Risk-Sensitive Portfolio Optimization and Down-Side Risk Minimization for Hidden Markov Factor Models (Financial Modeling and Analysis)

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Risk-Sensitive Portfolio Optimization and Down-Side Risk Minimization for Hidden Markov Factor Models

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1 Introduction

We consider a market model consisting of one bank account $S^0_t$ and $N$ risky securities $S^1_t, \ldots, S^N_t$. We assume that the mean returns of risky security prices depend nonlinearly upon “hidden economic factors,” which evolve as a continuous-time Markov chain with finite state space. “Hidden” means that the factors are only partially observable through the information of security prices.

Let $V_T(h)$ be an investor’s wealth at time $T$, corresponding to an investment strategy $h = (h_t)_{t \geq 0}$. Set

$$X_T(h) := \log \frac{V_T(h)}{S^0_T}.$$  

For a given level $k \in \mathbb{R}$, we want to minimize a down-side risk probability

$$P\left(\frac{X_T(h)}{T} \leq k\right)$$

over a large time interval $[0, T]$. More specifically, we consider the long-time average of a minimized down-side risk

$$\Pi_1(k) = \lim_{T \to \infty} \frac{1}{T} \inf_h \log P\left(\frac{X_T(h)}{T} \leq k\right),$$

and also the minimized long-time average of a down side risk

$$\Pi_2(k) = \inf_h \lim_{T \to \infty} \frac{1}{T} \log P\left(\frac{X_T(h)}{T} \leq k\right).$$

To treat these problems, we first consider the following risk-sensitive portfolio optimization problems (1) and (2), for a given “risk-averse” parameter $\gamma \in (-\infty, 0)$:

Finite time horizon problem:

$$\inf_h \log E[\exp\{\gamma X_T(h)\}],$$

and its long time average

$$\chi_1(\gamma) = \lim_{T \to \infty} \frac{1}{T} \inf_h \log E[\exp\{\gamma X_T(h)\}].$$

Infinite time horizon problem:
\[ \chi_2(\gamma) = \inf \lim_{h \to \infty} \frac{1}{T} \log E[\exp\{\gamma X_T(h)\}] . \]  

Suppose that we have "solved" the optimization problems (1) and (2). Then, in view of the large deviations principle, we expect that the following duality relation holds:

\[ \Pi_\nu(k) = -\inf_{k' \in (-\infty, k]} \chi_\nu^*(k') , \quad \nu = 1, 2 , \]

where \( \chi_\nu^*(\cdot) \) is the Legendre transform of \( \chi_\nu(\cdot) \):

\[ \chi_\nu^*(k) = \sup_{\gamma \in (-\infty, 0)} \{ k\gamma - \chi_\nu(\gamma) \} , \quad \nu = 1, 2. \]

### 2 The Model

We consider a market model with \( 1+N \) securities \( S^0_t, S^1_t, \ldots, S^N_t, N \in \{1, 2, 3, \ldots\} \), and an economic factor process \( x_t \). We assume that the factor process is a continuous-time Markov chain, whose state space is the unit vectors \( \mathcal{E}_d = \{e_1, e_2, \ldots, e_d\} \subset \mathbb{R}^d, d \in \{2,3,4,\ldots\} \). The bond price \( S^0_t \) and risky stock prices \( S^i_t, i = 1, \ldots, N \), are assumed to have the following dynamics:

\[ 
\begin{align*}
    dS^0_t &= rS^0_t dt , \quad S^0_0 = s^0 , \\
    dS^i_t &= S^i_t \{ g^i_0(x_t) dt + \sum_{j=1}^{N} \sigma^i_j dW^j_t \} , \quad S^i_0 = s^i , \quad i = 1, \ldots, N ,
\end{align*} \]

where \( W_t = (W^j_t)_{j=1,\ldots,N} \) is an \( N \)-dimensional standard Brownian motion independent of \( x_t \), defined on a probability space \((\Omega, \mathcal{F}, P)\). Here we assume that \( r \geq 0 \) is constant, \( g^i_0(\cdot) \) is an \( \mathbb{R}^N \)-valued function defined on \( \mathcal{E}_d \), and \( \sigma = (\sigma^i_j)_{i,j=1,\ldots,N} \) is a nonsingular constant matrix.

We recall that the dynamics of the Markov chain \( x_t \) can be written as

\[
\left\{ \begin{array}{l}
    dx_t = \Lambda^* x_t dt + dM_t , \\
    x_0 = \xi ,
\end{array} \right.
\]

where \( \Lambda = (\lambda_{ij})_{i,j=1,\ldots,d} \) is a Q-matrix, \( M_t \) is a martingale of pure jump type, and \( \xi \) is a random vector taking values in \( \mathcal{E}_d \). We set

\[ \beta^i := P(\xi = e_i) , \quad \beta := (\beta^1, \ldots, \beta^d)^* . \]

It will be convenient to consider the logarithmic prices of \( S^i_t \):

\[ Y^i_t := \log S^i_t - \log s^i_0 , \quad i = 0,1,\ldots,N , \quad Y_t = (Y^1_t, \ldots, Y^N_t)^* . \]

Then, by (3),

\[ Y^0_t = rt , \quad Y_t = \int_0^t g(x_s) ds + \sigma W_t , \]

where

\[ g^i(e) := g^i_0(e) - \frac{1}{2} (\sigma^* e)^{ii} , \quad g(e) := (g^1(e), \ldots, g^N(e))^* , \quad e \in \mathcal{E}_d . \]
We define
\[ \mathcal{F}_{t}^{0} := \sigma(x_{u}, W_{u}; u \leq t) = \sigma(x_{u}, Y_{u}; u \leq t), \]
\[ \mathcal{Y}_{t}^{0} := \sigma(Y_{u}; u \leq t), \]
and \( \mathcal{F}_{t}, \mathcal{Y}_{t} \) as the corresponding right-continuous, complete filtrations augmented by \( P \)-null sets.

Suppose that an investor invests, at time \( t \), a proportion \( h_{t}^{i} \) of his wealth in the \( i \)-th security \( S_{t}^{i}, i = 0, 1, \ldots, N \). Then, under the self-financing condition, the dynamics of the investor's wealth \( V_{t} = V_{t}(h) \) with initial value \( v_{0} \) is given by
\[
\frac{dV_{t}}{V_{t}} = (1 - h_{t} \cdot 1) \frac{dS_{t}^{0}}{S_{t}^{0}} + \sum_{i=1}^{N} h_{t}^{i} \frac{dS_{t}^{i}}{S_{t}^{i}} = \{r + \hat{g}_{0}(x_{t}) \cdot h_{t}\} dt + [\sigma^{*}h_{t}]^{*}dW_{t}, \tag{4}
\]
where \( h_{t} = (h_{t}^{1}, \ldots, h_{t}^{N})^{*}, 1 = (1, \ldots, 1)^{*} \) and
\[
\hat{g}_{0}(e) := g_{0}(e) - r1.
\]

**Definition 2.1.** \( h_{t} = (h_{t}^{1}, \ldots, h_{t}^{N})^{*} \) is said to be an investment strategy if the following conditions are satisfied:

(i) \( (h_{t})_{0 \leq t \leq T} \) is an \( \mathbb{R}^{N} \) valued \( \mathcal{Y}_{t} \)-progressively measurable process,

(ii) \( E \int_{0}^{T} |h_{t}|^{2} dt < \infty. \)

We denote by \( \mathcal{H}(T) \) the totality of all investment strategies.

For simplicity let us assume
\[
\frac{v_{0}}{s^{0}} = 1.
\]

Then, by (4), the process \( X_{t}(h) := \log \frac{V_{t}(h)}{s^{0}} \) has the dynamics
\[
X_{T}(h) = \int_{0}^{T} \left( \hat{g}_{0}(x_{t}) \cdot h_{t} - \frac{1}{2} |\sigma^{*}h_{t}|^{2} \right) dt + \int_{0}^{T} [\sigma^{*}h_{t}]^{*}dW_{t},
\]
for \( h \in \mathcal{H}(T) \).

### 3 The Results

**Assumptions**

(A1) \( \beta^{i} > 0 \) for all \( i \in \{1, \ldots, d\} \).

(A2) The \( N \times (d - 1) \)-matrix \( G \) defined by
\[
G := \left[ g_{0}^{\nu}(e_{i}) - g_{0}^{\nu}(e_{d}) \right]_{1 \leq \nu \leq N, 1 \leq i \leq d-1}
\]
has rank \( d - 1 \). In particular, \( d - 1 \leq N \).

(A3) Irreducibility: \( \forall i, j \exists i_{1}, \ldots, i_{n} \) s.t. \( \lambda_{i_{1}i_{2}} \cdots \lambda_{i_{n}j} \neq 0 \).

(A3)' "S-irreducibility": \( \lambda_{ij} \neq 0 \) for all \( i, j \in \{1, \ldots, d\} \).

Under (some of) these assumptions, we have the following results:
Theorem 1. For any $\gamma \in (-\infty, 0)$ and $T \in (0, \infty)$, there exist a subclass $A(T) \subset \mathcal{H}(T)$ and a strategy $\hat{h}^{(T, \gamma)} = (\hat{h}_t^{(T, \gamma)})_{t \in [0, T]} \in A(T)$ such that
\[
\inf_{h \in A(T)} \log E[\exp\{\gamma X_T(h)\}] = \log E[\exp\{\gamma X_T(\hat{h}^{(T, \gamma)})\}] = \log E[\exp\{\gamma X_T(\hat{h}^{(T, \gamma)})\}].
\]

Theorem 2. For any $\gamma \in (-\infty, 0)$, there exist a subclass $A \subset \mathcal{H}$ and a strategy $\hat{h}^{(\gamma)} = (\hat{h}_t^{(\gamma)})_{t \in [0, \infty)} \in A$ such that
\[
\inf_{h \in A} \lim_{T \to \infty} \frac{1}{T} \log E[\exp\{\gamma X_T(h)\}] = \lim_{T \to \infty} \frac{1}{T} \log E[\exp\{\gamma X_T(\hat{h}^{(\gamma)})\}] = \lim_{T \to \infty} \frac{1}{T} \log E[\exp\{\gamma X_T(\hat{h}^{(\gamma)})\}] = \lim_{T \to \infty} \frac{1}{T} \log E[\exp\{\gamma X_T(\hat{h}^{(\gamma)})\}].
\]

Theorem 3. Set
\[
\chi_1(\gamma) := \lim_{T \to \infty} \frac{1}{T} \inf_{h \in A(T)} \log E[\exp\{\gamma X_T(h)\}],
\]
\[
\chi_2(\gamma) := \inf_{h \in A} \lim_{T \to \infty} \frac{1}{T} \log E[\exp\{\gamma X_T(h)\}].
\]
Then we have
\[\chi_1(\gamma) = \chi_2(\gamma).\]

Theorem 4. $\chi(\gamma) := \chi_1(\gamma) = \chi_2(\gamma)$ is a convex and continuously differentiable function of $\gamma \in (-\infty, 0)$ and it satisfies $\chi'(-\infty) = 0$. In particular, for each $k \in (0, \chi'(0-))$, we can choose a number $\gamma_k \in (-\infty, 0)$ satisfying $\chi'(\gamma_k) = k$.

For $k \in (0, \chi'(0-))$, set $\chi^*(k) := \sup_{\gamma \in (-\infty, 0)} [k \gamma - \chi(\gamma)]$ and let $\gamma_k$ be the number specified in Theorem 4.

Theorem 5. We have
\[
\lim_{T \to \infty} \frac{1}{T} \log P\left(\frac{X_T(\hat{h}^{(T, \gamma_k)})}{T} \leq k\right) = \lim_{T \to \infty} \frac{1}{T} \inf_{h \in A(T)} \log P\left(\frac{X_T(h)}{T} \leq k\right) = -\inf_{k' \in (-\infty, k]} \chi^*(k'),
\]
where $\hat{h}^{(T, \gamma_k)}$ is an optimal strategy from Theorem 1.
We also have
\[
\lim_{T \to \infty} \frac{1}{T} \log P\left(\frac{X_T(\hat{h}^{(\gamma_k)})}{T} \leq k\right) = \inf_{h \in A} \lim_{T \to \infty} \frac{1}{T} \log P\left(\frac{X_T(h)}{T} \leq k\right) = -\inf_{k' \in (-\infty, k]} \chi^*(k'),
\]
where $\hat{h}^{(\gamma_k)}$ is an optimal strategy from Theorem 2.