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On A Hybrid Asymptotic Expansion Method [1]

Akihiko Takahashi and Kohta Takehara [2]
Graduate School of Economics, the University of Tokyo

Abstract

This paper explains the methodology called 'a hybrid asymptotic expansion technique' proposed by Takahashi and Takehara [4] in much simpler setting than in the original paper. To obtain accurate approximation formulas in closed form for option prices or risk sensitivities, the method can be applied under a broad class of models appearing in finance such as: stochastic volatility models, cross-currency (longterm-)Libor market models, models with a certain class of jumps.

1 Introduction

This paper explains a 'hybrid' scheme with an symptotic expnasion, developed by [4], under a much simeper setting than in the original paper without referring to regorous mathematical arguments. For details in the general setting, see Kunitmo and Takahashi[3], Takahashi and Takehara[4] and Takahashi, Takehara and Toda[5].

In this scheme, the option price will be derived via Fourier inversion of the characteristic function(henceforth sometimes called ch.f.) of the log-forward price of the terminal value of the underlying asset's price. Since in most of important applications in finance the underlying model is too complicated to obtain the closed-form solution of the ch.f., we approximate it with an asymptotic expansion technique. Moreover, in order to increase accuracy of our method, a certain change of the probability measure and a transformation of variable will be also applied, those are reasons why the method is called 'hybrid'. Finally, the asymptotic expansion will be used as a control variable in Monte Carlo simulations to accelerate their convergence.

2 A Hybrid Asymptotic Expansion Method

2.1 A Pricing Problem

Let (W, P) be a one-dimensional Wiener space. Hereafter P is considered as a risk-neutral equivalent martingale measure and a risk-free

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[2]Research Fellow of the Japan Society for the Promotion of Science. E-mail address: fin.tk.house@gmail.com

interest rate is set to be zero for simplicity. Then, let also assume that the underlying economy has only a ($\mathbf{R}_+$-valued)single risky asset $S = \{S_t; 0 \leq t\}$ satisfying

$$S_t = S_0 + \int_0^t S_{s-}\tilde{\sigma}(\omega, s)dW_s + \int_0^t S_{s-}d\tilde{A}_s \tag{1}$$

where $\tilde{\sigma}$: $\Omega \times \mathbf{R} \mapsto \mathbf{R}$ satisfies some regularity conditions; $\tilde{A} = \{\tilde{A}_t; 0 \leq t\}$ is some (possibly jumping) martingale independent of W. Then, We will consider the following pricing problem of a plain-vanilla call option;

$$V(0; K, T) = \mathbf{E}[(S_T - K)_+] \tag{2}$$

where $x_+ = \max(x, 0)$ and $\mathbf{E}[\ \cdot\ ]$ is an expectation operator under the probability measure P.

With a log-price of S_T, $s_t := \ln(\frac{S_t}{S_0})$, (2) can be rewritten as:

$$V(0; K, T) \quad = \quad S_0\, \mathbf{E}^P\left[(e^{s_T} - e^k)^+\right]$$

where $k := \ln(\frac{K}{S_0})$ denotes a log-strike rate. Here we note that $e^{s_t} = S_t$ is a martingale under the pricing measure.

Carr and Madan [1] proposed an expression of option prices alternative to (2) as some Fourier inversion of the characteristic function of the logarithm of the underlying asset.

Proposition 1 *Let $\Phi^P(u)$ denote a characteristic function of s_T under P. Then, $V(0; K, T)$ is given by:*

$$V(0; K, T) = \mathbf{\Psi}(\Phi^P; S_0, K, T) \tag{3}$$

where

$$\mathbf{\Psi}(\Phi; S, K, T) \quad := \quad S\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\gamma(u; \Phi)du + (S - K)_+, \tag{4}$$

$$\gamma(u; \Phi) \quad := \quad \frac{\Phi(u - i) - 1}{iu(1 + iu)} \quad \textit{and} \quad i := \sqrt{-1}. \tag{5}$$

Then, we need to know the characteristic function of s_T under the measure P for pricing the option. In particular, in our setting s_t satisfies

$$s_t = Z_t + A_t \tag{6}$$

where $Z = \{Z_t; 0 \leq t\}$ is an exponential martingele given by

$$Z_t = -\frac{1}{2}\int_0^t \tilde{\sigma}^2(\omega, s)ds + \int_0^t \tilde{\sigma}(\omega, s)dW_s \tag{7}$$

and $A = \{A_t; 0 \leq t\}$ is a exponential martingale obtained by appling Itô's formula to $s_t = \ln(S_t/S_0)$, which is independent of W due to the asssumption on $\tilde{A}$.

Further, we assume that the characteristic function of $A(t)$ is known in closed-form. e.g. $A(t)$ is a compound Poisson process, a variance gamma process, an inverse Gaussian process, a CGMY model or a Lévy process appearing in the Stochastic Skew Model(Carr and Wu [2]).

2.2 A Transformation of the Underlying Stochastic Differential Equation

Note that, due to independence of Z and A, $\Phi^P(u)$ can be decomposed as;

$$\Phi^P(u) = \Phi^P_Z(u)\Phi^P_A(u) \tag{8}$$

where $\Phi^P_Z(u)$ and $\Phi^P_A(u)$ denote the characteristic functions of Z_T and A_T under P, respectively.

For evaluation of the option, an explicit expression of $\Phi^P(u)$ is necessary. However, in most cases of in practical application, the process Z_t is too complicated to obtain the analytical expression of $\Phi^P_Z(u)$ while that of $\Phi^P_A(u)$ is assumed to be known. Then, later we will suggest to utilize the asymptotic expansion for the approximation of $\Phi^P_Z(u)$.

In (7), Z_t, the key process for evaluation of the option, has a nonzero drift. Thus, unless we provide the approximation which has not any error in the drift term, even the first moment(i.e. the expectation value) of that approximation will not match the target's. Contrarily, if we can eliminate its drift term by some means, that is the objective process will be a martingale, its first moment can be much easily kept by using a martingale process as an approximation. In this light, here we consider a certain change of measures so that the main objective process of our expansion will be martingale.

For a fixed u(an argument of $\Phi^P_Z(u)$) we define a new probability measure Q_u on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodym derivative of

$$\frac{dQ_u}{dP} = \exp\left(-\frac{1}{2}\int_0^T ||\lambda_u(s)||^2 ds - \int_0^T \lambda_u^{'}(s)dW_s\right) \tag{9}$$

where

$$\lambda_u(t) := \left((-iu) + i\sqrt{u^2 + iu}\right)\tilde{\sigma}(\omega, t) = \tilde{h}(u)\tilde{\sigma}(\omega, t)$$

and $\tilde{h}(u) := (-iu) + i\sqrt{u^2 + iu}$.

Then $\Phi_Z^P(u)$, the characteristic function of Z_T under the measure P, is expressed as that of another random variable $\hat{Z}_T$ under Q_u with a transformation of variable $h(\cdot)$:

$$\begin{aligned} \Phi_Z^P(u) &= \mathbf{E}^P\left[\exp\left(iuZ_T\right)\right] \\ &= \mathbf{E}^{Q_u}\left[\exp\left(ih(u)\int_0^T \tilde{\sigma}'(\omega,s)dW_s^{Q_u}\right)\right] \\ &=: \Phi_{\hat{Z}}^{Q_u}(h(u)) \end{aligned} \tag{10}$$

where $\mathbf{E}^{Q_u}[\cdot]$ is an expectation operator under Q_u; $W_t^{Q_u} := W_t + \int_0^t \lambda_u'(s)ds$ is now a Wiener process under that measure; $\Phi_{\hat{Z}}^{Q_u}(v)$ denotes the characteristic function of $\hat{Z}_T := \int_0^T \tilde{\sigma}'(\omega,s)dW_s^{Q_u}$ under Q_u and $h(u) := \sqrt{u^2+iu}$.

Now, we have the martingale objective process for the approximation. Then, in the following, we will apply the asymptotic expansion method to the process of the new underlying variable, $\hat{Z}$, under Q_u.

2.3 Approximating the Characteristic Function by an Asymptotic Expansion

Here, to fit the framework of the asymptotic expansion, the processes of $s_t^{(\epsilon)}$ is redefined under the measure Q_u with a parameter ϵ as follows;

$$S_t^{(\epsilon)} = S_0 + \epsilon\int_0^t S_{s-}^{(\epsilon)}\sigma(\epsilon,\omega,s)dW_s + \int_0^t S_{s-}^{(\epsilon)}d\tilde{A}_s \tag{11}$$

where $\epsilon \in (0,1]$ is a parameter for an expansion and σ satisfies $\epsilon\sigma(\epsilon,\omega,t) = \tilde{\sigma}(\omega,t)$. Further we assume that $\sigma(0,\omega,t)$ does not depend on ω.

Then $\hat{Z}_t^{(\epsilon)}$, the analogy of $\hat{Z}_t$, is given by

$$\hat{Z}_t^{(\epsilon)} = \epsilon\int_0^t \sigma(\epsilon,\omega,s)dW_s^{Q_u} \tag{12}$$

Then, followign tha way given by related papers such as Kunitomo and Takahashi [3], we can derive the following asymptotic expansion:

Proposition 2 *The asymptotic expansion of $G_{\hat{Z}}^{(\epsilon)} = \frac{1}{\epsilon}\hat{Z}_T^{(\epsilon)}$ up to ϵ^2 is expressed as follows:*

$$G_{\hat{Z}}^{(\epsilon)} = \hat{G}_T^{Q_u,\langle 1\rangle} + \frac{\epsilon}{2!}\hat{G}_T^{Q_u,\langle 2\rangle} + \frac{\epsilon^2}{3!}\hat{G}_T^{Q_u,\langle 3\rangle} + o(\epsilon^2) \tag{13}$$

where $\hat{G}_T^{Q_u,\langle k\rangle} := \frac{\partial^k \hat{Z}^{(\epsilon)}(T)}{\partial\epsilon^k}|_{\epsilon=0}$, $k = 1,2,3$.

Remark 1 *$\hat{G}_T^{Q_u,(k)}$ for any k is expressed as a certain (iterated) Itô integral. Since (iterated) Itô integrals always have zero means, the martingale property of $G_{\hat{Z}}^{(\epsilon)}$ (and hence $\hat{Z}^{(\epsilon)}(t)$) is kept at any order of this expansion. Especially, the first-order term $\hat{G}_T^{Q_u,(1)}$ follows a normal distribution with mean 0 and variance Σ:*

$$\Sigma := \int_0^{T_{N+1}} \|\sigma(0,\omega,s)\|^2 \, ds \tag{14}$$

(by the assumption, $\sigma(0,\omega,t)$ is deterministic function of t). Here it is assumed that $\Sigma > 0$.

Then, by the standard procedures of the asymptotic expasnion method given by [3] or [5], the desired characteristic function can be approximated with the following theorem.

Theorem 1 *An asymptotic expansion of $\Phi_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(v)$, the characteristic function of $G_{\hat{Z}}^{(\epsilon)}$ under Q_u, is given by*

$$\Phi_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(v) = \left[1 + \sum_{j=2}^{6} D_j^{Q_u,(\epsilon)}(iv)^j\right] \Phi_{0,\Sigma}(v) + o(\epsilon^2) \tag{15}$$

where $\Phi_{\mu,\Sigma}(v) := e^{i\mu v - \frac{\Sigma}{2}v^2}$.

$D_2^{Q_u,(\epsilon)}, D_3^{Q_u,(\epsilon)}, D_4^{Q_u,(\epsilon)}, D_5^{Q_u,(\epsilon)}$ and $D_6^{Q_u,(\epsilon)}$ are constants for pre-specified ϵ and u. Each subscript corresponds to the order of (iv) in the equation (15).

For details, see [4] and [5].

Finally, we provide an approximation formula for valuation of European call options written on $S_T^{(\epsilon)}$ by direct application of Theorem 1 to Proposition 1.

Theorem 2 *Let $\hat{V}(0;K,T)$ be an approximated value of $V(0;K,T)$ which denotes the exact value of the option with maturity T and strike rate K. Then, $\hat{V}(0;K,T)$ is given by:*

$$\hat{V}(0;K,T) \;\; := \;\; \Psi(\hat{\Phi}^{(\epsilon)}; S_0, K, T) \tag{16}$$

where the pricing functional $\Psi(\,\cdot\,; S, K, T)$ is given in (4), $\hat{\Phi}^{(\epsilon)}(u) := \hat{\Phi}_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(\epsilon h(u)) \times \Phi_A^P(u)$, and $k := ln(\frac{K}{S_0})$. Here, $\hat{\Phi}_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(v)$ is defined as;

$$\hat{\Phi}_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(v) = \left[1 + \sum_{j=2}^{6} D_j^{Q_u,(\epsilon)}(iv)^j\right] \times \Phi_{0,\Sigma}(v)$$

where $D_2^{Q_u,(\epsilon)}, D_3^{Q_u,(\epsilon)}, D_4^{Q_u,(\epsilon)}, D_5^{Q_u,(\epsilon)}$ *and* $D_6^{Q_u,(\epsilon)}$ *are the coefficients in Theorem 1.*

Remark 2 *Note that since* $h(-i) = 0$ *and* A *is assumed to be an exponential martingale,* $\mathbf{E}^P[e^{s_T^{(\epsilon)}}] = \Phi^{P,(\epsilon)}(u)$ *is approximated by* $\hat{\Phi}^{(\epsilon)}(-i) = \hat{\Phi}_{G_{\hat{Z}}}^{Q_{-i},(\epsilon)}(\epsilon h(-i)) \times \Phi_A^P(-i) = 1$*, which means that in our approximation the exponential-martingale property of* $s_T^{(\epsilon)}$ *is kept.*

Especially, when $A \equiv 0$ *the first-order approximation of the option price coincides* $BS(\Sigma^{\frac{1}{2}}; S_0, K, T)$ *which is the Black-Scholes price under the case where the stochastic interest rates and the stochastic volatility would be replaced by (their limiting-)deterministic processes:*

$$BS(\sigma; S, K, T) := SN(d_+) - KN(d_-) \tag{17}$$

where

$$d_\pm := \frac{\ln(S/K) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad N(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.$$

Moreover, in this case($A \equiv 0$*), the pricing functional can be modified so that the numerical inversion is stabilized as follows;*

$$V(0; K, T) = \tilde{\Psi}(\Phi_T^P; S_0, K, T) \tag{18}$$

where

$$\tilde{\Psi}(\Phi; S, K, T) \;:=\; S\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\Big(\gamma(u;\Phi) - \gamma(u;\Phi_{BS})\Big)du + BS(\Sigma^{\frac{1}{2}}; S, K, T),$$

and $\Phi_{BS}(u)$ *is the first-order-approximated characteristic function, or equivalently that of the (hypothetical)Gaussian underlying log-forward forex;*

$$\Phi_{BS}(u) := \Phi_{0,\Sigma}(h(u)) = \Phi_{-\frac{1}{2}\Sigma,\Sigma}(u).$$

Remark 3 *Using these approximation formulas, we can also provide analytical approximations of Greeks of the option, sensitivities of the option price to the factors. Note that our approximation for the underlying characteristic function does not depend upon the initial value of the spot price. Thus in particular,* Δ *and* Γ*, the first and second derivatives of the option value with respect to* S_0 *respectively, can be explicitly approximated with ease. For simplicity here we again assume* $A \equiv 0$*. Then* $\hat{\Delta}$ *and* $\hat{\Gamma}$*, the approximations of* Δ *and* Γ *respectively, are given by*

$$\hat{\Delta} \;:=\; \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)}) - \gamma(u;\Phi_{BS})\right)du\right.$$

$$
\begin{aligned}
& -\frac{1}{2\pi}\int_{-\infty}^{\infty}(-iu)e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)})-\gamma(u;\Phi_{BS})\right)du\Bigg\}+\Delta_{BS},\\
\hat{\Gamma} \;:=\; & -\frac{1}{S_0}\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}(-iu)e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)})-\gamma(u;\Phi_{BS})\right)du\right.\\
& -\frac{1}{2\pi}\int_{-\infty}^{\infty}(-iu)^2e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)})-\gamma(u;\Phi_{BS})\right)du\Bigg\}+\Gamma_{BS},
\end{aligned}
$$

where Δ_{BS} *and* Γ_{BS} *are the risk sensitivities of the Black-Scholes price* $BS(\Sigma^{\frac{1}{2}};S_0,K,T)$ *given by*

$$
\Delta_{BS}=N'(d_+) \quad and \quad \Gamma_{BS}=\frac{1}{S_0\sqrt{\Sigma T}}N'(d_+).
$$

For other risk parameters such as Θ, *sensitivities of the option price with respect to* t *respectively, their approximations are given in easy ways such as the difference quotient method, which needs few seconds for calculation with our closed-form formula and has satisfactory accuracies.*

3 A Characteristic-function-based Monte Carlo Simulation with the Asymptotic Expansion

Here we will introduce a Monte Carlo (henceforth sometimes called M.C.) simulation scheme which incorporates the analytically obtained characteristic function. Further, with the asymptotic expansion as a control variable, the variance of this characteristic-function-based(ch.f.-based) M.C. is reduced.

In a usual M.C. procedure, we discretize the stochastic differential equations (6) and (7), and generate $\{s^j\}_{j=1}^M$, M samples of $s_T^{(\epsilon)}$. Then the approximation for the option value, the discounted average of terminal payoffs, is obtained by;

$$
\hat{V}_{MC}^{payoff}(0,M;K,T):=\frac{1}{M}\sum_{j=1}^{M}(S_0e^{s^j}-K)^+. \tag{1}
$$

On the other hand, via the pricing formula (3) in Proposition 1, the option price can be expressed with the pricing functional $\mathbf{\Psi}(\,\cdot\,;S,K,T)$ substituted the characteristic function of the underlying log-process into:

$$
\begin{aligned}
V(0;K,T) &= \mathbf{\Psi}(\Phi^{P,(\epsilon)};S_0,K,T)\\
\text{where}\quad \mathbf{\Psi}(\Phi;S,K,T) &= S\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iuk}\gamma(u;\Phi)du+(S-K)^+.
\end{aligned}
$$

Since $\Phi^{P,(\epsilon)}(u)$ is defined by $\mathbf{E}^P\left[e^{ius_T^{(\epsilon)}}\right] = \mathbf{E}^P\left[e^{iuZ_T^{(\epsilon)}}\right] \times \mathbf{E}^P\left[e^{iuA_T}\right]$, the alternative approximation with M.C. can be constructed;

$$\hat{V}_{MC}^{chf}(0, M; K, T) := \Psi(\hat{\Phi}_{MC}^P(\,\cdot\,; M); S_0, K, T) \tag{2}$$

$$\hat{\Phi}_{MC}^P(u; M) = \hat{\Phi}_{Z,MC}^P(u; M) \times \Phi_A^P(u) := \left(\frac{1}{M}\sum_{j=1}^{M} e^{iuZ^j}\right)\Phi_A^P(u) \tag{3}$$

where $\{Z^j\}_{j=1}^M$ are samples of $Z_T^{(\epsilon)}$. Here it is stressed that in this approximation there does not exist any error caused by M.C. for the (jump or continuous) part A.

Further, this ch.f.-based scheme can be much refined through the better estimation for $\Phi_Z^{P,(\epsilon)}(u)$ by M.C., achieved with our asymptotic expansion of the first order. Since $\Phi_Z^{P,(\epsilon)}(u)$ is expressed as $\Phi_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(\epsilon h(u))$, it is done by the approximation of $\Phi_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(\epsilon h(u))$ with M.C.. In what follows in this section, we abbreviate ϵ(or set $\epsilon = 1$) for simplicity and use the notation $g_1 = \hat{G}_T^{Q_u,(1)}$, the first order coefficient of the expansion (13).

Here, in order to avoid the influence appearing in this variance reduction procedure caused by the variable transformation $h(\cdot)$, we use the following relationship

$$\mathbf{E}^{Q_u}\left[e^{ih(u)g_1}\right] = \exp\left(-\frac{1}{2}iu\Sigma\right)\mathbf{E}^{Q_u}\left[e^{iug_1}\right], \tag{4}$$

i.e. $\Phi_{g_1}^{Q_u}(h(u)) = \exp\left(-\frac{1}{2}iu\Sigma\right) \times \Phi_{g_1}^{Q_u}(u)$. $\Phi_{g_1}^{Q_u}(v)$ is the characteristic function of g_1, which is equivalent to $\hat{\Phi}_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(v)$ in Theorem 1 if the expansion were made only up to the first order. This equation can be easily checked with recalling $\Phi_{g_1}^{Q_u}(v) = \Phi_{0,\Sigma}(v) = \exp(-\frac{\Sigma}{2}v^2)$.

Thus on the one hand, the closed-form characteristic function of g_1 evaluated at $v = h(u)$ is given by

$$\Phi_{g_1}^{Q_u}(h(u)) = \exp\left(-\frac{1}{2}iu\Sigma\right)\Phi_{0,\Sigma}(u). \tag{5}$$

But on the other hand, generating samples of g_1 following $N(0,\Sigma)$, $\{g^j\}_{j=1}^M$, we can further approximate the right hand side of (4) by

$$\hat{\Phi}_{g_1,MC}^{Q_u}(u; M) := \exp\left(-\frac{1}{2}iu\Sigma\right)\frac{1}{M}\sum_{j=1}^{M}\left(e^{iug^j}\right). \tag{6}$$

Note that because only the distribution of g_1 matters here, we can simulate samples of $\tilde{g}_1 := \int_0^T \sigma(0,\omega,s)dW_s$ following $N(0,\Sigma)$ under P instead of those of g_1, not under the measure Q_u but under P as well as other random variables simulated for (3).

Using two functions in (5) and (6), which both are the first-order approximations for $\Phi_Z^{Q_u,(\epsilon)}(h(u))$, define two following estimators for the option price.

$$\hat{V}_{ana}^{AE}(0;K.T) := \Psi\left(\Phi_{g_1}^{Q_u}(h(\cdot)) \times \Phi_A^P; S_0, K, T\right) \quad (7)$$

$$\hat{V}_{MC}^{AE}(0,M;K,T) := \Psi\left(\hat{\Phi}_{g_1,MC}^{Q_u}(\,\cdot\,;M) \times \Phi_A^P; S_0, K, T\right) \quad (8)$$

Finally, using $\Phi_{g_1}^{Q_u}(h(u))$ as a control variable, we can construct the more sophisticated estimator $\hat{V}^{CV}(0,M;K,T)$ for the option price $V(0;K,T)$ as

$$\begin{aligned}\hat{V}^{CV}(0,M;K,T) &:= \hat{V}_{MC}^{chf}(0,M;K,T) + \left(\hat{V}_{ana}^{AE}(0;K,T) - \hat{V}_{MC}^{AE}(0,M;K,T)\right) \quad (9)\\ &= \Psi\left(\left\{\hat{\Phi}_{Z,MC}^P(\,\cdot\,;M) + \left(\Phi_{g_1}^{Q_u}(h(\cdot)) - \hat{\Phi}_{g_1,MC}^{Q_u}(\,\cdot\,;M)\right)\right\} \times \Phi_A^P; S_0, K, T\right)\end{aligned}$$

where $T = T_{N+1}$ and

$$\begin{aligned}\hat{\Phi}_{Z,MC}^P(u;M) &= \frac{1}{M}\sum_{j=1}^{M} e^{iuZ^j},\\ \Phi_{g_1}^{Q_u}(h(u)) &= \exp\left(-\frac{1}{2}iu\Sigma\right) \times \Phi_{0,\Sigma}(u),\\ \hat{\Phi}_{g_1,MC}^{Q_u}(u;M) &= \exp\left(-\frac{1}{2}iu\Sigma\right) \times \frac{1}{M}\sum_{j=1}^{M}\left(e^{iug^j}\right).\end{aligned}$$

Remark 4 *Here we note the following fact.*

$$\begin{aligned}&V(0;K,T) - \hat{V}^{CV}(0,M;K,T)\\ =\ &\left(V(0;K,T) - \hat{V}_{MC}^{chf}(0,M;K,T)\right) - \left(\hat{V}_{ana}^{AE}(0;K,T) - \hat{V}_{MC}^{AE}(0,M;K,T)\right)\\ =\ &\Psi\left(\left\{\left(\Phi_Z^{P,(\epsilon)} - \hat{\Phi}_{Z,MC}^P(\,\cdot\,;M)\right) - \left(\Phi_{g_1}^{Q_u}(h(\cdot)) - \hat{\Phi}_{g_1,MC}^{Q_u}(h(\cdot);M)\right)\right\} \times \Phi_A^P; S_0, K, T\right)\end{aligned}$$

where $\Phi_Z^{P,(\epsilon)}$ is the exact characteristic function of $Z_T^{(\epsilon)}$. The former in the first parentheses is the exact characteristic function of $Z_T^{(\epsilon)}$ and the latter is its approximation by Monte Carlo simulations. Similarly, the former in the second parentheses is the exact one of g_1, the first-order expansion for $Z_T^{(\epsilon)}$, and the latter is its approximation. Thus, in the case where the first and second term in the braces cancel each other out, the error of our hybrid estimator is expected to be small.

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