SOME RESULTS ON GENERALIZED QUADRATIC OPERATORS

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ABSTRACT. A bounded linear operator acting on a Hilbert space is a generalized quadratic operator if it has an operator matrix of the form

\[
\begin{bmatrix}
    aI & cT \\
    dT^* & bI
\end{bmatrix}.
\]

It reduces to a quadratic operator if \( d = 0 \). In this paper, norms and numerical ranges of generalized quadratic operators are determined. Some operator inequalities are also obtained. Moreover we consider \( q \)-numerical range.

1. INTRODUCTION

Let \( \mathcal{B}(\mathcal{H}) \) be the algebra of bounded linear operators acting on a Hilbert space \( \mathcal{H} \). We identify \( \mathcal{B}(\mathcal{H}) \) with \( M_n \) if \( \mathcal{H} \) has dimension \( n \). An operator \( A \in \mathcal{B}(\mathcal{H}) \) is a generalized quadratic operators if it has an operator matrix of the form

\[
\begin{bmatrix}
    aI & cT \\
    dT^* & bI
\end{bmatrix}
\]

(1.1)

where \( T \) is an operator from \( \mathcal{K}_2 \) to \( \mathcal{K}_1 \) (\( \mathcal{K}_1, \mathcal{K}_2 \): Hilbert spaces), and \( a, b, c, d \) are complex numbers. [In the following discussion, we will not distinguish the operator and its operator matrix if there is no ambiguity.] When \( d = 0 \), such an operator \( A \) satisfies condition

\[
(aI - A)(bI - A) = 0
\]

(1.2)

and is known as a \textit{quadratic operator}. In fact, it is known that an operator \( A \) satisfies (1.2) if and only if it has an operator matrix of the form (1.1) with \( d = 0 \).

In this paper, a complete description is given to the norm and ranges of an operator of the form (1.1). In particular, the norm of \( A \) is the same as that of \( A_p \) with \( p = \|T\| \). \textbf{We always assume that} \( cdT \neq 0 \) in the following discussion.

In Section 2, we obtain a different operator matrix for an generalized quadratic operator \( A \). In Section 3, we determine the numerical range and the norm of generalized quadratic operators. Furthermore, we obtain some operator inequalities concerning generalized quadratic operators that extend some results of Furuta [1] and Garcia [2]. We then give the description of \( q \)-numerical ranges of \( A \) in Section 4.

We will use the following notations in our discussion. For \( S \subseteq \mathbb{C} \), denote by \textbf{int}(\( S \)), \textbf{cl}(\( S \)) and \textbf{conv}(\( S \)) the relative interior, the closure and the convex hull of \( S \), respectively.

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2. A DIFFERENT OPERATOR MATRIX REPRESENTATION

First, we obtain a different operator matrix for $A$ of the form (1.1). The special form reduces to that of quadratic operators in [8, Theorem 1.1] if $d = 0$.

**Theorem 2.1.** Let $A \in \mathcal{B}(\mathcal{H})(\mathcal{H} = \mathcal{K}_{1} \oplus \mathcal{K}_{2})$ be an operator with an operator matrix

\[
\begin{bmatrix}
    aI & cT \\
    dT^* & bI
\end{bmatrix}
\]

where $a, b, c, d \in \mathbb{C}$ and $T \in \mathcal{B}(\mathcal{K}_{2}, \mathcal{K}_{1})$ with $cdT \neq 0$. Let $\mathcal{H}_{1} = \text{range}T^*$ (the closure of range$T^*$), $\mathcal{H}_{2} = \text{range}T$, $\mathcal{H}_{3} = \ker T$. Let $T_{0}$ be a restriction of $T$ to $\mathcal{H}_{1}$ with the polar decomposition $T_{0} = U|T_{0}|$ where $U \in \mathcal{B}(\mathcal{H}_{1}, \mathcal{H}_{1})$ is a unitary. Then the operator matrix (1.1) is unitarily similar to

\[
\begin{bmatrix}
    aI_{\mathcal{H}_{2}} & cT_{0} & dT_{0}^{*} & bI_{\mathcal{H}_{3}}
\end{bmatrix}
\]

by the unitary $I_{\mathcal{H}_{2}} \oplus (U \oplus I_{\mathcal{H}_{1}}) \oplus I_{\mathcal{H}_{3}}$ from $\mathcal{H}_{2} \oplus (\mathcal{H}_{1} \oplus \mathcal{H}_{1}) \oplus \mathcal{H}_{3}$.

**Proof.** The operator matrix (1.1) has the following form by the direct sum decomposition $\mathcal{H}(=\mathcal{K}_{1} \oplus \mathcal{K}_{2}) = (\mathcal{H}_{2} \oplus \mathcal{H}_{1}) \oplus (\mathcal{H}_{1} \oplus \mathcal{H}_{3})$

\[
\begin{bmatrix}
    aI_{\mathcal{H}_{2}} & 0 & 0 & 0 \\
    0 & aI_{\mathcal{H}_{1}} & cT_{0} & 0 \\
    0 & dT_{0}^{*} & bI_{\mathcal{H}_{1}} & 0 \\
    0 & 0 & 0 & bI_{\mathcal{H}_{3}}
\end{bmatrix}.
\]

So we may only consider the part $[aI_{\mathcal{H}_{2}} \ cT_{0} \ dT_{0}^{*} \ bI_{\mathcal{H}_{1}}]$. Indeed, we have

\[
\begin{bmatrix}
    U^{*} & 0 \\
    0 & I_{r_{1}}
\end{bmatrix}^{*} \begin{bmatrix}
    aI_{r_{1}} & cT_{0} \\
    dT_{0}^{*} & bI_{r_{1}}
\end{bmatrix} \begin{bmatrix}
    U^{*} & 0 \\
    0 & I_{r_{1}}
\end{bmatrix} = \begin{bmatrix}
    aI_{r_{1}} & cT_{0} \\
    dT_{0}^{*} & bI_{r_{1}}
\end{bmatrix}.
\]

It completes this theorem. \qed

**Remark 2.2.** We have $\langle |T_{0}|x, x \rangle \neq 0$ for all nonzero $x \in \mathcal{H}_{1}$. That is, $|T_{0}|$ is injection.

By Theorem 2.1, we can focus on an operator $A$ with an operator matrix of the form (2.1) with $cd|T_{0}| \neq 0$. Also, the family of matrices

\[
A_{p} = \begin{bmatrix}
    a & cp \\
    dp & b
\end{bmatrix}, \quad p \geq 0,
\]

will be very useful in our discussion.
3. Numerical Range and Operator Inequalities

Recall that the numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(A) = \{(Ax, x) : x \in \mathcal{H}, \|x\| = 1\};$$

see [3], [4], [5]. The numerical range is useful in studying matrices and operators. One of the basic properties of the numerical range is that $W(A)$ is always convex; for example, see [4]. In particular, we have the following result, e.g., see [5, Theorem 1.3.6] and [6].

**Elliptical Range Theorem.** If $A \in M_2$ has eigenvalues $\mu_1$ and $\mu_2$, then $W(A)$ is an elliptical disk with $\mu_1$, $\mu_2$ as foci and $\sqrt{\text{tr}(A^*A) - |\mu_1|^2 - |\mu_2|^2}$ as the length of minor axis. Furthermore, if $\tilde{A} = A - (\text{tr} A)I/2$, then the lengths of minor and major axis of $W(A)$ are, respectively,

$$\{\text{tr}(\tilde{A}^*\tilde{A}) - 2|\det \tilde{A}|\}^{1/2} \quad \text{and} \quad \{\text{tr}(\tilde{A}^*\tilde{A}) + 2|\det \tilde{A}|\}^{1/2}.$$

Using this theorem, one can deduce the convexity of the numerical range of a general operator; e.g., see [6]. It turns out that for an operator $A$ in Theorem 2.1, $W(A)$ is also an elliptical disk with all the boundary points, two boundary points, or none of its boundary points as shown in the following.

**Theorem 3.1.** Suppose $A \in \mathcal{B}(\mathcal{H})$ has the operator matrix in Theorem 2.1. Let $\tilde{p} = \|T_0\|$.

$$\tilde{A} = \begin{bmatrix} a & cp \\ dp & b \end{bmatrix}$$ so that $\tilde{A}$ has eigenvalues $\mu_\pm = \frac{1}{2} \left\{ (a+b) \pm \sqrt{(a-b)^2 + 4cd\tilde{p}^2} \right\}$ and $W(\tilde{A})$ is the elliptical disk with foci $\mu_+, \mu_-$ and minor axis of length

$$\sqrt{|a|^2 + |b|^2 + \tilde{p}^2(|c|^2 + |d|^2) - |\mu_+|^2 - |\mu_-|^2}.$$

If $\|T_0x\| = \|T_0\|$ for some unit vector $x \in \mathcal{H}_1$, then

$$W(A) = W(\tilde{A}).$$

Otherwise, $W(A) = \text{int}(W(\tilde{A})) \cup \{a, b\}$. More precisely, one of the following holds:

1. If $|c| = |d|$ and $d(a-b) = c(\overline{a} - \overline{b})$, then both $A$ and $\tilde{A}$ are normal, and

$$W(A) = W(\tilde{A}) \setminus \sigma(\tilde{A}) = \text{conv}\{\mu_+, \mu_-\} \setminus \{\mu_+, \mu_-\}.$$

2. If $|c| = |d|$ and there is $\zeta \in (0, \pi)$ such that $d(a-b) = e^{i2\zeta}c(\overline{a}-\overline{b}) \neq 0$, then both numbers $a, b$ lie on the boundary $\partial W(A)$ of $W(A)$, and

$$W(A) = \text{int}(W(\tilde{A})) \cup \{a, b\}.$$  

3. If $|c| \neq |d|$, then $W(A) = \text{int}(W(\tilde{A}))$.

To prove Theorem 3.1, we need the following lemma, which will also be useful for later discussion.

**Lemma 3.2.** Let $A_p = \begin{bmatrix} a & cp \\ dp & b \end{bmatrix}$ for $p \geq 0$ so that $W(A_p)$ is the closed elliptical disk with foci $\mu_\pm = \frac{1}{2} \left\{ (a+b) \pm \sqrt{(a-b)^2 + 4cd\tilde{p}^2} \right\}$ and minor axis of length

$$\sqrt{|a|^2 + |b|^2 + p^2(|c|^2 + |d|^2) - |\mu_+|^2 - |\mu_-|^2}.$$
Then

$$W(A_p) \subseteq W(A_q) \quad \text{for } p < q.$$ 

More precisely, one of the following holds:

1. If $|c| = |d|$ and $d(a - b) = c(\overline{a} - \overline{b})$, then $W(A_p) = \text{conv}\sigma(A_p)$ and $W(A_q) = \text{conv}\sigma(A_q)$ are line segments such that $W(A_p)$ is a subset of the relative interior of $W(A_q)$.

2. If $|c| = |d|$ and there is $\zeta \in (0, \pi)$ such that $d(a - b) = e^{i2\zeta}c(\overline{a} - \overline{b}) \neq 0$, then 

$$\{a, b\} = \partial W(A_p) \cap \partial W(A_q), \quad \text{and} \quad W(A_p) \subseteq \text{int}(W(A_q)) \cup \{a, b\}.$$ 

3. If $|c| \neq |d|$, then $W(A_p) \subseteq \text{int}(W(A_q))$.

Proof. All numerical ranges $W(A_p)$ have the same center $\alpha = (a + b)/2$. Suppose $\beta = (a - b)/2$. Denote by $\lambda_1(X)$ the largest eigenvalue of a self-adjoint matrix $X$. Then

$$W(A_p) = \bigcap_{\xi \in [0,2\pi)} \Pi_{\xi}(A_p)$$

where

$$\Pi_{\xi}(A_p) = \{\mu \in \mathbb{C} : e^{i\xi}\mu + e^{-i\xi}\overline{\mu} \leq \lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*)\}$$

is a half space in $\mathbb{C}$. Since

$$\lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*) = e^{i\xi}\alpha + e^{-i\xi}\overline{\alpha} + \sqrt{|e^{i\xi}\beta + e^{-i\xi}\overline{\beta}|^2 + p^2|e^{i\xi}c + e^{-i\xi}d|^2}$$

is an increasing function of $p$, we see that $\Pi_{\xi}(A_p) \subseteq \Pi_{\xi}(A_q)$ and hence $W(A_p) \subseteq W(A_q)$ if $p \leq q$.

Case 1. Suppose $a, b, c, d$ satisfy condition (1). Then $A_p$ is normal and $A_p = \alpha I_2 + B_p$, where $W(B_p) = \text{conv}\{\pm \sqrt{-\det(B_p)}\}$ is a line segment of length $2\sqrt{|\beta|^2 + p^2|c|^2} = 2\sqrt{|\beta|^2 + p^2|d|^2}$. Thus, the conclusion of (1) holds.

Case 2. Suppose $a, b, c, d$ satisfy condition (2). Then $A_p = \alpha I_2 + \beta B_p$ with

$$e^{i\xi}B_p = \begin{bmatrix} e^{i\xi} & \delta p \\ \overline{\delta p} & -e^{-i\xi} \end{bmatrix}, \quad \delta = e^{i\xi}\frac{2c}{a - b} = e^{-i\xi}\frac{2\overline{d}}{\overline{a} - \overline{b}}.$$ 

Using the elliptical range theorem, one readily checks that $W(e^{i\xi}B_p)$ is a nondegenerate elliptical disk. Since $B_p = \begin{bmatrix} 1 & \delta p e^{-i\xi} \\ \delta p e^{-i\xi} & -1 \end{bmatrix}$ and

$$e^{i\xi}B_p + e^{-i\xi}B_p^* = 2 \begin{bmatrix} \cos\xi & \delta p \cos(\xi - \zeta) \\ \overline{\delta p \cos(\xi - \zeta)} & -\cos\xi \end{bmatrix},$$

we have

$$\lambda_1(e^{i\xi}B_p + e^{-i\xi}B_p^*) = 2\sqrt{\cos^2\xi + |\delta|^2p^2\cos^2(\xi - \zeta)} \geq \pm 2\cos\xi = \pm (e^{i\xi} + e^{-i\xi})$$

where equality holds only for $\xi = \zeta \pm \pi/2$. Therefore $\lambda_1(e^{i\xi}B_p + e^{-i\xi}B_p^*)$ is a strictly increasing function for $p \geq 0$, except for $\xi = \zeta \pm \pi/2$. Moreover $1$ and $-1$ are on the boundary of $W(B_p)$ for $\xi = \zeta \pm \pi/2$. From this, we get the conclusion of (2).

Case 3. Suppose $a, b, c, d$ do not satisfy the conditions in (1) or (2). Since $|c| \neq |d|$, for every $\xi \in [0,2\pi)$,

$$\lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*) = e^{i\xi}\alpha + e^{-i\xi}\overline{\alpha} + \sqrt{|e^{i\xi}\beta + e^{-i\xi}\overline{\beta}|^2 + p^2|e^{i\xi}c + e^{-i\xi}d|^2}.$$
is a strictly increasing function for $p \geq 0$. Thus, the conclusion of (3) holds. \hfill \Box

**Proof of Theorem 3.1.** Since $W(X \oplus Y) = \text{conv}\{W(X) \cup W(Y)\} = W(X)$ if $W(Y) \subseteq W(X)$, we may assume that $\gamma I_s$ is vacuous. Let $P = [T_0]$.  

Suppose $x \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ is a unit vector and $\mu = \langle Ax, x \rangle \in W(A)$. Let $x = \begin{bmatrix} \cos \theta x_1 \\ \sin \theta x_2 \end{bmatrix}$ for some unit vectors $x_1, x_2 \in \mathcal{H}_1$. Let $\langle Px_1, x_2 \rangle = pe^{-i\phi}$ with $p \in [0, \tilde{p}]$ and $\phi \in [0, 2\pi)$. Then 

$$\mu = [\cos \theta \mid e^{-i\phi} \sin \theta] A_p \begin{bmatrix} \cos \theta \\ e^{i\phi} \sin \theta \end{bmatrix} \in W(A_p) \subseteq W(\tilde{A})$$

by Lemma 3.2.

If there is a unit vector $x \in \mathcal{H}_1$ such that $\|P\| = \|Px\|$, then 

$$\|P\|^2 = \langle P^2 x, x \rangle \leq \|P^2 x\| \|x\| \leq \|P\|^2 = \|P\|^2.$$

Thus, $P^2 x = \|P\|^2 x$ and hence $Px = \|P\| x$ as $P$ is positive semi-definite. Then the operator matrix of $A$ with respect to $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, where 

$$\mathcal{H}_0 = \text{span} \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix} \right\}$$

has the form $\tilde{A} \oplus \tilde{A}' \in \mathcal{B}(\mathcal{H})$. Thus, $W(\tilde{A}) \subseteq W(A)$, and the equality holds.

Suppose there is no unit vector $z \in \mathcal{H}_1$ such that $\|P\| = \|Pz\|$. Then for any unit vector $x \in \mathcal{H}$, let $x = \begin{bmatrix} \cos \theta x_1 \\ \sin \theta x_2 \end{bmatrix}$ for some unit vectors $x_1, x_2 \in \mathcal{H}_1$. If $\langle Px_1, x_2 \rangle = pe^{i\phi}$ with $p \in [0, \tilde{p}]$ and $\phi \in [0, 2\pi)$, then $p < \tilde{p}$. By Lemma 3.2, we see that $\mu \in \text{int}(W(\tilde{A}))$ if (a) or (c) holds, and $\mu \in \text{int}(W(\tilde{A})) \cup \{a, b\}$ if (b) holds.

To prove the reverse set equalities, note that there is a sequence of unit vectors $\{x_m\}$ in $\mathcal{H}_1$ such that $\langle Px_m, x_m \rangle = p_m$ converges to $\tilde{p}$. Then the compression of $A$ on the subspace 

$$V_m = \text{span} \left\{ \begin{bmatrix} x_m \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_m \end{bmatrix} \right\} \subseteq \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$$

has the form $A_{p_m}$. Since $W(A_{p_m}) \rightarrow W(\tilde{A})$, we see that $\text{int}(W(\tilde{A})) \subseteq W(A)$. It is also clear that $\{a, b\} \subseteq W(A)$. Thus, the set equalities in (1) – (3) hold. \hfill \Box

We consider some operator inequalities. Denote by 

$$w(A) = \sup\{|\mu| : \mu \in W(A)\}$$

the numerical radius of $A \in \mathcal{B}(\mathcal{H})$. It follows readily from Theorem 3.1 that $w(A) = w(\tilde{A})$ if $A$ and $\tilde{A}$ are defined as in Theorem 3.1. Since $A$ has a dilation of the form $\tilde{A} \otimes I$, we have $\|A\| \leq \|\tilde{A}\|$. As shown in the proof of Theorem 3.1, there is a sequence of two dimensional subspaces $\{V_m\}$ such that the compression of $A$ on $V_m$ is $A_{p_m}$ which converges to $\tilde{A}$. Thus, we have $\|A\| = \|\tilde{A}\|$. Suppose $\tilde{A}$ has singular values $s_1 \geq s_2$. Then $\|\tilde{A}\| = s_1$, $\text{tr}(\tilde{A}^* \tilde{A}) = s_1^2 + s_2^2$ and $|\det(\tilde{A})| = s_1 s_2$. Hence, for $\tilde{p} = \|P\|$, 

$$\|\tilde{A}\| = \frac{1}{2} \left\{ \sqrt{\text{tr}(\tilde{A}^* \tilde{A})} + 2|\det(\tilde{A})| + \sqrt{\text{tr}(\tilde{A}^* \tilde{A})} - 2|\det(\tilde{A})| \right\}$$

$$= \frac{1}{2} \left\{ \sqrt{|a|^2 + |b|^2 + (|c|^2 + |d|^2)p^2 + 2|ab - cd\tilde{p}|} \right\}.$$
$+\sqrt{|a|^2 + |b|^2 + (|c|^2 + |d|^2)p^2 - 2|ab - cd|p^2}\}.

By the fact that \( s_1^2 \) is the larger zero of \( \det(\lambda I - \tilde{A}^* \tilde{A}) \) and that \( \det(\tilde{A}^* \tilde{A}) = |\det(\tilde{A})|^2 \), we have

\[
\|\tilde{A}\| = \frac{1}{\sqrt{2}} \left\{ \sqrt{\text{tr}(\tilde{A}^* \tilde{A}) + \sqrt{\text{tr}(\tilde{A}^* \tilde{A})^2 - 4|\det(\tilde{A})|^2}} \right\} = \frac{1}{\sqrt{2}} \sqrt{|a|^2 + |b|^2 + (|c|^2 + |d|^2)p^2 + \sqrt{(|a|^2 + |b|^2 + (|c|^2 + |d|^2)p^2)^2 - 4|ab - cd|p^2}}.
\]

We summarize the above discussion in the following corollary, which also covers the result of Furuta [1] on \( w(A) \) for \( A \) of the form (1.1) for \( a, b, c, d \geq 0 \).

**Corollary 3.3.** Suppose \( A \) and \( \tilde{A} \) satisfy the hypothesis of Theorem 3.1. Then \( w(A) = w(\tilde{A}) \) and \( \|A\| = \|\tilde{A}\|. \) In particular, if \( a, b \in \mathbb{R} \) and \( c, d \in \mathbb{C} \) satisfy \( cd \geq 0 \), then \( \text{cl}(W(A)) = W(\tilde{A}) \) is symmetric about the real axis, and

\[
w(A) = w((A + A^*)/2) = w(\tilde{A}) = w((\tilde{A} + \tilde{A}^*)/2)
= \frac{1}{2} \left\{ |a + b| + \sqrt{(a-b)^2 + (|c|-|d|)^2\|P\|^2} \right\}
\]

and

\[
\|A\| = \|\tilde{A}\| = \frac{1}{2} \left\{ \sqrt{(a+b)^2 + (|c|-|d|)^2\|P\|^2} + \sqrt{(a-b)^2 + (|c|+|d|)^2\|P\|^2} \right\}.
\]

**Proof.** The first assertion follows readily from Theorem 3.1. Suppose \( a, b \in \mathbb{R} \) and \( c, d \in \mathbb{C} \) with \( cd \geq 0 \). Then there is a diagonal unitary matrix \( D = \text{diag}(1, \mu) \) such that \( D^* \tilde{A} D = \begin{bmatrix} a & |c||P||b| \\ |d||P| & b \end{bmatrix} \). It is then easy to get the equalities. \( \square \)

**Corollary 3.4.** Let \( A_i \) be self-adjoint operators on \( \mathcal{H}_i \) with \( \sigma(A_i) \subseteq [m, M] \) for \( i = 1, 2 \), and let \( T \) be an operator from \( \mathcal{H}_2 \) to \( \mathcal{H}_1 \). Then

\[
(3.1) \quad w \left( \begin{bmatrix} A_1 & T \\ T^* & -A_2 \end{bmatrix} \right) \leq \frac{1}{2}(M-m) + \frac{1}{2}\sqrt{(M+m)^2 + 4\|T\|^2}.
\]

**Proof.** For two self-adjoint operators \( X, Y \in \mathcal{B}(\mathcal{H}) \), we write \( X \leq Y \) if \( Y - X \) is positive semidefinite. Since \( mI \leq A_i \leq MI \) for \( i = 1, 2 \), we have

\[
\begin{bmatrix} mI & T \\ T^* & -MI \end{bmatrix} \leq \begin{bmatrix} A_1 & T \\ T^* & -A_2 \end{bmatrix} \leq \begin{bmatrix} MI & T \\ T^* & -mI \end{bmatrix}.
\]

By Theorem 3.1,

\[
\left\| \begin{bmatrix} mI & T \\ T^* & -MI \end{bmatrix} \right\| = \left\| \begin{bmatrix} MI & T \\ T^* & -mI \end{bmatrix} \right\| = \frac{1}{2}(M-m) + \frac{1}{2}\sqrt{(M+m)^2 + 4\|T\|^2}.
\]

The desired inequality holds. \( \square \)
Note that if \( X, Y \in \mathcal{B}(\mathcal{H}) \), then we have the unitary similarity relations
\[
\begin{bmatrix}
X + iY & 0 \\
0 & X - iY
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
I & iI \\
iI & I
\end{bmatrix} \begin{bmatrix}
X & -Y \\
Y & X
\end{bmatrix} \begin{bmatrix}
I & -iI \\
iI & I
\end{bmatrix} \frac{1}{\sqrt{2}}
\]
\[
= \frac{1}{\sqrt{2}} \begin{bmatrix}
I & I \\
-I & I
\end{bmatrix} \begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix} \begin{bmatrix}
I & -I \\
I & I
\end{bmatrix} \frac{1}{\sqrt{2}}
\]

Thus,
\[
\max(\|X + iY\|, \|X - iY\|) = \|\begin{bmatrix}
X & -Y \\
Y & X
\end{bmatrix}\| = \|\begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix}\|
\]

Consequently, if \( X, Y \in \mathcal{B}(\mathcal{H}) \) are self-adjoint with \( \sigma(X) \subseteq [m, M] \), then using Corollary 3.4, we have
\[
\|X + iY\| = \|X - iY\| = \|\begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix}\| = \|\begin{bmatrix}
X & Y \\
-Y & X
\end{bmatrix}\| \leq \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|Y\|^2}.
\]

This covers a result in [2].

4. q-NUMERICAL RANGE

For \( q \in [0, 1] \), the \( q \)-numerical range of \( A \) is the set
\[
W_q(A) := \{\langle Ax, y \rangle : x, y \in \mathcal{H}, \|x\| = \|y\| = 1, \langle x, y \rangle = q\}.
\]

It is known [7], [9] that
\[
W_q(A) = \left\{q\langle Ax, x \rangle + \sqrt{1 - q^2}\langle Ax, y \rangle : \exists \text{ orthonormal \{x, y\} \subseteq \mathcal{H}}\right\},
\]

and also
\[
W_q(A) = \left\{q\mu + \sqrt{1 - q^2}\nu : \exists x \in \mathcal{H} \text{ with } \|x\| = 1, \mu = \langle Ax, x \rangle, |\mu|^2 + |\nu|^2 \leq \|Ax\|^2\right\}.
\]

If \( q = 1 \), then \( W_q(A) = W(A) \). For \( 0 \leq q < 1 \), we have the following description of \( W_q(A) \) for a generalized quadratic operator \( A \in \mathcal{B}(\mathcal{H}) \). In particular, \( W_q(A) \) will always be an open or closed elliptical disk, which may degenerate to a line segment or a point.

**Theorem 4.1.** Suppose \( A \) and \( \tilde{A} \) satisfy the condition in Theorem 3.1. For any \( q \in [0, 1] \), if there is a unit vector \( z \in \mathcal{H}_1 \) such that \( \| T_0 z \| = \| T_0 \| \), then \( W_q(A) = W_q(\tilde{A}) \); otherwise \( W_q(A) = \text{int} \left( W_q(\tilde{A}) \right) \).

We need the following lemma:

**Lemma 4.2.** Let \( A_p \) be defined as in (2.2). If \( p < q \), then for any unit vector \( x \in \mathbb{C}^2 \) there is a unit vector \( x' \in \mathbb{C}^2 \) such that \( \langle A_p x, x \rangle = \langle A_q x', x' \rangle \) and \( \|A_p x\| < \|A_q x'\| \).
Proof. Choose a unit vector $y$ orthogonal to $x$ such that $A_{p}x = \mu_{1}x + \nu_{1}y$. Let $U = [x | y]$. Then $U$ is a unitary in $M_{2}(\mathbb{C})$. So $A_{p}$ is unitarily similar to a matrix of the following form by $U$

$$
\hat{A}_{p} = \begin{bmatrix} \mu_{1} & \mu_{2} \\
\nu_{1} & \nu_{2} \end{bmatrix} \quad \left(= U^{*}A_{p}U = \begin{bmatrix} x^{*} \\
y^{*} \end{bmatrix} A_{p} [x | y] = \begin{bmatrix} x^{*} \\
y^{*} \end{bmatrix} [A_{p}x | A_{p}y] = \begin{bmatrix} (A_{p}x, x) \\
(A_{p}x, y) \end{bmatrix} \right).$$

Here we remark that $\mu_{1} = \langle A_{p}x, x \rangle$ and $||A_{p}x||^{2} = |\mu_{1}|^{2} + |\nu_{1}|^{2}$. Since the condition $p < q$ implies $W(A_{p}) \subseteq W(A_{q})$ by Lemma 3.2, there exists a unit vector $x' \in W_{q}(A)$ such that $\langle A_{p}x, x \rangle = \langle A_{q}x', x' \rangle$. Moreover there exists a unit vector $y'$ orthogonal to $x'$ such that $A_{q}x' = \mu_{1}x' + \nu_{1}y'$. Then $V = [x' | y']$ is a unitary in $M_{2}(\mathbb{C})$. Since $tr A_{p} = tr A_{q} (= a + b = tr(U^{*}A_{p}U) = tr(V^{*}A_{q}V))$ and $V^{*}A_{q}V = \begin{bmatrix} \langle A_{q}x', x' \rangle & \langle A_{q}y', x' \rangle \\
\langle A_{q}x', y' \rangle & \langle A_{q}y', y' \rangle \end{bmatrix}$, we have $\langle A_{p}x, x \rangle + \langle A_{p}y, y \rangle = \langle A_{q}x', x' \rangle + \langle A_{q}y', y' \rangle$. It implies $\nu_{2} = \langle A_{p}y, y \rangle = \langle A_{q}y', y' \rangle$. Hence $A_{q}$ is unitarily similar to a matrix of the following form by $V$

$$
\hat{A}_{q} = \begin{bmatrix} \mu_{1} & \hat{\mu}_{2} \\
\hat{\nu}_{1} & \nu_{2} \end{bmatrix} = V^{*}A_{q}V.
$$

Since $||A_{q}x'||^{2} = |\mu_{1}|^{2} + |\nu_{1}|^{2}$, we may show $|\nu_{1}| < |\hat{\nu}_{1}|$ for this lemma.

Since a matrix $X \in M_{2}$ is unitarily similar to $X^{t}$ in general, we may assume that $|\nu_{1}| \geq |\hat{\nu}_{1}|$. By basic calculations we have

$$
|\hat{\nu}_{1}|^{2} + |\hat{\mu}_{2}|^{2} - |\nu_{1}|^{2} - |\mu_{2}|^{2} = tr(A_{q}^{*}\hat{A}_{q} - \hat{A}_{q}^{*}\hat{A}_{p}) = tr(A_{q}^{*}A_{q} - A_{p}^{*}A_{p})
$$

(4.4)

and

$$
||\hat{\nu}_{1}\hat{\mu}_{2} - \nu_{1}\mu_{2}|| \leq |\hat{\nu}_{1}\hat{\mu}_{2} - \nu_{1}\mu_{2}| = |\det(\hat{A}_{p}) - \det(\hat{A}_{q})| = |\det(A_{p}) - \det(A_{q})| = |cd|(q^{2} - p^{2}).
$$

(4.5)

The above two inequalities (4.4) and (4.5) implies

$$(|\hat{\nu}_{1}| + |\hat{\mu}_{2}|)^{2} - (|\nu_{1}| + |\mu_{2}|)^{2} \geq (|c| - |d|)(q^{2} - p^{2}) \geq 0$$

and

$$(||\hat{\nu}_{1}|| - |\hat{\mu}_{2}|)^{2} - (||\nu_{1}|| - |\mu_{2}|)^{2} \geq (||c|| - ||d||)(q^{2} - p^{2}) \geq 0.$$ 

So we have

$$
|\hat{\nu}_{1}| + |\hat{\mu}_{2}| \geq |\nu_{1}| + |\mu_{2}| \quad \text{and} \quad ||\hat{\nu}_{1}|| - |\hat{\mu}_{2}| \geq ||\nu_{1}|| - |\mu_{2}|| \geq |\nu_{1}| - |\mu_{2}|.
$$

(4.6)

which implies that $|\hat{\nu}_{1}| \geq |\nu_{1}|$. From the proof, we can see that if $|\hat{\nu}_{1}| = |\nu_{1}|$, then we have $|\hat{\mu}_{2}| = |\mu_{2}|$ by (4.6). Then the left hand side of (4.4) is 0, a contradiction. Therefore, we must have $|\hat{\nu}_{1}| > |\nu_{1}|$ and the result follows.$\square$

Proof of Theorem 4.1. Since the operator $A$ has a dilation of the form $\tilde{A} \otimes I$, we have

$$W_{q}(A) \subseteq W_{q}(\tilde{A} \otimes I) = W_{q}(\tilde{A}).$$
Let $P = |T_0|$ and $\{z_m\}$ be a sequence of unit vectors in $\mathcal{H}_1$ such that $\langle P z_m, z_m \rangle = p_m \to \|P\| = p$. The compression of $A$ on the subspace $V_m = \text{span}\left\{ \begin{bmatrix} z_m \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z_m \end{bmatrix} \right\}$ equals $A_{p_m}$ as defined in (2.2). Indeed, we have $\left\langle A \begin{bmatrix} \alpha z_m \\ \beta z_m \end{bmatrix}, \begin{bmatrix} \alpha z_m \\ \beta z_m \end{bmatrix} \right\rangle = \left\langle A_{p_m} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle$ for any $\begin{bmatrix} \alpha z_m \\ \beta z_m \end{bmatrix} \in V_m$. Thus, $W_q(A_{p_m}) \subseteq W_q(A)$ for all $m$.

Suppose that there is a unit vector $z \in \mathcal{H}_1$ such that $\|Pz\| = \|P\| = p$. Then we may assume that $z_m = z$ for each $m$ so that $W_q(\tilde{A}) (= W_q(A_p)) \subseteq W_q(A)$. So we have $W_q(\tilde{A}) = W_q(A)$.

Suppose there is no unit vector $z \in \mathcal{H}_1$ such that $\|Pz\| = \|P\|$. Since $A_{p_m} \to \tilde{A}$, we see that $\text{int}(W_q(\tilde{A})) \subseteq W_q(A)$. For any unit vectors $x, y \in \mathcal{H}$ with $\langle x, y \rangle = q$, we put $x = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $y = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_1$ such that $u_1, u_2, v_1, v_2 \in \mathcal{H}_1$ are unit vectors with $u_i \perp v_i$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for $i = 1, 2$. Then the compression of $A$ on

$$V = \text{span}\left\{ \begin{bmatrix} u_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \right\}$$

has the form

$$B = \begin{bmatrix} aI_2 \\ cS^* \\ dS \\ bI_2 \end{bmatrix}$$

where $S \in M_2$ satisfies $\|S\| < \|P\|$. Let $\tilde{B} \equiv A_{\|S\|}$. Since $W(B) \subseteq W(\tilde{B})$ by Theorem 3.1, $B$ has a dilation $\tilde{B} \otimes I$. Therefore, $W_q(B) \subseteq W_q(\tilde{B} \otimes I) = W_q(\tilde{B})$. Let $\zeta = \langle Ax, y \rangle \in W_q(A)$. Since $B$ is a compression of $A$ on $V$, we have $\zeta \in W_q(B)(\subseteq W_q(\tilde{B}))$. By the inequality (4.2), there exist orthogonal vectors $x', y' \in \mathbb{C}^2$ such that $\zeta = q\langle \tilde{B}x', \tilde{x}' \rangle + \sqrt{1 - q^2}\langle \tilde{B}x', y' \rangle$. Moreover there exist $\mu_1, \nu_1$ in $\mathbb{C}$ such that $\tilde{B}x' = \mu_1 x' + \nu_1 y'$. We see $\mu_1 = \langle \tilde{B}x', x' \rangle, \nu_1 = \langle \tilde{B}x', y' \rangle$ and so $\zeta = q\mu_1 + \sqrt{1 - q^2}\nu_1$. Let $U = [x'|y']$ be a unitary. Hence $\tilde{B}$ is unitarily similar to a matrix of the form

$$\tilde{B} = \begin{bmatrix} \mu_1 \\ \nu_1 \end{bmatrix} \begin{bmatrix} \mu_2 \\ \nu_2 \end{bmatrix} \left( = U^* \tilde{B} U = \begin{bmatrix} \langle \tilde{B}x, x \rangle & \langle \tilde{B}y, x \rangle \\ \langle \tilde{B}x, y \rangle & \langle \tilde{B}y, y \rangle \end{bmatrix} \right).$$

Hence we remark that $\tilde{B} = A_{\|S\|}$ and $\tilde{B} = A_{\|P\|}$ ($\|S\| < \|P\|$). By Lemma 4.2, there exists a unit vector $y''$ in $\mathbb{C}^2$ that $(\mu_1) = \langle \tilde{B}x', x' \rangle = \langle \tilde{A}y'', y'' \rangle$ and $\|\tilde{B}x'\| < \|\tilde{A}y''\|$. Let $z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then we have $\|\tilde{B}z\| = \|\tilde{B}x'\| = \sqrt{|\mu_1|^2 + |\nu_1|^2}$ and $\langle \tilde{B}z, z \rangle = \langle \tilde{B}z, z \rangle = \mu_1$, and so

$$\zeta = q\mu_1 + \sqrt{1 - q^2}\nu_1 \in \left\{ q\mu_1 + \sqrt{1 - q^2}\nu : \mu_1 = \langle \tilde{B}z, z \rangle, |\mu_1|^2 + |\nu|^2 \leq \|\tilde{B}z\|^2 \right\} = \left\{ q\mu_1 + \sqrt{1 - q^2}\nu : \mu_1 = \langle \tilde{B}x', x' \rangle, |\mu_1|^2 + |\nu|^2 \leq \|\tilde{B}x'\|^2 \right\} \subseteq \left\{ q\mu_1 + \sqrt{1 - q^2}\nu : \mu_1 = \langle \tilde{A}y'', y'' \rangle, |\mu_1|^2 + |\nu|^2 < \|\tilde{A}y''\|^2 \right\} \subseteq \text{int} W_q(\tilde{A}).$$
In above, we remark that
\[
\left\{ (\mu_1, \nu) : |\mu_1|^2 + |\nu|^2 < \|\tilde{A}y''\|^2 \right\} \subset \left\{ (\mu, \nu) : |\mu|^2 + |\nu|^2 < \|\tilde{A}y''\|^2 \right\}
\subset \text{int} \left\{ (\mu, \nu) : |\mu|^2 + |\nu|^2 \leq \|\tilde{A}y''\|^2 \right\}.
\]
Hence the proof is completed. 

REFERENCES

1. T. Furuta, Applications of polar decompositions of idempotent and 2-nilpotent operators, Linear and Multilinear Algebra 56(2008), 69–79.