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Remarks on perturbation of defect operators on Hilbert function spaces

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1 Introduction

Let $(\mathcal{H}, k_\lambda, \Omega)$ be a reproducing kernel Hilbert space consisting of analytic functions on a domain Ω in $\mathbb{C}^n$ with the variable $z = (z_1, \ldots, z_n)$ and the reproducing kernel $k_\lambda = k(\lambda, \cdot)$, where λ is a point in Ω. Without loss of generality, we may assume that Ω contains the origin. Moreover, we assume that $\mathcal{H}$ is invariant under pointwise multiplication of any polynomial in $\mathbb{C}[z_1, \ldots, z_n]$. Then a family of operators encoding structure of $(\mathcal{H}, k_\lambda, \Omega)$ is obtained under appropriate conditions. In this note, these operators will be denoted by Δ_λ. We should mention that $\Delta = \Delta_0$ has been studied already by many researchers on some Hilbert function spaces.

This note has been organized as follows. In Section 2 and Section 3, we will give a partial announcement of results obtained in [8], where we dealt with Δ_λ's of submodules in Hardy space over the bidisk. In Section 4, we revisit the Hardy space over the unit disk from our point of view. In Section 5, we studies Δ_λ's of submodules in the Bergman space over the unit disk.

2 Rudin's module

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $H^2(\mathbb{D})$ be the Hardy space over $\mathbb{D}$. The Hardy space over the bidisk $\mathbb{D}^2$ will be denoted by $H^2(\mathbb{D}^2)$, or H^2 for short. Then $z = (z_1, z_2)$ will denote the variable of functions in H^2. We note that H^2 can be defined as the tensor product Hilbert space $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$. A will denote the bidisk algebra. Then, under pointwise multiplication, H^2 becomes a Hilbert module over A. A closed subspace $\mathcal{M}$ of H^2 is called a submodule if $\mathcal{M}$ is invariant under the module action, that is, a submodule is an invariant subspace of H^2 under multiplication of each function in A. $[S]$ denotes the submodule generated by a set S. The rank of a submodule $\mathcal{M}$ is the least cardinality of a generating set of $\mathcal{M}$ as a Hilbert module, and which will be denoted by rank $\mathcal{M}$, and the following inequality is well known:

$$\dim \mathcal{M}/[(z_1-\lambda_1)\mathcal{M}+(z_2-\lambda_2)\mathcal{M}] \leq \operatorname{rank} \mathcal{M} \quad ((\lambda_1,\lambda_2)\in \mathbb{D}). \tag{2.1}$$

Set $\alpha_n = 1-n^{-3}$ $(n\in\mathbb{N})$, and let b_{α_n} be the Blaschke factor whose zero is α_n. Then

$$\mathcal{M}=\sum_{j=0}^{\infty} q_j H^2(\mathbb{D})\otimes z_2^j \quad \left(\text{where } q_j=\prod_{n=j}^{\infty} b_{\alpha_n}^{n-j}\right)$$

has been called Rudin's module (cf. Rudin [7]). The striking fact on Rudin's module is that the module rank is infinity. Indeed, for any $\lambda=(\lambda_1,\lambda_2)$ in $\mathbb{D}^2$, we have

$$\dim \mathcal{M}/[(z_1-\lambda_1)\mathcal{M}+(z_2-\lambda_2)\mathcal{M}] = \begin{cases} n+1 & (\lambda=(\alpha_n,0)) \\ 1 & (\text{otherwise}). \end{cases}$$

As n tends to infinity, we have rank $\mathcal{M}=\infty$ by (2.1).

Therefore we are interested in the following family of quotient vector spaces.

$$\mathcal{M}/[(z_1-\lambda_1)\mathcal{M}+(z_2-\lambda_2)\mathcal{M}] \quad ((\lambda_1,\lambda_2)\in\mathbb{D}^2).$$

3 $H^2(\mathbb{D}^2)$ case

Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{D}^2)$. Then R_f denotes the compression of a Toeplitz operator T_f into $\mathcal{M}$, that is, we set $R_f = P_{\mathcal{M}}T_f|_{\mathcal{M}}$ where $P_{\mathcal{M}}$ is the orthogonal projection of H^2 onto a submodule $\mathcal{M}$. The following operator is called the defect operator of a submodule $\mathcal{M}$.

$$\Delta = I_{\mathcal{M}} - R_{z_1}R_{z_1}^* - R_{z_2}R_{z_2}^* + R_{z_1}R_{z_2}R_{z_1}^*R_{z_2}^*,$$

which has been introduced by Yang in [9, 10] (see, also Guo [3] and Guo-Yang [5]). Moreover, we introduce the following operator valued function:

$$\Delta_\lambda = I_{\mathcal{M}} - R_{b_{\lambda_1}(z_1)}R_{b_{\lambda_1}(z_1)}^* - R_{b_{\lambda_2}(z_2)}R_{b_{\lambda_2}(z_2)}^* + R_{b_{\lambda_1}(z_1)}R_{b_{\lambda_2}(z_2)}R_{b_{\lambda_1}(z_1)}^*R_{b_{\lambda_2}(z_2)}^*,$$

where

$$(b_{\lambda_1}(z_1), b_{\lambda_2}(z_2)) = \left(\frac{z_1 - \lambda_1}{1 - \overline{\lambda_1}z_1}, \frac{z_2 - \lambda_2}{1 - \overline{\lambda_2}z_2}\right) \quad (\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

Since $(b_{\lambda_1}(z_1), b_{\lambda_2}(z_2))$ defines an automorphism of $\mathbb{D}^2$ (i.e. a biholomorphic map acting on $\mathbb{D}^2$) , Δ_λ can be seen as a defect operator perturbed by an automorphism. The following theorem is the reason why we are interested in Δ_λ, which was shown in Guo-Yang [5] for the case where $\lambda = 0$ (see also Guo-Wang [4]), and their proof can be applied to the general case.

Theorem 3.1 (Guo-Yang [5], Guo-Wang [4]) *Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{D}^2)$. Then for any $\lambda \in \mathbb{D}^2$,*

$$\ker(I_{\mathcal{M}} - \Delta_\lambda) = \mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}].$$

Yang defined a new class of submodules in $H^2(\mathbb{D}^2)$.

Definition 3.1 ([10]) *A submodule $\mathcal{M}$ in H^2 is said to be Hilbert-Schmidt if Δ is Hilbert-Schmidt.*

Yang showed that Hilbert-Schmidt class includes Rudin's module and submodules generated by polynomials.

Theorem 3.2 (S [8]) Let $\mathcal{M}$ be a submodule of H^2.

(i) If Δ_μ is Hilbert-Schmidt for some μ in $\mathbb{D}^2$, then Δ_λ is Hilbert-Schmidt for any λ in $\mathbb{D}^2$.

(ii) If $\mathcal{M}$ is Hilbert-Schmidt then $\|\Delta_\lambda - \Delta_\mu\|_2 \to 0$ $(\lambda \to \mu)$.

Theorem 3.3 (S [8]) *Let $\mathcal{M}$ be a Hilbert-Schmidt submodule such that $\dim\ker(I - \Delta_\mu) = n > 1$ for some μ in $\mathbb{D}^2$. Then, for any neighborhood U_1 of 1 such that $\sigma(\Delta_\mu) \cap \overline{U_1} = \{1\}$, there exists a neighborhood U_μ of μ such that $\sigma(\Delta_\lambda) \cap U_1 = \{1, \sigma_1(\lambda), \ldots, \sigma_{n-1}(\lambda)\}$ for any λ in U_μ, counting multiplicity.*

Example 3.1 (Yang [9], S [8]) Let $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ be one variable inner functions, and let $\mathcal{M}$ be the submodule generated by q_1 and q_2 in $H^2(\mathbb{D}^2)$. Then we have

$$\dim\ker(I_{\mathcal{M}} - \Delta_\lambda) = \begin{cases} 2 & (\text{if } q_1(\lambda_1) = q_2(\lambda_2) = 0) \\ 1 & (\text{otherwise}). \end{cases}$$

and

$$\sigma(\Delta_\lambda) = \{0, 1, \pm\sigma(\lambda)\},$$

where we set

$$\sigma(\lambda) = \sqrt{(1 - |q_1(\lambda_1)|^2)(1 - |q_2(\lambda_2)|^2)}.$$

This calculation has been done already in the case where $(\lambda_1, \lambda_2) = (0, 0)$ by Yang in [9]. If $\sigma(\lambda) \neq 1$ then the eigenfunction corresponding to $\sigma(\lambda)$ is

$$e(\lambda) = \left(\sqrt{1 - |q_2(\lambda_2)|^2} - \sqrt{1 - |q_1(\lambda_1)|^2}\right) \frac{q_1(z_1)q_2(z_2)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}$$

$$- \frac{q_2(\lambda_2)}{\sqrt{1 - |q_2(\lambda_2)|^2}} \frac{q_1(z_1)(1 - \overline{q_2(\lambda_2)}q_2(z_2))}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}$$

$$+ \frac{q_1(\lambda_1)}{\sqrt{1 - |q_1(\lambda_1)|^2}} \frac{q_2(z_2)(1 - \overline{q_1(\lambda_1)}q_1(z_1))}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}$$

If $\sigma(\lambda) = 1$ then the eigenfunctions corresponding to $\sigma(\lambda)$ are

$$\frac{q_1(z_1)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}, \quad \frac{q_2(z_2)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}.$$

Note that $e(\lambda)$ converges to 0 as $\sigma(\lambda)$ tends to 1.

4 $H^2(\mathbb{D})$ case

The defect operator of a submodule $\mathcal{M}$ in $H^2(\mathbb{D})$ is as follows:

$$\Delta = I_{\mathcal{M}} - R_z R_z^* = \mathrm{Proj}(\mathcal{M}/z\mathcal{M}) = q \otimes q,$$

where q is the inner function corresponding to a submodule $\mathcal{M}$ by Beurling's theorem. The definition of Δ_λ is similar to that given in Section 3, and we have

$$\Delta_\lambda = I_{\mathcal{M}} - R_{b_\lambda} R_{b_\lambda}^* = \mathrm{Proj}(\mathcal{M}/(z-\lambda)\mathcal{M}) = qK_\lambda \otimes qK_\lambda,$$

where we set $b_\lambda = (z-\lambda)/(1-\overline{\lambda}z)$ and K_λ denotes the normalized Szegö kernel. These facts are well known.

5 $L_a^2(\mathbb{D})$ case

In this section, we deal with the defect operator of a submodule in Bergman space over $\mathbb{D}$. The Bergman space over $\mathbb{D}$ is defined as follows:

$$L_a^2(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, dxdy < \infty \ (z = x + iy) \right\}.$$

The reproducing kernel is

$$k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^2} \quad \text{(the Bergman kernel)},$$

and the operator $S_z : f \mapsto zf$ acting on $L_a^2(\mathbb{D})$ is called the Bergman shift. The definition of submodules in $L_a^2(\mathbb{D})$ is the same as that of $H^2(\mathbb{D}^2)$. We summarize well known facts on submodules of $L_a^2(\mathbb{D})$.

Theorem 5.1 Let $\mathcal{M}$ be a submodule of $L_a^2(\mathbb{D})$.

(i) $\dim \mathcal{M}/(z-\lambda)\mathcal{M}$ is independent of choice of λ in $\mathbb{D}$ (Richter [6]).

(ii) For every n in $\{1, 2, \ldots, \infty\}$, there exists a submodule $\mathcal{M}$ such that $\dim \mathcal{M}/z\mathcal{M} = n$ (Apostol-Bercovici-Foiaş-Pearcy [1]).

(iii) $\mathcal{M}/z\mathcal{M}$ is a generating set of $\mathcal{M}$ (Aleman-Richter-Sundberg [2]).

The defect operator of a submodule of $L_a^2(\mathbb{D})$ is as follows:

$$\Delta = I_{\mathcal{M}} - 2R_z R_z^* + R_z^2 R_z^{*2},$$

which was introduced by Yang-Zhu [11] (they called this the root operator of $\mathcal{M}$). The definition of Δ_λ is similar to that given in Section 3,

$$\Delta_\lambda = I_{\mathcal{M}} - 2R_{b_\lambda} R_{b_\lambda}^* + R_{b_\lambda}^2 R_{b_\lambda}^{*\,2},$$

where we set $b_\lambda = (z - \lambda)/(1 - \overline{\lambda} z)$. The following theorem was shown in Yang-Zhu [11] in the case where $\lambda = 0$, and their proof can be applied to the general case.

Theorem 5.2 (Yang-Zhu [11])

$$\ker(I_{\mathcal{M}} - \Delta_\lambda) = \mathcal{M}/(z - \lambda)\mathcal{M}.$$

The Hilbert-Schmidt class of submodules in $L_a^2(\mathbb{D})$ is defined as same as that given in Section 3.

Theorem 5.3 (S) Let $\mathcal{M}$ be a Hilbert-Schmidt submodule of $L_a^2(\mathbb{D})$. Then

(i) Δ_λ is Hilbert-Schmidt for any λ in $\mathbb{D}$,

(ii) $\|\Delta_\lambda - \Delta_\mu\|_2 \to 0 \ (\lambda \to \mu)$.

Proof First, we shall show (i). Setting $k_z^{\mathcal{M}} = P_{\mathcal{M}} k_z$, we have

$$\begin{aligned}
(\Delta_\lambda f)(z) &= \langle \Delta_\lambda f, k_z^{\mathcal{M}} \rangle \\
&= \langle (I_{\mathcal{M}} - 2R_{b_\lambda} R_{b_\lambda}^* + R_{b_\lambda}^2 R_{b_\lambda}^{*\,2}) f, k_z^{\mathcal{M}} \rangle \\
&= \langle f, (I_{\mathcal{M}} - 2R_{b_\lambda} R_{b_\lambda}^* + R_{b_\lambda}^2 R_{b_\lambda}^{*\,2}) k_z^{\mathcal{M}} \rangle \\
&= \langle f, (1 - 2\overline{b_\lambda(z)} R_{b_\lambda} + \overline{b_\lambda(z)}^2 R_{b_\lambda}^2) k_z^{\mathcal{M}} \rangle \\
&= \langle f, (1 - 2\overline{b_\lambda(z)} b_\lambda + \overline{b_\lambda(z)}^2 b_\lambda^2) k_z^{\mathcal{M}} \rangle \\
&= \int_{\mathbb{D}} f(w) \overline{(1 - \overline{b_\lambda(z)} b_\lambda(w))^2 k_z^{\mathcal{M}}(w)} \, dA(w),
\end{aligned}$$

where $dA(w) = \pi^{-1}dxdy$ $(w = x + iy)$. Hence Δ_λ is Hilbert-Schmidt if and only if

$$(1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w)$$

is square integrable with respect to the Lebesgue measure on $\mathbb{D}^2$. We note that

$$\frac{(1 - \overline{b_\lambda(z)}b_\lambda(w))^2}{(1 - \overline{z}w)^2} = \left(\frac{1 - |\lambda|^2}{(1 - \lambda\overline{z})(1 - \overline{\lambda}w)}\right)^2. \tag{5.1}$$

Hence we have

$$\begin{aligned}(1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w) &= \frac{(1 - \overline{b_\lambda(z)}b_\lambda(w))^2}{(1 - \overline{z}w)^2}(1 - \overline{z}w)^2 k_z^{\mathcal{M}} \\ &= \left(\frac{1 - |\lambda|^2}{(1 - \lambda\overline{z})(1 - \overline{\lambda}w)}\right)^2 (1 - \overline{z}w)^2 k_z^{\mathcal{M}}. \end{aligned} \tag{5.2}$$

Since trivially (5.1) is bounded on $\mathbb{D}^2$, (5.2) is square integrable on $\mathbb{D}^2$. This concludes (i).

Next, we shall show (ii). Since the integral kernel of Δ_λ is

$$(1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w),$$

and using (5.2), we have

$$\begin{aligned}&\|\Delta_\lambda - \Delta_\mu\|_2^2 \\ &= \int_{\mathbb{D}}\int_{\mathbb{D}} |(1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w) - (1 - \overline{b_\mu(z)}b_\mu(w))^2 k_z^{\mathcal{M}}(w)|^2 \, dA(z)dA(w) \\ &= \int_{\mathbb{D}}\int_{\mathbb{D}} \left|\left(\frac{1 - |\lambda|^2}{(1 - \lambda\overline{z})(1 - \overline{\lambda}w)}\right)^2 - \left(\frac{1 - |\mu|^2}{(1 - \mu\overline{z})(1 - \overline{\mu}w)}\right)^2\right|^2 \\ &\qquad\qquad \times |(1 - \overline{z}w)^2 k_z^{\mathcal{M}}|^2 \, dA(z)dA(w) \\ &\to 0 \ (\lambda \to \mu)\end{aligned}$$

by the Lebesgue dominated convergence theorem. This concludes (ii).

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