数理解析研究所講究録
第 1737 巻 2011 年 133-139

A new look at Gamma function

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The Euler form of the gamma function $\Gamma(x)$ is given by

$$\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt$$

for $x>0$. The Weierstrass form

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\prod_{n=1}^{\infty}(1+\frac{x}{n})e^{-\frac{x}{n}} \tag{1}$$

extend it to $\mathbf{R}\setminus\{0,-1,-2,\cdots\}$, where γ is the Euler constant defined by

$$\gamma=\lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)=0.57721\cdots .$$

It is clear that $\Gamma(1)=\Gamma(2)=1,\ \Gamma'(1)=-\gamma,\ \Gamma'(2)=-\gamma+1$. Denote the unique zero in $(0,\infty)$ of $\Gamma'(x)$ by α. It is known that $\alpha=1.4616\cdots$ and $\Gamma(\alpha)=0.8856\cdots$. We call the inverse function of the restriction of $\Gamma(x)$ to (α,∞) *the principal inverse function* and write Γ^{-1}. $\Gamma^{-1}(x)$ is an increasing and concave function defined on $(\Gamma(\alpha),\infty)$. (1) guarantees that $\Gamma(x)$ has the holomorphic extension which is a meromorphic function with poles at non-positive integers and (3) holds there. This implies that $\Gamma'(z)$ does not vanish on $\mathbf{C}\setminus(-\infty,\alpha]$.

$$\log\Gamma(x) = -\log x-\gamma x+\sum_{n=1}^{\infty}\left(\frac{x}{n}-\log(1+\frac{x}{n})\right), \tag{2}$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma+\sum_{n=0}^{\infty}(\frac{1}{n+1}-\frac{1}{n+x}). \tag{3}$$

Let Π_+ and Π_- be respectively the upper half plane and the lower half plane.

We will show

Theorem 1 The principal inverse $\Gamma^{-1}(x)$ of $\Gamma(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ to $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$, which satisfies

(i) $\Gamma^{-1}(\Pi_+) \subset \Pi_+$ and $\Gamma^{-1}(\Pi_-) \subset \Pi_-$,

(ii) $\Gamma^{-1}(z)$ is univalent,

(iii) $\Gamma(\Gamma^{-1}(z)) = z$ for $z \in \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$.

Let $K(x, y)$ be a *real* continuous function defined on $I \times I$, and suppose $K(x, y) = K(y, x)$. Then $K(x, y)$ is said to be a *positive semidefinite* - abbreviated to *p.s.d.* - kernel function on an interval $I \times I$ if

$$\iint_{I \times I} K(x, y)\phi(x)\phi(y)dxdy \geqq 0 \tag{4}$$

for every real continuous function ϕ with compact support in I.
In this case (4) holds for complex valued functions $\phi(x)$ as well.
It is clear that $K(x, y)$ is p.s.d. if and only if for each n and for all n points $x_i \in I$, the $n \times n$ matrices

$$(K(x_i, x_j))_{i,j=1}^{n}$$

are positive semidefinite matrices. Suppose $K(x, y) \geqq 0$ for every x, y in I. Then $K(x, y)$ is said to be *infinitely divisible* if $K(x, y)^a$ is p.s.d. for every $a > 0$.

$K(x, y)$ is said to be *conditionally (or almost) positive semidefinite* - abbreviated to *c.p.s.d.* - on I if (4) holds for every continuous function

ϕ on I such that the support of ϕ is compact and the integral of ϕ over I vanishes. One can see $K(x,y)$ is c.p.s.d. if and only if

$$\sum_{i,j=1}^{n} K(x_i, x_j) z_i \overline{z_j} \geqq 0 \tag{5}$$

for each n, for all n points $x_i \in I$ and for n complex numbers z_i with $\sum_{i=1}^{n} z_i = 0$.

Let $f(x)$ be a C^1-functions on I. Then the *Löwner kernel* function is defined by

$$K_f(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y} & (x \neq y) \\ f'(x) & (x = y). \end{cases}$$

We make use of the following excellent theorem by Löwner[6] (also see Koranyi[5] and [7]).

Theorem A Let $f(x)$ be a C^1-functions on I. Then the Löwner kernel function $K_f(x,y)$ is p.s.d. if and only if $f(x)$ has a holomorphic extension $f(z)$ to Π_+ and it is a Pick function.

Lemma 2

$$K_1(x,y) := \begin{cases} \frac{\log x - \log y}{x-y} & (x \neq y) \\ \frac{1}{x} & (x = y) \end{cases}$$

is p.s.d. on $(0,\infty) \times (0,\infty)$

Proof. This is wellknown. However we give direct proof.

By the formula

$$\log x = \int_0^\infty \left(\frac{-1}{x+t} + \frac{t}{t^2+1}\right) dt \quad (x > 0),$$

we obtain

$$K_1(x,y) = \int_0^\infty \frac{1}{(x+t)(y+t)} dt$$

for $x, y > 0$.

Suppose the support of $\phi(x)$ is included in $[m, M]$ with $m > 0$. Since the above infinite integral converges uniformly with respect to $x, y \in [m, M]$, we have

$$\int_0^\infty \int_0^\infty K_1(x,y)\phi(x)\phi(y)dxdy =$$

$$\int_m^M \int_m^M \left(\int_0^\infty \frac{1}{(x+t)(y+t)} dt \right) \phi(x)\phi(y)dxdy$$

$$= \int_0^\infty \left(\int_m^M \int_m^M \frac{1}{(x+t)(y+t)} \phi(x)\phi(y)dxdy \right) dt =$$

$$\int_0^\infty \left(\int_m^M \frac{1}{x+t}\phi(x)dx \right)^2 dt \geqq 0. \quad \square$$

Lemma 3 Let $K_2(x,y)$ be the function defined on $(0,\infty) \times (0,\infty)$ by

$$K_2(x,y) := \begin{cases} \frac{\log\Gamma(x) - \log\Gamma(y)}{x-y} & (x \neq y) \\ \frac{\Gamma'(x)}{\Gamma(x)} & (x = y). \end{cases}$$

Then $-K_2(x,y)$ is c.p.s.d. on $(0,\infty)$.

Proof. Suppose the support of $\phi(x)$ is included in $[m, M]$ with $m > 0$ and $\int_m^M \phi(x)dx = 0$. From (2) it follows that $-K_2(x,y) = K_1(x,y) + \gamma - K_g(x,y)$, where K_g is a Löwner kernel function of g defined by

$$g(x) = \sum_{k=1}^{\infty} \left(\frac{x}{k} - \log(1 + \frac{x}{k}) \right).$$

Since $K_1(x,y)$ is p.s.d. and $\int_0^\infty \int_0^\infty \gamma\phi(x)\phi(y)dxdy = 0$, we have only to show $-K_g(x,y)$ is c.p.s.d. Put

$$g_n(x) = \sum_{k=1}^{n} \left(\frac{x}{k} - \log(1 + \frac{x}{k}) \right).$$

Then

$$g_n'(x) = \sum_{k=1}^{n} \frac{x}{k(k+x)}$$

converges uniformly to $\sum_{k=1}^{\infty} \frac{x}{k(k+x)} = g'(x)$ on $[0, M]$. The sequence of Löwner kernel functions $K_{g_n}(x, y)$ converges uniformly to $K_g(x, y)$; indeed,

$$K_{g_n}(x,y) - K_g(x,y) = \begin{cases} \frac{1}{x-y} \int_y^x (g_n'(t) - g'(t))\, dt & (x \neq y) \\ \\ g_n'(x) - g'(x) & (x = y) \end{cases}$$

converges uniformly to 0 on $[0, M] \times [0, M]$. Since

$$-K_{g_n}(x,y) = \sum_{k=1}^{n} \left(-\frac{1}{k} + \frac{1}{k} K_1(1 + \frac{x}{k}, 1 + \frac{y}{k}) \right)$$

is c.p.s.d., so is $-K_g(x, y)$. □

The following is known (p.152 of [7], [8] and [9]).

Lemma 4 Let $K(x, y) > 0$ for $x, y \in I$. If $-K(x, y)$ is c.p.s.d. on $I \times I$, then the reciprocal function $\frac{1}{K(x,y)}$ is infinitely divisible there.

Lemma 5 Let $K_3(x, y)$ be the kernel function defined on $(\alpha, \infty) \times (\alpha, \infty)$ by

$$K_3(x,y) = \begin{cases} \frac{x-y}{\Gamma(x) - \Gamma(y)} & (x \neq y) \\ \\ \frac{1}{\Gamma'(x)} & (x = y). \end{cases}$$

Then $K_3(x, y)$ is p.s.d.

Proof.

$$K_3(x,y) = K_1(\Gamma(x),\Gamma(y)) \cdot \frac{1}{K_2(x,y)}$$

$$K_1(\Gamma(x),\Gamma(y)) = \begin{cases} \frac{\log\Gamma(x)-\log\Gamma(y)}{\Gamma(x)-\Gamma(y)} & (x \neq y) \\ \\ \frac{1}{\Gamma(x)} & (x = y) \end{cases}$$

$$K_2(x,y) := \begin{cases} \frac{\log\Gamma(x)-\log\Gamma(y)}{x-y} & (x \neq y) \\ \\ \frac{\Gamma'(x)}{\Gamma(x)} & (x = y). \end{cases}$$

□

Proof of Theorem1 The Löwner kernel $K_{\Gamma^{-1}}(x,y)$ defined on $(\Gamma(\alpha),\infty)\times(\Gamma(\alpha),\infty)$ by

$$K_{\Gamma^{-1}}(x,y) = \begin{cases} \frac{\Gamma^{-1}(x)-\Gamma^{-1}(y)}{x-y} & (x \neq y) \\ \\ (\Gamma^{-1})'(x) & (x = y) \end{cases}$$

coincides with $K_3(\Gamma^{-1}(x),\Gamma^{-1}(y))$, which is p.s.d. Thus by Theorem A, $\Gamma^{-1}(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ onto Π_+, which is a Pick function. By reflection $\Gamma^{-1}(x)$ has also holomorphic extension to Π_- and the range is in it. We thus get (i). $\Gamma(\Gamma^{-1}(z))$ is thus holomorphic on the simply connected domain $\mathbf{C}\setminus(-\infty,\Gamma(\alpha)]$, and $\Gamma(\Gamma^{-1}(x)) = x$ for $\Gamma(\alpha) < x < \infty$. By the uniqueness theorem, $\Gamma(\Gamma^{-1}(z)) = z$ for $z \in \mathbf{C}\setminus(-\infty,\Gamma(\alpha)]$. This means (iii), which clearly yields (ii). □

Corollary 6

$$\Gamma^{-1}(x) = a + bx + \int_{\Gamma(\alpha)}^{\infty}\left(-\frac{1}{x+t} + \frac{1}{t^2+1}\right)d\mu(t), \tag{6}$$

where $\int_{\Gamma(\alpha)}^{\infty} \frac{1}{t^2+1} d\mu(t) < \infty$, and a, b are real numbers and $b \geqq 0$.

Corollary 7 The principal inverse $\Gamma^{-1}(x)$ of $\Gamma(x)$ is operator monotone on $[\Gamma(\alpha), \infty)$; and hence for bounded self-adjoint operators A, B whose spectra are in $[\alpha, \infty)$

$$\Gamma(A) \leqq \Gamma(B) \quad \Rightarrow A \leqq B.$$

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