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Remarks on perturbation of defect operators on Hilbert function spaces

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1 Introduction

Let \((\mathcal{H}, k_{\lambda}, \Omega)\) be a reproducing kernel Hilbert space consisting of analytic functions on a domain \(\Omega\) in \(\mathbb{C}^n\) with the variable \(z = (z_1, \ldots, z_n)\) and the reproducing kernel \(k_{\lambda} = k(\lambda, \cdot)\), where \(\lambda\) is a point in \(\Omega\). Without loss of generality, we may assume that \(\Omega\) contains the origin. Moreover, we assume that \(\mathcal{H}\) is invariant under pointwise multiplication of any polynomial in \(\mathbb{C}[z_1, \ldots, z_n]\). Then a family of operators encoding structure of \((\mathcal{H}, k_{\lambda}, \Omega)\) is obtained under appropriate conditions. In this note, these operators will be denoted by \(\Delta_{\lambda}\). We should mention that \(\Delta = \Delta_0\) has been studied already by many researchers on some Hilbert function spaces.

This note has been organized as follows. In Section 2 and Section 3, we will give a partial announcement of results obtained in [8], where we dealt with \(\Delta_{\lambda}\)'s of submodules in Hardy space over the bidisk. In Section 4, we revisit the Hardy space over the unit disk from our point of view. In Section 5, we studies \(\Delta_{\lambda}\)'s of submodules in the Bergman space over the unit disk.

2 Rudin's module

Let \(\mathbb{D}\) denote the open unit disk in the complex plane \(\mathbb{C}\), and let \(H^2(\mathbb{D})\) be the Hardy space over \(\mathbb{D}\). The Hardy space over the bidisk \(\mathbb{D}^2\) will be denoted by \(H^2(\mathbb{D}^2)\), or \(H^2\) for short. Then \(z = (z_1, z_2)\) will denote the variable of functions in \(H^2\). We note that \(H^2\) can be defined as the tensor product Hilbert space \(H^2(\mathbb{D}) \otimes H^2(\mathbb{D})\). \(A\) will denote the bidisk algebra. Then,
under pointwise multiplication, $H^2$ becomes a Hilbert module over $A$. A closed subspace $\mathcal{M}$ of $H^2$ is called a submodule if $\mathcal{M}$ is invariant under the module action, that is, a submodule is an invariant subspace of $H^2$ under multiplication of each function in $A$. $[S]$ denotes the submodule generated by a set $S$. The rank of a submodule $\mathcal{M}$ is the least cardinality of a generating set of $\mathcal{M}$ as a Hilbert module, and which will be denoted by rank $\mathcal{M}$, and the following inequality is well known:

$$\dim \mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}] \leq \text{rank } \mathcal{M} \quad ((\lambda_1, \lambda_2) \in \mathbb{D}). \quad (2.1)$$

Set $\alpha_n = 1 - n^{-3}$ $(n \in \mathbb{N})$, and let $b_{\alpha_n}$ be the Blaschke factor whose zero is $\alpha_n$. Then

$$\mathcal{M} = \sum_{j=0}^{\infty} q_j H^2(\mathbb{D}) \otimes z_2^j \left(\text{where } q_j = \prod_{n=j}^{\infty} b_{\alpha_n}^{n-j}\right)$$

has been called Rudin's module (cf. Rudin [7]). The striking fact on Rudin's module is that the module rank is infinity. Indeed, for any $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{D}^2$, we have

$$\dim \mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}] = \begin{cases} n + 1 & (\lambda = (\alpha_n, 0)) \\ 1 & \text{(otherwise)} \end{cases}.$$}

As $n$ tends to infinity, we have rank $\mathcal{M} = \infty$ by (2.1).

Therefore we are interested in the following family of quotient vector spaces.

$$\mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}] \quad ((\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

### 3 $H^2(\mathbb{D}^2)$ case

Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{D}^2)$. Then $R_f$ denotes the compression of a Toeplitz operator $T_f$ into $\mathcal{M}$, that is, we set $R_f = P_{\mathcal{M}}T_f|_{\mathcal{M}}$ where $P_{\mathcal{M}}$ is the
orthogonal projection of $H^2$ onto a submodule $\mathcal{M}$. The following operator is called the defect operator of a submodule $\mathcal{M}$.

$$\Delta = I_{\mathcal{M}} - R_{z_1}R^*_{z_1} - R_{z_2}R^*_{z_2} + R_{z_1}R_{z_2}R^*_{z_1}R^*_{z_2},$$

which has been introduced by Yang in [9, 10] (see also Guo [3] and Guo-Yang [5]). Moreover, we introduce the following operator valued function:

$$\Delta_\lambda = I_{\mathcal{M}} - R_{b_{\lambda_1}(z_1)}R^*_{b_{\lambda_1}(z_1)} - R_{b_{\lambda_2}(z_2)}R^*_{b_{\lambda_2}(z_2)} + R_{b_{\lambda_1}(z_1)}R_{b_{\lambda_2}(z_2)}R^*_{b_{\lambda_1}(z_1)}R^*_{b_{\lambda_2}(z_2)},$$

where

$$(b_{\lambda_1}(z_1), b_{\lambda_2}(z_2)) = \left(\frac{z_1 - \lambda_1}{1 - \overline{\lambda_1}z_1}, \frac{z_2 - \lambda_2}{1 - \overline{\lambda_2}z_2}\right) \quad (\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

Since $(b_{\lambda_1}(z_1), b_{\lambda_2}(z_2))$ defines an automorphism of $\mathbb{D}^2$ (i.e. a biholomorphic map acting on $\mathbb{D}^2$), $\Delta_\lambda$ can be seen as a defect operator perturbed by an automorphism. The following theorem is the reason why we are interested in $\Delta_\lambda$, which was shown in Guo-Yang [5] for the case where $\lambda = 0$ (see also Guo-Wang [4]), and their proof can be applied to the general case.

**Theorem 3.1 (Guo-Yang [5], Guo-Wang [4])** Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{D}^2)$. Then for any $\lambda \in \mathbb{D}^2$,

$$\ker(I_{\mathcal{M}} - \Delta_\lambda) = \mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}].$$

Yang defined a new class of submodules in $H^2(\mathbb{D}^2)$.

**Definition 3.1 ([10])** A submodule $\mathcal{M}$ in $H^2$ is said to be Hilbert-Schmidt if $\Delta$ is Hilbert-Schmidt.

Yang showed that Hilbert-Schmidt class includes Rudin's module and submodules generated by polynomials.

**Theorem 3.2 (S [8])** Let $\mathcal{M}$ be a submodule of $H^2$.

(i) If $\Delta_\mu$ is Hilbert-Schmidt for some $\mu$ in $\mathbb{D}^2$, then $\Delta_\lambda$ is Hilbert-Schmidt for any $\lambda$ in $\mathbb{D}^2$. 
(ii) If $\mathcal{M}$ is Hilbert-Schmidt then $\|\Delta_\lambda - \Delta_\mu\|_2 \to 0$ ($\lambda \to \mu$).

**Theorem 3.3 (S [8])** Let $\mathcal{M}$ be a Hilbert-Schmidt submodule such that $\dim \ker (I - \Delta_\mu) = n > 1$ for some $\mu$ in $\mathbb{D}^2$. Then, for any neighborhood $U_1$ of 1 such that $\sigma(\Delta_\mu) \cap \overline{U}_1 = \{1\}$, there exists a neighborhood $U_\mu$ of $\mu$ such that $\sigma(\Delta_\lambda) \cap U_1 = \{1, \sigma_1(\lambda), \ldots, \sigma_{n-1}(\lambda)\}$ for any $\lambda$ in $U_\mu$, counting multiplicity.

**Example 3.1 (Yang [9], S [8])** Let $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ be one variable inner functions, and let $\mathcal{M}$ be the submodule generated by $q_1$ and $q_2$ in $H^2(\mathbb{D}^2)$. Then we have

$$\dim \ker (I_{\mathcal{M}} - \Delta_\lambda) = \begin{cases} 2 & \text{(if } q_1(\lambda_1) = q_2(\lambda_2) = 0) \\ 1 & \text{(otherwise).} \end{cases}$$

and

$$\sigma(\Delta_\lambda) = \{0, 1, \pm \sigma(\lambda)\},$$

where we set

$$\sigma(\lambda) = \sqrt{(1 - |q_1(\lambda_1)|^2)(1 - |q_2(\lambda_2)|^2)}.$$

This calculation has been done already in the case where $(\lambda_1, \lambda_2) = (0, 0)$ by Yang in [9]. If $\sigma(\lambda) \neq 1$ then the eigenfunction corresponding to $\sigma(\lambda)$ is

$$e(\lambda) = \left( \sqrt{1 - |q_2(\lambda_2)|^2} - \sqrt{1 - |q_1(\lambda_1)|^2} \right) \frac{q_1(z_1)q_2(z_2)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}$$

$$- \frac{q_2(\lambda_2)}{\sqrt{1 - |q_2(\lambda_2)|^2}} \frac{q_1(z_1)(1 - \overline{q_2(\lambda_2)}q_2(z_2))}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}$$

$$+ \frac{q_1(\lambda_1)}{\sqrt{1 - |q_1(\lambda_1)|^2}} \frac{q_2(z_2)(1 - \overline{q_1(\lambda_1)}q_1(z_1))}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}$$

If $\sigma(\lambda) = 1$ then the eigenfunctions corresponding to $\sigma(\lambda)$ are

$$\frac{q_1(z_1)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}, \quad \frac{q_2(z_2)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}.$$
Note that \( e(\lambda) \) converges to 0 as \( \sigma(\lambda) \) tends to 1.

4 \( H^2(\mathbb{D}) \) case

The defect operator of a submodule \( \mathcal{M} \) in \( H^2(\mathbb{D}) \) is as follows:

\[
\Delta = I_{\mathcal{M}} - R_zR_z^* = \text{Proj}(\mathcal{M}/z\mathcal{M}) = q \otimes q,
\]

where \( q \) is the inner function corresponding to a submodule \( \mathcal{M} \) by Beurling's theorem. The definition of \( \Delta_\lambda \) is similar to that given in Section 3, and we have

\[
\Delta_\lambda = I_{\mathcal{M}} - R_{b_\lambda} R_{b_\lambda}^* = \text{Proj}(\mathcal{M}/(z - \lambda)\mathcal{M}) = qK_\lambda \otimes qK_\lambda,
\]

where we set \( b_\lambda = (z - \lambda)/(1 - \overline{\lambda}z) \) and \( K_\lambda \) denotes the normalized Szegö kernel. These facts are well known.

5 \( L_a^2(\mathbb{D}) \) case

In this section, we deal with the defect operator of a submodule in Bergman space over \( \mathbb{D} \). The Bergman space over \( \mathbb{D} \) is defined as follows:

\[
L_a^2(\mathbb{D}) = \left\{ f \in \text{Hol}(\mathbb{D}) : \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, dx \, dy < \infty \quad (z = x + iy) \right\}.
\]

The reproducing kernel is

\[
k_\lambda(z) = \frac{1}{(1 - \overline{\lambda}z)^2} \quad \text{(the Bergman kernel)},
\]

and the operator \( S_z : f \mapsto zf \) acting on \( L_a^2(\mathbb{D}) \) is called the Bergman shift. The definition of submodules in \( L_a^2(\mathbb{D}) \) is the same as that of \( H^2(\mathbb{D}^2) \). We summarize well known facts on submodules of \( L_a^2(\mathbb{D}) \).

**Theorem 5.1** Let \( \mathcal{M} \) be a submodule of \( L_a^2(\mathbb{D}) \).

(i) \( \dim \mathcal{M}/(z - \lambda)\mathcal{M} \) is independent of choice of \( \lambda \) in \( \mathbb{D} \) (Richter [6]).
(ii) For every $n$ in $\{1, 2, \ldots, \infty\}$, there exists a submodule $\mathcal{M}$ such that $\dim \mathcal{M}/z\mathcal{M} = n$ (Apostol-Bercovici-Foiaș-Pearcy [1]).

(iii) $\mathcal{M}/z\mathcal{M}$ is a generating set of $\mathcal{M}$ (Aleman-Richter-Sundberg [2]).

The defect operator of a submodule of $L_a^2(\mathbb{D})$ is as follows:

$$\Delta = I_{\mathcal{M}} - 2R_zR_z^* + R_z^2R_z^{*2},$$

which was introduced by Yang-Zhu [11] (they called this the root operator of $\mathcal{M}$). The definition of $\Delta_{\lambda}$ is similar to that given in Section 3,

$$\Delta_{\lambda} = I_{\mathcal{M}} - 2R_{b_{\lambda}}R_{b_{\lambda}}^* + R_{b_{\lambda}}^2R_{b_{\lambda}}^{*2},$$

where we set $b_{\lambda} = (z - \lambda)/(1 - \overline{\lambda}z)$. The following theorem was shown in Yang-Zhu [11] in the case where $\lambda = 0$, and their proof can be applied to the general case.

**Theorem 5.2 (Yang-Zhu [11])**

$$\ker(I_{\mathcal{M}} - \Delta_{\lambda}) = \mathcal{M}/(z - \lambda)\mathcal{M}.$$ 

The Hilbert-Schmidt class of submodules in $L_a^2(\mathbb{D})$ is defined as same as that given in Section 3.

**Theorem 5.3 (S) Let $\mathcal{M}$ be a Hilbert-Schmidt submodule of $L_a^2(\mathbb{D})$. Then**

(i) $\Delta_{\lambda}$ is Hilbert-Schmidt for any $\lambda$ in $\mathbb{D}$,

(ii) $\|\Delta_{\lambda} - \Delta_{\mu}\|_2 \rightarrow 0$ ($\lambda \rightarrow \mu$).

**Proof** First, we shall show (i). Setting $k_z^{\mathcal{M}} = P_{\mathcal{M}}k_z$, we have

$$(\Delta_{\lambda}f)(z) = \langle \Delta_{\lambda}f, k_z^{\mathcal{M}} \rangle$$

$$= \langle (I_{\mathcal{M}} - 2R_{b_{\lambda}}R_{b_{\lambda}}^* + R_{b_{\lambda}}^2R_{b_{\lambda}}^{*2})f, k_z^{\mathcal{M}} \rangle$$

$$= \langle f, (I_{\mathcal{M}} - 2R_{b_{\lambda}}R_{b_{\lambda}}^* + R_{b_{\lambda}}^2R_{b_{\lambda}}^{*2})k_z^{\mathcal{M}} \rangle$$

$$= \langle f, (1 - 2\overline{b_{\lambda}(z)}R_{b_{\lambda}} + \overline{b_{\lambda}(z)}^2R_{b_{\lambda}}^2)k_z^{\mathcal{M}} \rangle$$

$$= \langle f, (1 - 2\overline{b_{\lambda}(z)}b_{\lambda} + \overline{b_{\lambda}(z)}^2b_{\lambda}^2)k_z^{\mathcal{M}} \rangle$$

$$= \int_{\mathbb{D}} f(w)(1 - \overline{b_{\lambda}(z)b_{\lambda}(w)})^2k_z^{\mathcal{M}}(w) \, dA(w),$$
where \( dA(w) = \pi^{-1}dxdy \ (w = x + iy) \). Hence \( \Delta_{\lambda} \) is Hilbert-Schmidt if and only if

\[
(1 - \overline{b_{\lambda}(z)b_{\lambda}(w)})^2 k_{z}^{M}(w)
\]

is square integrable with respect to the Lebesgue measure on \( \mathbb{D}^2 \). We note that

\[
\frac{(1 - \overline{b_{\lambda}(z)b_{\lambda}(w)})^2}{(1 - \overline{z}w)^2} = \left( \frac{1 - |\lambda|^2}{(1 - \lambda \overline{z})(1 - \overline{\lambda}w)} \right)^2.
\]

(5.1)

Hence we have

\[
(1 - \overline{b_{\lambda}(z)b_{\lambda}(w)})^2 k_{z}^{M}(w) = \frac{(1 - \overline{b_{\lambda}(z)b_{\lambda}(w)})^2}{(1 - \overline{z}w)^2} (1 - \overline{z}w)^2 k_{z}^{\mathcal{M}}
\]

\[
= \left( \frac{1 - |\lambda|^2}{(1 - \lambda \overline{z})(1 - \overline{\lambda}w)} \right)^2 (1 - \overline{z}w)^2 k_{z}^{\mathcal{M}}.
\]

(5.2)

Since trivially (5.1) is bounded on \( \mathbb{D}^2 \), (5.2) is square integrable on \( \mathbb{D}^2 \). This concludes (i).

Next, we shall show (ii). Since the integral kernel of \( \Delta_{\lambda} \) is

\[
(1 - \overline{b_{\lambda}(z)b_{\lambda}(w)})^2 k_{z}^{M}(w),
\]

and using (5.2), we have

\[
\| \Delta_{\lambda} - \Delta_{\mu} \|_2^2 = \int_{\mathbb{D}} \int_{\mathbb{D}} |(1 - \overline{b_{\lambda}(z)b_{\lambda}(w)})^2 k_{z}^{M}(w) - (1 - \overline{b_{\mu}(z)b_{\mu}(w)})^2 k_{z}^{\mathcal{M}}(w)|^2 \, dA(z)\,dA(w)
\]

\[
= \int_{\mathbb{D}} \int_{\mathbb{D}} \left| \left( \frac{1 - |\lambda|^2}{(1 - \lambda \overline{z})(1 - \overline{\lambda}w)} \right)^2 - \left( \frac{1 - |\mu|^2}{(1 - \mu \overline{z})(1 - \overline{\mu}w)} \right)^2 \right|^2 \times |(1 - \overline{z}w)^2 k_{z}^{\mathcal{M}}|^2 \, dA(z)\,dA(w)
\]

\[
\rightarrow 0 \quad (\lambda \rightarrow \mu)
\]

by the Lebesgue dominated convergence theorem. This concludes (ii).
References


