lowest and the *ex ante* firm value is the largest under mixed debt structure. Therefore, mixed debt structure is the optimal structure when $\eta > 0$. Under exclusive market debt structure, the investment threshold and the *ex ante* firm value are certainly independent of the bargaining power. Under exclusive bank debt structure and mixed debt structure, the investment threshold increases and the *ex ante* firm value decreases with the equityholders' bargaining power. In other words, stronger equityholders' bargaining power reduces the *ex ante* firm value and discourages growth option exercising even under mixed debt structure. This result extends that in Sundaresan and Wang (2007a), which examined exclusive bank debt structure only.

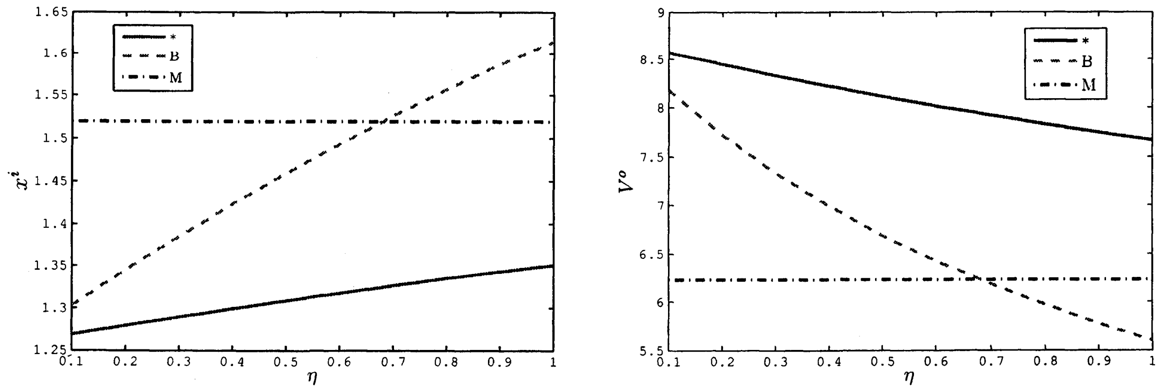

Figure 1: Investment threshold and *ex ante* firm value with positive bargaining power.

Figure 2 displays that the optimal market debt ratio under mixed debt structure. The market debt ratio increases with the equityholders' bargaining power. If we consider the equityholders' bargaining power as a proxy for firm size and age, our results are consistent with the empirical findings in Houston and James (1996), Johnson (1997), Krishnaswami *et al.* (1999),and Denis and Mihov (2003), who find that the percentage of market debt in total debt is increasing in firm size and age.

According to our computation, the results above are robust across a wide range of parameter values. Therefore, we summarize the results in the following proposition.

Proposition 5.1 (Strong firm)

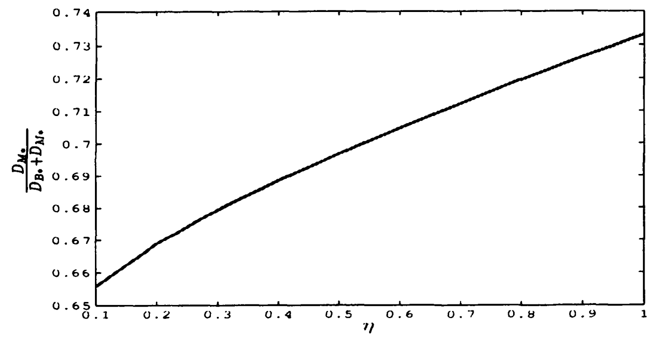

Figure 2: Market debt ratio with positive bargaining power.

For strong firms where equityholders have bargaining power ($\eta > 0$), mixed debt structure is the optimal debt structure. Moreover, the market debt ratio increases with the equityholders' bargaining power.

6 Conclusions

In this paper, we examined firm's financing and investment decisions under different debt structures. Our results demonstrate that: (i) For strong (i.e., large/mature) firms where equityholders have bargaining power, mixed debt structure is optimal, because investment occurs the earliest and the *ex ante* firm value is the largest. The ratio of market debt to the total mixed debt increases with the equityholders' bargaining power. (ii) For weak (i.e., small/emergent) firms where the bank has full bargaining power, exclusive bank debt structure is optimal, since bank debt dominates market debt. The results that the optimal debt structure depends on firms' characteristics are consistent with the empirical finding in Blackwell and Kidwell (1988), which report that while small firms issue privately placed debt almost exclusively, large firms are more likely to issue market debt.

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