SOME RESULTS ON GENERALIZED QUADRATIC OPERATORS

Noncommutative Structure in Operator Theory and its Application

Author(s)
Tominaga, Masaru

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SOME RESULTS ON GENERALIZED QUADRATIC OPERATORS

Masaru Tominaga (富永 雅)
Hiroshima Institute of Technology
(広島工業大学)
m.tominaga.3n@it-hiroshima.ac.jp

ABSTRACT. A bounded linear operator acting on a Hilbert space is a generalized quadratic operator if it has an operator matrix of the form

$$\begin{bmatrix} aI & cT \\ dT^* & bI \end{bmatrix}.$$ 

It reduces to a quadratic operator if $d = 0$. In this paper, norms and numerical ranges of generalized quadratic operators are determined. Some operator inequalities are also obtained. Moreover we consider $q$-numerical range.

1. INTRODUCTION

Let $B(H)$ be the algebra of bounded linear operators acting on a Hilbert space $H$. We identify $B(H)$ with $M_n$ if $H$ has dimension $n$. An operator $A \in B(H)$ is a generalized quadratic operators if it has an operator matrix of the form

$$(1.1) \quad \begin{bmatrix} aI & cT \\ dT^* & bI \end{bmatrix}$$

where $T$ is an operator from $K_2$ to $K_1$ ($K_1, K_2$: Hilbert spaces), and $a, b, c, d$ are complex numbers. [In the following discussion, we will not distinguish the operator and its operator matrix if there is no ambiguity.] When $d = 0$, such an operator $A$ satisfies condition

$$(1.2) \quad (aI - A)(bI - A) = 0$$

and is known as a quadratic operator. In fact, it is known that an operator $A$ satisfies (1.2) if and only if it has an operator matrix of the form (1.1) with $d = 0$.

In this paper, a complete description is given to the norm and ranges of an operator of the form (1.1). In particular, the norm of $A$ is the same as that of $A_p$ with $p = \|T\|$. We always assume that $cdT \neq 0$ in the following discussion.

In Section 2, we obtain a different operator matrix for an generalized quadratic operator $A$. In Section 3, we determine the numerical range and the norm of generalized quadratic operators. Furthermore, we obtain some operator inequalities concerning generalized quadratic operators that extend some results of Furuta [1] and Garcia [2]. We then give the description of $q$-numerical ranges of $A$ in Section 4.

We will use the following notations in our discussion. For $S \subseteq \mathbb{C}$, denote by int$(S)$, cl$(S)$ and conv$(S)$ the relative interior, the closure and the convex hull of $S$, respectively.

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Note that in our discussion, it may happen that $S = \text{conv}\{\mu_1, \mu_2\}$ is a line segment in $\mathbb{C}$ so that $\text{int}(S) = S \setminus \{\mu_1, \mu_2\}$.

For $A \in \mathcal{B}(\mathcal{H})$, let $\ker A$ and $\text{range } A$ denote the null space and range space of $A$, respectively. Let $V$ be a closed subspace of $\mathcal{H}$ and $Q$ the embedding of $V$ into $\mathcal{H}$. Then $B = Q^* AQ$ is the compression of $A$ onto $V$.

2. A DIFFERENT OPERATOR MATRIX REPRESENTATION

First, we obtain a different operator matrix for $A$ of the form (1.1). The special form reduces to that of quadratic operators in [8, Theorem 1.1] if $d = 0$.

**Theorem 2.1.** Let $A \in \mathcal{B}(\mathcal{H}) (\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2)$ be an operator with an operator matrix

\[
\begin{bmatrix}
    aI & cT \\
    dT^* & bI
\end{bmatrix}
\]

where $a, b, c, d \in \mathbb{C}$ and $T \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ with $cdT \neq 0$. Let $\mathcal{H}_1 = \overline{\text{range } T}$ (the closure of range $T$), $\tilde{\mathcal{H}}_1 = \text{range } T$, $\mathcal{H}_2 = \ker T^*$, $\mathcal{H}_3 = \ker T$. Let $T_0$ be a restriction of $T$ to $\mathcal{H}_1$ with the polar decomposition $T_0 = U|T_0|$ where $U \in \mathcal{B}(\mathcal{H}_1, \tilde{\mathcal{H}}_1)$ is a unitary. Then the operator matrix (1.1) is unitarily similar to

\[
\begin{bmatrix}
    aI_{\mathcal{H}_2} & 0 & 0 & 0 \\
    0 & aI_{\mathcal{H}_1} & c|T_0| & 0 \\
    0 & d|T_0| & bI_{\tilde{\mathcal{H}}_1} & 0 \\
    0 & 0 & 0 & bI_{\mathcal{H}_3}
\end{bmatrix}
\]

by the unitary

\[
I_{\mathcal{H}_2} \oplus (U \oplus I_{\tilde{\mathcal{H}}_1}) \oplus I_{\mathcal{H}_3}
\]

from $\mathcal{H}_2 \oplus (\mathcal{H}_1 \oplus \tilde{\mathcal{H}}_1) \oplus \mathcal{H}_3$ to $\mathcal{H}_2 \oplus \tilde{\mathcal{H}}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_3$.

**Proof.** The operator matrix (1.1) has the following form by the direct sum decomposition $\mathcal{H}(=\mathcal{K}_1 \oplus \mathcal{K}_2) = (\mathcal{H}_1 \oplus \tilde{\mathcal{H}}_1) \oplus (\mathcal{H}_2 \oplus \mathcal{H}_3)$

\[
\begin{bmatrix}
    aI_{\mathcal{H}_2} & 0 & 0 & 0 \\
    0 & aI_{\mathcal{H}_1} & c|T_0| & 0 \\
    0 & d|T_0| & bI_{\tilde{\mathcal{H}}_1} & 0 \\
    0 & 0 & 0 & bI_{\mathcal{H}_3}
\end{bmatrix}
\]

So we may only consider the part

\[
\begin{bmatrix}
    aI_{r_1} & c|T_0| \\
    d|T_0| & bI_{r_1}
\end{bmatrix}
\]

Indeed, we have

\[
\begin{bmatrix}
    U^* & 0 \\
    0 & I_{r_1}
\end{bmatrix}
\begin{bmatrix}
    aI_{r_1} & c|T_0| \\
    d|T_0| & bI_{r_1}
\end{bmatrix}
\begin{bmatrix}
    U^* & 0 \\
    0 & I_{r_1}
\end{bmatrix} = \begin{bmatrix}
    aI_{r_1} & c|T_0| \\
    d|T_0| & bI_{r_1}
\end{bmatrix}.
\]

It completes this theorem. \( \square \)

**Remark 2.2.** We have $\langle |T_0|x, x \rangle \neq 0$ for all nonzero $x \in \mathcal{H}_1$. That is, $|T_0|$ is injection.

By Theorem 2.1, we can focus on an operator $A$ with an operator matrix of the form (2.1) with $cd|T_0| \neq 0$. Also, the family of matrices

\[
A_p = \begin{bmatrix}
    a & cp \\
    dp & b
\end{bmatrix}, \quad p \geq 0,
\]

will be very useful in our discussion.
3. Numerical Range and Operator Inequalities

Recall that the numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(A) = \{(Ax, x) : x \in \mathcal{H}, \|x\| = 1\};$$

see [3], [4], [5]. The numerical range is useful in studying matrices and operators. One of the basic properties of the numerical range is that $W(A)$ is always convex; for example, see [4]. In particular, we have the following result, e.g., see [5, Theorem 1.3.6] and [6].

**Elliptical Range Theorem.** If $A \in M_2$ has eigenvalues $\mu_1$ and $\mu_2$, then $W(A)$ is an elliptical disk with $\mu_1, \mu_2$ as foci and $\sqrt{\text{tr}(A^*A) - |\mu_1|^2 - |\mu_2|^2}$ as the length of minor axis. Furthermore, if $\tilde{A} = A - (\text{tr} A)I/2$, then the lengths of minor and major axis of $W(A)$ are, respectively,

$$\{\text{tr}(\tilde{A}^*\tilde{A}) - 2|\det \tilde{A}|\}^{1/2} \quad \text{and} \quad \{\text{tr}(\tilde{A}^*\tilde{A}) + 2|\det \tilde{A}|\}^{1/2}.$$

Using this theorem, one can deduce the convexity of the numerical range of a general operator; e.g., see [6]. It turns out that for an operator $A$ in Theorem 2.1, $W(A)$ is also an elliptical disk with all the boundary points, two boundary points, or none of its boundary points as shown in the following.

**Theorem 3.1.** Suppose $A \in \mathcal{B}(\mathcal{H})$ has the operator matrix in Theorem 2.1. Let $\tilde{p} = \|T_0\|$, 

$$\tilde{A} = \begin{bmatrix} a & cp \\ dp & b \end{bmatrix}$$

so that $\tilde{A}$ has eigenvalues $\mu_\pm = \frac{1}{2} \left\{ (a + b) \pm \sqrt{(a - b)^2 + 4cdp^2} \right\}$ and $W(\tilde{A})$ is the elliptical disk with foci $\mu_+, \mu_-$ and minor axis of length

$$\sqrt{|a|^2 + |b|^2 + p^2(|c|^2 + |d|^2) - |\mu_+|^2 - |\mu_-|^2}.$$

If $\|T_0x\| = \|T_0\|$ for some unit vector $x \in \mathcal{H}_1$, then

$$W(A) = W(\tilde{A}).$$

Otherwise, $W(A) = \text{int}(W(\tilde{A})) \cup \{a, b\}$. More precisely, one of the following holds:

1. If $|c| = |d|$ and $d(a - b) = c(\bar{a} - \bar{b})$, then both $A$ and $\tilde{A}$ are normal, and

$$W(A) = W(\tilde{A}) \setminus \sigma(\tilde{A}) = \text{conv}\{\mu_+, \mu_-\} \setminus \{\mu_+, \mu_-\}.$$

2. If $|c| = |d|$ and there is $\zeta \in (0, \pi)$ such that $\overline{d}(a - b) = e^{i2\zeta}c(\overline{a} - \overline{b}) \neq 0$, then both numbers $a, b$ lie on the boundary $\partial W(A)$ of $W(A)$, and

$$W(A) = \text{int}(W(\tilde{A})) \cup \{a, b\}.$$

3. If $|c| \neq |d|$, then $W(A) = \text{int}(W(\tilde{A}))$.

To prove Theorem 3.1, we need the following lemma, which will also be useful for later discussion.

**Lemma 3.2.** Let $A_p = \begin{bmatrix} a & cp \\ dp & b \end{bmatrix}$ for $p \geq 0$ so that $W(A_p)$ is the closed elliptical disk with foci $\mu_\pm = \frac{1}{2} \left\{ (a + b) \pm \sqrt{(a - b)^2 + 4cdp^2} \right\}$ and minor axis of length

$$\sqrt{|a|^2 + |b|^2 + p^2(|c|^2 + |d|^2) - |\mu_+|^2 - |\mu_-|^2}.$$
Then
\[ W(A_p) \subseteq W(A_q) \quad \text{for } p < q. \]

More precisely, one of the following holds:

1. If \(|c| = |d|\) and \(\overline{d}(a-b) = c(\overline{a}-\overline{b})\), then \(W(A_p) = \text{conv} \sigma(A_p)\) and \(W(A_q) = \text{conv} \sigma(A_q)\) are line segments such that \(W(A_p)\) is a subset of the relative interior of \(W(A_q)\).

2. If \(|c| = |d|\) and there is \(\zeta \in (0, \pi)\) such that \(\overline{d}(a-b) = e^{i2\zeta}c(\overline{a}-\overline{b}) \neq 0\), then \(\{a, b\} = \partial W(A_p) \cap \partial W(A_q)\), and
   \[ W(A_p) \subseteq \text{int}(W(A_q)) \cup \{a, b\}. \]

3. If \(|c| \neq |d|\), then \(W(A_p) \subseteq \text{int}(W(A_q))\).

Proof. All numerical ranges \(W(A_p)\) have the same center \(\alpha = (a+b)/2\). Suppose \(\beta = (a-b)/2\). Denote by \(\lambda_1(X)\) the largest eigenvalue of a self-adjoint matrix \(X\). Then
\[ W(A_p) = \bigcap_{\xi \in [0,2\pi)} \Pi_\xi(A_p) \]
where
\[ \Pi_\xi(A_p) = \{ \mu \in \mathbb{C} : e^{i\xi}\mu + e^{-i\xi}\overline{\mu} \leq \lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*) \} \]
is a half space in \(\mathbb{C}\). Since
\[ \lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*) = e^{i\xi}\alpha + e^{-i\xi}\overline{\alpha} + \sqrt{|e^{i\xi}\beta + e^{-i\xi}\overline{\beta}|^2 + p^2|e^{i\xi}c + e^{-i\xi}d|^2} \]
is an increasing function of \(p\), we see that \(\Pi_\xi(A_p) \subseteq \Pi_\xi(A_q)\) and hence \(W(A_p) \subseteq W(A_q)\) if \(p \leq q\).

Case 1. Suppose \(a, b, c, d\) satisfy condition (1). Then \(A_p\) is normal and \(A_p = \alpha I_2 + B_p\), where \(W(B_p) = \text{conv}\{\pm \sqrt{-\det(B_p)}\}\) is a line segment of length \(2\sqrt{|\beta|^2 + p^2|c|^2} = 2\sqrt{|\beta|^2 + p^2|d|^2}\). Thus, the conclusion of (1) holds.

Case 2. Suppose \(a, b, c, d\) satisfy condition (2). Then \(A_p = \alpha I_2 + \beta B_p\) with
\[ e^{i\xi}B_p = \begin{bmatrix} e^{i\xi} & \delta p \\ \delta p & -e^{i\xi} \end{bmatrix}, \quad \delta = e^{i\xi} \frac{2c}{a-b} = e^{-i\xi} \frac{2\overline{d}}{\overline{a}-\overline{b}}. \]
Using the elliptical range theorem, one readily checks that \(W(e^{i\xi}B_p)\) is a nondegenerate elliptical disk. Since \(B_p = \begin{bmatrix} 1 & \delta p e^{-i\xi} \\ \delta p e^{-i\xi} & -1 \end{bmatrix}\) and
\[ e^{i\xi}B_p + e^{-i\xi}B_p^* = 2 \begin{bmatrix} \cos \xi & \delta p \cos(\xi - \zeta) \\ \delta p \cos(\xi - \zeta) & -\cos \xi \end{bmatrix}, \]
we have
\[ \lambda_1(e^{i\xi}B_p + e^{-i\xi}B_p^*) = 2\sqrt{\cos^2 \xi + |\delta|^2 p^2 \cos^2 (\xi - \zeta)} \geq \pm 2\cos \xi = \pm (e^{i\xi} + e^{-i\xi}) \]
where equality holds only for \(\xi = \zeta \pm \pi/2\). Therefore \(\lambda_1(e^{i\xi}B_p + e^{-i\xi}B_p^*)\) is a strictly increasing function for \(p \geq 0\), except for \(\xi = \zeta \pm \pi/2\). Moreover 1 and \(-1\) are on the boundary of \(W(B_p)\) for \(\xi = \zeta \pm \pi/2\). From this, we get the conclusion of (2).

Case 3. Suppose \(a, b, c, d\) do not satisfy the conditions in (1) or (2). Since \(|c| \neq |d|\), for every \(\xi \in [0,2\pi)\),
\[ \lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*) = e^{i\xi}\alpha + e^{-i\xi}\overline{\alpha} + \sqrt{|e^{i\xi}\beta + e^{-i\xi}\overline{\beta}|^2 + p^2|e^{i\xi}c + e^{-i\xi}d|^2} \]
is a strictly increasing function for $p \geq 0$. Thus, the conclusion of (3) holds. \qed

**Proof of Theorem 3.1.** Since $W(X \oplus Y) = \text{conv}\{W(X) \cup W(Y)\} = W(X)$ if $W(Y) \subseteq W(X)$, we may assume that $\gamma I_s$ is vacuous. Let $P = [I_0]$.

Suppose $x \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ is a unit vector and $\mu = \langle Ax, x \rangle \in W(A)$. Let $x = \begin{bmatrix} \cos \theta x_1 \\ \sin \theta x_2 \end{bmatrix}$ for some unit vectors $x_1, x_2 \in \mathcal{H}_1$. Let $\langle Px_1, x_2 \rangle = pe^{-i\phi}$ with $p \in [0, \tilde{p}]$ and $\phi \in [0, 2\pi)$. Then

$$\mu = [\cos \theta \mid e^{-i\phi} \sin \theta] A_p \begin{bmatrix} \cos \theta \\ e^{i\phi} \sin \theta \end{bmatrix} \in W(A_p) \subseteq W(\tilde{A})$$

by Lemma 3.2.

If there is a unit vector $x \in \mathcal{H}_1$ such that $\|P\| = \|Px\|$, then

$$\|P\|^2 = \langle P^2 x, x \rangle \leq \|P^2 x\| \|x\| \leq \|P^2\| = \|P\|^2.$$

Thus, $P^2 x = \|P\|^2 x$ and hence $Px = \|P\| x$ as $P$ is positive semi-definite. Then the operator matrix of $A$ with respect to $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, where

$$\mathcal{H}_0 = \text{span} \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix} \right\}$$

has the form $\tilde{A} \oplus \tilde{A}^\prime \in \mathcal{B}(\mathcal{H})$. Thus, $W(\tilde{A}) \subseteq W(A)$, and the equality holds.

Suppose there is no unit vector $z \in \mathcal{H}_1$ such that $\|P\| = \|Pz\|$. Then for any unit vector $x \in \mathcal{H}$, let $x = \begin{bmatrix} \cos \theta x_1 \\ \sin \theta x_2 \end{bmatrix}$ for some unit vectors $x_1, x_2 \in \mathcal{H}_1$. If $\langle Px_1, x_2 \rangle = pe^{i\phi}$ with $p \in [0, \tilde{p}]$ and $\phi \in [0, 2\pi)$, then $p < \tilde{p}$. By Lemma 3.2, we see that $\mu \in \text{int}(W(\tilde{A}))$ if (a) or (c) holds, and $\mu \in \text{int}(W(\tilde{A})) \cup \{a, b\}$ if (b) holds.

To prove the reverse set equalities, note that there is a sequence of unit vectors $\{x_m\}$ in $\mathcal{H}_1$ such that $\langle Px_m, x_m \rangle = p_m$ converges to $\tilde{p}$. Then the compression of $A$ on the subspace

$$V_m = \text{span} \left\{ \begin{bmatrix} x_m \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_m \end{bmatrix} \right\} \subseteq \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$$

has the form $A_{p_m}$. Since $W(A_{p_m}) \to W(\tilde{A})$, we see that $\text{int}(W(\tilde{A})) \subseteq W(A)$. It is also clear that $\{a, b\} \subseteq W(A)$. Thus, the set equalities in (1) – (3) hold. \qed

We consider some operator inequalities. Denote by

$$w(A) = \sup\{\|\mu\| : \mu \in W(A)\}$$

the numerical radius of $A \in \mathcal{B}(\mathcal{H})$. It follows readily from Theorem 3.1 that $w(A) = w(\tilde{A})$ if $A$ and $\tilde{A}$ are defined as in Theorem 3.1. Since $A$ has a dilation of the form $\tilde{A} \otimes I$, we have $\|A\| \leq \|\tilde{A}\|$. As shown in the proof of Theorem 3.1, there is a sequence of two dimensional subspaces $\{V_m\}$ such that the compression of $A$ on $V_m$ is $A_{p_m}$ which converges to $\tilde{A}$. Thus, we have $\|A\| = \|\tilde{A}\|$. Suppose $\tilde{A}$ has singular values $s_1 \geq s_2$. Then $\|\tilde{A}\| = s_1$, $\text{tr}(\tilde{A}^* \tilde{A}) = s_1^2 + s_2^2$ and $|\det(\tilde{A})| = s_1 s_2$. Hence, for $\tilde{p} = \|P\|$, \begin{align*}
\|\tilde{A}\| &= \frac{1}{2} \left\{ \sqrt{\text{tr}(\tilde{A}^* \tilde{A})} + 2|\det(\tilde{A})| \right\} \\
&= \frac{1}{2} \left\{ \sqrt{|a|^2 + |b|^2 + (|c|^2 + |d|^2) \tilde{p}^2 + 2|ab - cd\tilde{p}^2|} \right\}
\end{align*}
By the fact that $s_1^2$ is the larger zero of $\det(\lambda I - \tilde{A}^* \tilde{A})$ and that $\det(\tilde{A}^* \tilde{A}) = |\det(\tilde{A})|^2$, we have

\[
\|\tilde{A}\| = \frac{1}{\sqrt{2}} \left\{ \sqrt{\text{tr}(\tilde{A}^* \tilde{A})} + \sqrt{\text{tr}(\tilde{A}^* \tilde{A})^2 - 4|\det(\tilde{A})|^2} \right\}.
\]

We summarize the above discussion in the following corollary, which also covers the result of Furuta [1] on $w(A)$ for $A$ of the form (1.1) for $a, b, c, d \geq 0$.

**Corollary 3.3.** Suppose $A$ and $\tilde{A}$ satisfy the hypothesis of Theorem 3.1. Then $w(A) = w(\tilde{A})$ and $\|A\| = \|\tilde{A}\|$. In particular, if $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ satisfy $cd \geq 0$, then $\text{cl}(W(A)) = W(\tilde{A})$ is symmetric about the real axis, and

\[
w(A) = w((A + A^*)/2) = w(\tilde{A}) = w((\tilde{A} + \tilde{A}^*)/2)
= \frac{1}{2} \left\{ |a + b| + \sqrt{(a - b)^2 + (|c| + |d|)^2 \|P\|^2} \right\}
\]

and

\[
\|A\| = \|\tilde{A}\| = \frac{1}{2} \left\{ \sqrt{(a + b)^2 + (|c| - |d|)^2 \|P\|^2} + \sqrt{(a - b)^2 + (|c| + |d|)^2 \|P\|^2} \right\}.
\]

**Proof.** The first assertion follows readily from Theorem 3.1. Suppose $a, b \in \mathbb{R}$ and $c, d \in \mathbb{C}$ with $cd \geq 0$. Then there is a diagonal unitary matrix $D = \text{diag}(1, \mu)$ such that $D^* \tilde{A} = \left[ \begin{array}{cc} a & |c|\|P\| \\ |d|\|P\| & b \end{array} \right]$. It is then easy to get the equalities. \qed

**Corollary 3.4.** Let $A_i$ be self-adjoint operators on $\mathcal{H}_i$ with $\sigma(A_i) \subseteq [m, M]$ for $i = 1, 2$, and let $T$ be an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$. Then

\[
w\left( \begin{bmatrix} A_1 & T \\ T^* & -A_2 \end{bmatrix} \right) \leq \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|T\|^2}.
\]

**Proof.** For two self-adjoint operators $X, Y \in \mathcal{B}(\mathcal{H})$, we write $X \leq Y$ if $Y - X$ is positive semidefinite. Since $mI \leq A_i \leq MI$ for $i = 1, 2$, we have

\[
\begin{bmatrix} mI & T \\ T^* & -mI \end{bmatrix} \leq \begin{bmatrix} A_1 & T \\ T^* & -A_2 \end{bmatrix} \leq \begin{bmatrix} MI & T \\ T^* & -mI \end{bmatrix}.
\]

By Theorem 3.1,

\[
\left\| \begin{bmatrix} mI & T \\ T^* & -mI \end{bmatrix} \right\| = \left\| \begin{bmatrix} MI & T \\ T^* & -mI \end{bmatrix} \right\| = \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|T\|^2}.
\]

The desired inequality holds. \qed
Note that if $X, Y \in \mathcal{B}(\mathcal{H})$, then we have the unitary similarity relations

$$
\begin{bmatrix}
X + iY & 0 \\
0 & X - iY
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
I & iI \\
iI & I
\end{bmatrix} \begin{bmatrix}
X & -Y \\
Y & X
\end{bmatrix} \begin{bmatrix}
I & -iI \\
-iI & I
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
I & iI \\
iI & I
\end{bmatrix} \begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix} \begin{bmatrix}
I & -iI \\
-iI & I
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
I & iI \\
iI & I
\end{bmatrix} \begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix} \begin{bmatrix}
I & -iI \\
-iI & I
\end{bmatrix} = \frac{1}{\sqrt{2}}.
$$

Thus,

$$
\max\{\|X + iY\|, \|X - iY\|\} = \|\begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix}\| = \|\begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix}\|.
$$

Consequently, if $X, Y \in \mathcal{B}(\mathcal{H})$ are self-adjoint with $\sigma(X) \subseteq [m, M]$, then using Corollary 3.4, we have

$$
\|X + iY\| = \|X - iY\| = \|\begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix}\| = \|\begin{bmatrix}
X & iY \\
iY & X
\end{bmatrix}\| = \|\begin{bmatrix}
X & Y \\
Y & X
\end{bmatrix}\| \leq \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|Y\|^2}.
$$

This covers a result in [2].

4. $q$-NUMERICAL RANGE

For $q \in [0, 1]$, the $q$-numerical range of $A$ is the set

$$
W_q(A) := \{\langle Ax, y \rangle : x, y \in \mathcal{H}, \|x\| = \|y\| = 1, \langle x, y \rangle = q\}.
$$

It is known [7], [9] that

$$
W_q(A) = \left\{q\langle Ax, x \rangle + \sqrt{1 - q^2}\langle Ax, y \rangle : \exists \text{ orthonormal } \{x, y\} \subseteq \mathcal{H}\right\},
$$

and also

$$
W_q(A) = \left\{q\mu + \sqrt{1 - q^2}\nu : \exists x \in \mathcal{H} \text{ with } \|x\| = 1, \mu = \langle Ax, x \rangle, |\mu|^2 + |\nu|^2 \leq \|Ax\|^2\right\}.
$$

If $q = 1$, then $W_q(A) = W(A)$. For $0 \leq q < 1$, we have the following description of $W_q(A)$ for a generalized quadratic operator $A \in \mathcal{B}(\mathcal{H})$. In particular, $W_q(A)$ will always be an open or closed elliptical disk, which may degenerate to a line segment or a point.

**Theorem 4.1.** Suppose $A$ and $\tilde{A}$ satisfy the condition in Theorem 3.1. For any $q \in [0, 1)$, if there is a unit vector $x \in \mathcal{H}_1$ such that $\|T_0x\| = \|T_0\|$, then $W_q(A) = W_q(\tilde{A})$; otherwise $W_q(A) = \text{int}(W_q(\tilde{A}))$.

We need the following lemma:

**Lemma 4.2.** Let $A_p$ be defined as in (2.2). If $p < q$, then for any unit vector $x \in \mathbb{C}^2$ there is a unit vector $x' \in \mathbb{C}^2$ such that $\langle A_px, x \rangle = \langle A_qx', x' \rangle$ and $\|A_px\| < \|A_qx'\|$.
Proof. Choose a unit vector \( y \) orthogonal to \( x \) such that \( A_p x = \mu_1 x + \nu_1 y \). Let \( U = [ x \mid y ] \). Then \( U \) is a unitary in \( M_2(\mathbb{C}) \). So \( A_p \) is unitarily similar to a matrix of the following form by \( U \):

\[
\hat{A}_p = \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix}
\]

\[
= U^* A_p U = \begin{bmatrix} x^* \\ y^* \end{bmatrix} A_p \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} [ A_p x \mid A_p y ] = \begin{bmatrix} \langle A_p x, x \rangle & \langle A_p y, x \rangle \\ \langle A_p x, y \rangle & \langle A_p y, y \rangle \end{bmatrix}.
\]

Here we remark that \( \mu_1 = \langle A_p x, x \rangle \) and \( \| A_p x \|^2 = |\mu_1|^2 + |\nu_1|^2 \). Since the condition \( p < q \) implies \( W(A_p) \subseteq W(A_q) \) by Lemma 3.2, there exists a unit vector \( x' \in W_q(\mathcal{A}) \) such that \( \langle A_p x, x \rangle = \langle A_q x', x' \rangle \). Moreover, there exists a unit vector \( y' \) orthogonal to \( x' \) such that \( A_q x' = \mu_1 x' + \nu_1 y' \). Then \( V = [ x' \mid y' ] \) is a unitary in \( M_2(\mathbb{C}) \). Since \( \text{tr} \, A_p = \text{tr} \, A_q = a + b = \text{tr} (U^* A_p U) = \text{tr} (V^* A_q V) \), we have

\[
\langle A_p x, x \rangle + \langle A_p y, y \rangle = \langle A_q x', x' \rangle + \langle A_q y', y' \rangle.
\]

It implies \( \nu_2 = \langle A_p y, y \rangle = \langle A_q y', y' \rangle \). Hence \( A_q \) is unitarily similar to a matrix of the following form by \( V \):

\[
\hat{A}_q = \begin{bmatrix} \mu_1 & \hat{\mu}_2 \\ \hat{\nu}_1 & \nu_2 \end{bmatrix} = V^* A_q V.
\]

Since \( \| A_q x' \|^2 = |\mu_1|^2 + |\nu_1|^2 \), we may show \( |\nu_1| < |\hat{\nu}_1| \) for this lemma.

Since a matrix \( X \in M_2 \) is unitarily similar to \( ^t X \) in general, we may assume that \( |\hat{\nu}_1| \geq |\hat{\nu}_2| \). By basic calculations we have

\[
|\hat{\nu}_1|^2 + |\hat{\mu}_2|^2 - |\nu_1|^2 - |\mu_2|^2 = \text{tr} (A_q^* \hat{A}_q - \hat{A}_p^* \hat{A}_p) = \text{tr} (A_q^* A_q - A_p^* A_p)
\]

\[
= (|c|^2 + |d|^2)(q^2 - p^2) > 0,
\]

and

\[
||\hat{\nu}_1 \hat{\mu}_2| - |\nu_1 \mu_2|| \leq |\hat{\nu}_1 \hat{\mu}_2 - \nu_1 \mu_2| = |\det(\hat{A}_p) - \det(\hat{A}_q)|
\]

\[
= |\det(A_p) - \det(A_q)| = |cd|(q^2 - p^2).
\]

The above two inequalities (4.4) and (4.5) implies

\[
(|\hat{\nu}_1| + |\hat{\mu}_2|)^2 - (|\nu_1| + |\mu_2|)^2 \geq (|c| - |d|)(q^2 - p^2) \geq 0
\]

and

\[
(|\hat{\nu}_1| - |\hat{\mu}_2|)^2 - (|\nu_1| - |\mu_2|)^2 \geq (|c| - |d|)(q^2 - p^2) \geq 0.
\]

So we have

\[
(\hat{\nu}_1 + \hat{\mu}_2 \geq |\nu_1| + |\mu_2| \quad \text{and} \quad |\hat{\nu}_1| - |\hat{\mu}_2| \geq |\nu_1| - |\mu_2| \geq |\nu_1| - |\mu_2|)
\]

which implies that \( |\hat{\nu}_1| \geq |\nu_1| \). From the proof, we can see that if \( |\hat{\nu}_1| = |\nu_1| \), then we have \( |\hat{\mu}_2| = |\mu_2| \) by (4.6). Then the left hand side of (4.4) is 0, a contradiction. Therefore, we must have \( |\hat{\nu}_1| > |\nu_1| \) and the result follows. \( \Box \)

Proof of Theorem 4.1. Since the operator \( \mathcal{A} \) has a dilation of the form \( \tilde{\mathcal{A}} \otimes I \), we have

\[
W_q(\mathcal{A}) \subseteq W_q(\tilde{\mathcal{A}} \otimes I) = W_q(\tilde{\mathcal{A}}).
\]
Let $P = |T_0|$ and $\{z_m\}$ be a sequence of unit vectors in $\mathcal{H}_1$ such that $\langle Pz_m, z_m \rangle = p_m \to \|P\| = p$. The compression of $A$ on the subspace $V_m = \text{span} \left\{ \begin{bmatrix} z_m \\ 0 \\ 0 \end{bmatrix} \right\}$ equals $A_{p_m}$ as defined in (2.2). Indeed, we have $\left\langle A \begin{bmatrix} \alpha z_m \\ \beta z_m \end{bmatrix}, \begin{bmatrix} \alpha z_m \\ \beta z_m \end{bmatrix} \right\rangle = A_{p_m} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle$ for any $\begin{bmatrix} \alpha z_m \\ \beta z_m \end{bmatrix} \in V_m$. Thus, $W_q(A_{p_m}) \subseteq W_q(A)$ for all $m$.

Suppose that there is a unit vector $z \in \mathcal{H}_1$ such that $\|Pz\| = \|P\| = p$. Then we may assume that $z_m = z$ for each $m$ so that $W_q(\hat{A})(= W_q(A_p)) \subseteq W_q(A)$. So we have $W_q(A) = W_q(\hat{A})$.

Suppose there is no unit vector $z \in \mathcal{H}_1$ such that $\|Pz\| = \|P\|$. Since $A_{p_m} \to \hat{A}$, we see that $\text{int}(W_q(\hat{A})) \subseteq W_q(A)$. For any unit vectors $x, y \in \mathcal{H}$ with $\langle x, y \rangle = q$, we put $x = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $y = \begin{bmatrix} \beta_1 u_1 + \gamma_1 v_1 \\ \beta_2 u_2 + \gamma_2 v_2 \end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_1$ such that $u_1, u_2, v_1, v_2 \in \mathcal{H}_1$ are unit vectors with $u_i \perp v_i$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for $i = 1, 2$. Then the compression of $A$ on

$$V = \text{span} \left\{ \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

has the form

$$B = \begin{bmatrix} aI_2 & cS \\ dS^* & bI_2 \end{bmatrix}$$

where $S \in M_2$ satisfies $\|S\| < \|P\|$. Let $\hat{B} \equiv A_{\|S\|}$. Since $W(B) \subseteq W(\hat{B})$ by Theorem 3.1, $B$ has a dilation $\hat{B} \otimes I$. Therefore, $W_q(B) \subseteq W_q(\hat{B} \otimes I) = W_q(\hat{B})$. Let $\zeta = \langle Ax, y \rangle \in W_q(A)$. Since $B$ is a compression of $A$ on $V$, we have $\zeta \in W_q(B)(\subseteq W_q(\hat{B}))$. By the inequality (4.2), there exist orthogonal vectors $x', y' \in \mathbb{C}^2$ such that $\zeta = q \langle \hat{B}x', x' \rangle + \sqrt{1 - q^2} \langle \hat{B}x', y' \rangle$. Moreover there exist $\mu_1, \nu_1$ in $\mathbb{C}$ such that $\hat{B}x' = \mu_1 x' + \nu_1 y'$. We see $\mu_1 = \langle \hat{B}x', x' \rangle$, $\nu_1 = \langle \hat{B}x', y' \rangle$ and so $\zeta = q\mu_1 + \sqrt{1 - q^2} \nu_1$. Let $U = \begin{bmatrix} x' \\ y' \end{bmatrix}$ be a unitary. Hence $\hat{B}$ is unitarily similar to a matrix of the form

$$\hat{B} = \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix} = U^* \tilde{B} U = \begin{bmatrix} \langle \hat{B}x, x \rangle & \langle \hat{B}y, x \rangle \\ \langle \hat{B}x, y \rangle & \langle \hat{B}y, y \rangle \end{bmatrix}.$$

Hence we remark that $\hat{B} = A_{\|S\|}$ and $\hat{A} = A_{\|P\|}$ ($\|S\| < \|P\|$). By Lemma 4.2, there exists a unit vector $y''$ in $\mathbb{C}^2$ that $(\mu_1 =) \langle \hat{B}x', x' \rangle = \langle \hat{A}y'', y'' \rangle$ and $\|\hat{B}x'\| < \|\hat{A}y''\|$. Let $z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then we have $\|\hat{B}z\| = \|\hat{B}x'\| = \sqrt{\mu_1^2 + 2 \nu_1^2}$ and $\langle \hat{B}z, z \rangle = \langle \hat{B}z, z \rangle = \mu_1$, and so

$$\zeta = q\mu_1 + \sqrt{1 - q^2} \nu_1 \in \left\{ q\mu_1 + \sqrt{1 - q^2} \nu_1 : \mu_1 = \langle \hat{B}z, z \rangle, \ |\mu_1|^2 + |\nu_1|^2 \leq ||\hat{B}z||^2 \right\}$$

$$= \left\{ q\mu_1 + \sqrt{1 - q^2} \nu_1 : \mu_1 = \langle \hat{B}x', x' \rangle, \ |\mu_1|^2 + |\nu_1|^2 \leq ||\hat{B}x'||^2 \right\}$$

$$\subseteq \left\{ q\mu_1 + \sqrt{1 - q^2} \nu_1 : \mu_1 = \langle \hat{A}y'', y'' \rangle, \ |\mu_1|^2 + |\nu_1|^2 < ||\hat{A}y''||^2 \right\}$$

(by $||\hat{B}x'|| < ||\hat{A}y''||$)

$$\subseteq \text{int} W_q(\hat{A}).$$
In above, we remark that
\[
\left\{ (\mu_1, \nu) : |\mu_1|^2 + |\nu|^2 < \|\tilde{A}y''\|^2 \right\} \subset \left\{ (\mu, \nu) : |\mu|^2 + |\nu|^2 < \|\tilde{A}y''\|^2 \right\} \\
\subset \text{int} \left\{ (\mu, \nu) : |\mu|^2 + |\nu|^2 \leq \|\tilde{A}y''\|^2 \right\}.
\]
Hence the proof is completed. \qed

REFERENCES

1. T. Furuta, Applications of polar decompositions of idempotent and 2-nilpotent operators, Linear and Multilinear Algebra 56(2008), 69–79.