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Risk-Sensitive Portfolio Optimization and Down-Side Risk Minimization for Hidden Markov Factor Models

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1 Introduction

We consider a market model consisting of one bank account S_t^0 and N risky securities $S_t^1, \ldots, S_t^N$. We assume that the mean returns of risky security prices depend nonlinearly upon "hidden economic factors," which evolve as a continuous-time Markov chain with finite state space. "Hidden" means that the factors are only partially observable through the information of security prices.

Let $V_T(h)$ be an investor's wealth at time T, correponding to an investment strategy $h = (h_t)_{t\geq 0}$. Set

$$X_T(h) := \log \frac{V_T(h)}{S_T^0}.$$

For a given level $k \in \mathbb{R}$, we want to minimize a down-side risk probability

$$P\Big(\frac{X_T(h)}{T} \leq k\Big)$$

over a large time interval $[0, T]$. More specifically, we consider the long-time average of a minimized down-side risk

$$\Pi_1(k) = \varliminf_{T\to\infty} \frac{1}{T} \inf_h \log P\Big(\frac{X_T(h)}{T} \leq k\Big),$$

and also the minimized long-time average of a down side risk

$$\Pi_2(k) = \inf_h \varliminf_{T\to\infty} \frac{1}{T} \log P\Big(\frac{X_T(h)}{T} \leq k\Big).$$

To treat these problems, we first consider the following risk-sensitive portfolio optimization problems (1) and (2), for a given "risk-averse" parameter $\gamma \in (-\infty, 0)$:

Finite time horizon problem:

$$\inf_h \log E[\exp\{\gamma X_T(h)\}], \tag{1}$$

and its long time average

$$\chi_1(\gamma) = \varliminf_{T\to\infty} \frac{1}{T} \inf_h \log E \exp\{\gamma X_T(h)\}].$$

Infinite time horizon problem:

$$\chi_2(\gamma) = \inf_h \varliminf_{T\to\infty} \frac{1}{T} \log E[\exp\{\gamma X_T(h)\}]. \tag{2}$$

Suppose that we have "solved" the optimization problems (1) and (2). Then, in view of the large deviations principle, we expect that the following duality relation holds:

$$\Pi_\nu(k) = -\inf_{k'\in(-\infty,k]} \chi_\nu^*(k'), \qquad \nu = 1,2,$$

where $\chi_\nu^*(\cdot)$ is the Legendre transform of $\chi_\nu(\cdot)$:

$$\chi_\nu^*(k) = \sup_{\gamma\in(-\infty,0)} \{k\gamma - \chi_\nu(\gamma)\}, \qquad \nu = 1,2.$$

2 The Model

We consider a market model with $1+N$ securities $S_t^0, S_t^1, \ldots, S_t^N$, $N \in \{1,2,3,\ldots\}$, and an economic factor process $\mathbf{x}_t$. We assume that the factor process is a continuous-time Markov chain, whose state space is the unit vectors $\mathcal{E}_d = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\} \subset \mathbb{R}^d$, $d \in \{2,3,4,\ldots\}$. The bond price S_t^0 and risky stock prices S_t^i, $i = 1,\ldots,N$, are assumed to have the following dynamics:

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \qquad S_0^0 = s^0, \\ dS_t^i &= S_t^i\{g_0^i(\mathbf{x}_t)dt + \sum_{j=1}^N \sigma_j^i dW_t^j\}, \qquad S_0^i = s^i, \qquad i = 1,\ldots,N, \end{aligned} \tag{3}$$

where $W_t = (W_t^j)_{j=1,\ldots,N}$ is an N-dimensional standard Brownian motion independent of $\mathbf{x}_t$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Here we assume that $r \geq 0$ is constant, $g_0(\cdot) = (g_0^i(\cdot))_{i=1,\ldots,N}$ is an $\mathbb{R}^N$-valued function defined on $\mathcal{E}_d$, and $\sigma = (\sigma_j^i)_{i,j=1,\ldots,N}$ is a nonsingular constant matrix.

We recall that the dynamics of the Markov chain $\mathbf{x}_t$ can be written as

$$\begin{cases} d\mathbf{x}_t = \Lambda^* \mathbf{x}_t dt + dM_t, \\ \mathbf{x}_0 = \xi, \end{cases}$$

where $\Lambda = (\lambda_{ij})_{i,j=1,\ldots,d}$ is a Q-matrix, M_t is a martingale of pure jump type, and ξ is a random vector taking values in $\mathcal{E}_d$. We set

$$\beta^i := P(\xi = \mathbf{e}_i), \qquad \beta := (\beta^1, \ldots, \beta^d)^*.$$

It will be convenient to consider the logarithmic prices of S_t^i:

$$Y_t^i := \log S_t^i - \log s_0^i, \qquad i = 0,1,\ldots,N, \qquad Y_t = (Y_t^1,\ldots,Y_t^N)^*.$$

Then, by (3),

$$Y_t^0 = rt, \qquad Y_t = \int_0^t g(\mathbf{x}_s)ds + \sigma W_t,$$

where

$$g^i(\mathbf{e}) := g_0^i(\mathbf{e}) - \frac{1}{2}(\sigma\sigma^*)^{ii}, \quad g(\mathbf{e}) := (g^1(\mathbf{e}),\ldots,g^N(\mathbf{e}))^*, \quad \mathbf{e} \in \mathcal{E}_d.$$

We define

$$\mathcal{F}_t^0 := \sigma(\mathbf{x}_u, W_u; u \le t) = \sigma(\mathbf{x}_u, Y_u; u \le t),$$
$$\mathcal{Y}_t^0 := \sigma(Y_u; u \le t),$$

and $\mathcal{F}_t$, $\mathcal{Y}_t$ as the corresponding right-continuous, complete filtrations augmented by P-null sets.

Suppose that an investor invests, at time t, a proportion h_t^i of his wealth in the i-th security S_t^i, $i = 0, 1, \dots, N$. Then, under the self-financing condition, the dynamics of the investor's wealth $V_t = V_t(h)$ with initial value v_0 is given by

$$\frac{dV_t}{V_t} = (1 - h_t \cdot \mathbf{1})\frac{dS_t^0}{S_t^0} + \sum_{i=1}^N h_t^i \frac{dS_t^i}{S_t^i} = \{r + \hat{g}_0(\mathbf{x}_t) \cdot h_t\}dt + [\sigma^* h_t]^* dW_t, \tag{4}$$
$$V_0 = v_0,$$

where $h_t = (h_t^1, \dots, h_t^N)^*$, $\mathbf{1} = (1, \dots, 1)^*$ and

$$\hat{g}_0(\mathbf{e}) := g_0(\mathbf{e}) - r\mathbf{1}.$$

Definition 2.1. $h_t = (h_t^1, \dots, h_t^N)^*$ *is said to be an investment strategy if the following conditions are satisfied:*

(i) $(h_t)_{0 \le t \le T}$ *is an* $\mathbb{R}^N$ *valued* $\mathcal{Y}_t$*-progressively measurable process,*

(ii) $E \int_0^T |h_t|^2 dt < \infty$.

We denote by $\mathcal{H}(T)$ *the totality of all investment strategies.*

For simplicity let us assume

$$\frac{v_0}{s^0} = 1.$$

Then, by (4), the process $X_t(h) := \log \frac{V_t(h)}{S_t^0}$ has the dynamics

$$X_T(h) = \int_0^T \left\{\hat{g}_0(\mathbf{x}_t) \cdot h_t - \frac{1}{2}|\sigma^* h_t|^2\right\}dt + \int_0^T [\sigma^* h_t]^* dW_t,$$

for $h \in \mathcal{H}(T)$.

3 The Results

Assumptions

(A1) $\beta^i > 0$ for all $i \in \{1, \dots, d\}$.

(A2) The $N \times (d-1)$-matrix G defined by

$$G := \left[g_0^\nu(\mathbf{e}_i) - g_0^\nu(\mathbf{e}_d)\right]_{1 \le \nu \le N, 1 \le i \le d-1}$$

has rank $d-1$. In particular, $d - 1 \le N$.

(A3) Irreducibility: $\forall i, j\ \exists i_1, \dots, i_n$ s.t. $\lambda_{ii_1}\lambda_{i_1 i_2} \cdots \lambda_{i_n j} \ne 0$.

(A3)' "S-irreducibility": $\lambda_{ij} \ne 0$ for all $i, j \in \{1, \dots, d\}$.

Under (some of) these assumptions, we have the following results:

Theorem 1. *For any $\gamma \in (-\infty, 0)$ and $T \in (0, \infty)$, there exist a subclass $\mathcal{A}(T) \subset \mathcal{H}(T)$ and a strategy $\hat{h}^{(T,\gamma)} = (\hat{h}_t^{(T,\gamma)})_{t\in[0,T]} \in \mathcal{A}(T)$ such that*

$$\inf_{h\in\mathcal{A}(T)} \log E[\exp\{\gamma X_T(h)\}] = \log E[\exp\{\gamma X_T(\hat{h}^{(T,\gamma)})\}].$$

Theorem 2. *For any $\gamma \in (-\infty, 0)$, there exist a subclass $\mathcal{A} \subset \mathcal{H}$ and a strategy $\hat{h}^{(\gamma)} = (\hat{h}_t^{(\gamma)})_{t\in[0,\infty)} \in \mathcal{A}$ such that*

$$\inf_{h\in\mathcal{A}} \varlimsup_{T\to\infty} \frac{1}{T} \log E[\exp\{\gamma X_T(h)\}] = \varlimsup_{T\to\infty} \frac{1}{T} \log E[\exp\{\gamma X_T(\hat{h}^{(\gamma)})\}].$$

Theorem 3. *Set*

$$\chi_1(\gamma) := \varlimsup_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} \log E[\exp\{\gamma X_T(h)\}],$$

$$\chi_2(\gamma) := \inf_{h\in\mathcal{A}} \varlimsup_{T\to\infty} \frac{1}{T} \log E[\exp\{\gamma X_T(h)\}].$$

Then we have

$$\chi_1(\gamma) = \chi_2(\gamma).$$

Theorem 4. *$\chi(\gamma) := \chi_1(\gamma) = \chi_2(\gamma)$ is a convex and continuously differentiable function of $\gamma \in (-\infty, 0)$ and it satisfies $\chi'(-\infty) = 0$. In particular, for each $k \in (0, \chi'(0-))$, we can choose a number $\gamma_k \in (-\infty, 0)$ satisfying $\chi'(\gamma_k) = k$.*

For $k \in (0, \chi'(0-))$, set $\chi^*(k) := \sup_{\gamma\in(-\infty,0)}\{k\gamma - \chi(\gamma)\}$ and let γ_k be the number specified in Theorem 4.

Theorem 5. *We have*

$$\begin{aligned}\varlimsup_{T\to\infty} \frac{1}{T} \log P\Big(\frac{X_T(\hat{h}^{(T,\gamma_k)})}{T} \le k\Big) &= \varlimsup_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} \log P\Big(\frac{X_T(h)}{T} \le k\Big) \\ &= -\inf_{k'\in(-\infty,k]} \chi^*(k'),\end{aligned}$$

where $\hat{h}^{(T,\gamma_k)}$ is an optimal strategy from Theorem 1.
We also have

$$\begin{aligned}\varlimsup_{T\to\infty} \frac{1}{T} \log P\Big(\frac{X_T(\hat{h}^{(\gamma_k)})}{T} \le k\Big) &= \inf_{h\in\mathcal{A}} \varlimsup_{T\to\infty} \frac{1}{T} \log P\Big(\frac{X_T(h)}{T} \le k\Big) \\ &= -\inf_{k'\in(-\infty,k]} \chi^*(k'),\end{aligned}$$

where $\hat{h}^{(\gamma_k)}$ is an optimal strategy from Theorem 2.