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Loewner matrices of matrix convex and monotone functions
(joint work with F. Hiai)

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Some results in [3, 6] were reported. Here we collect results from them. For the detail, please see the papers.

1 Characterisations by Bhatia-Sano

In this section, we consider a $C^1$ function $f$ from the interval $(0, \infty)$ into itself, with $f(0) = \lim_{t \to 0^+} f(t) = 0$. Given any $n$ distinct points $p_1, \ldots, p_n$ in $(0, \infty)$, let $L_f(p_1, \ldots, p_n)$ be the $n \times n$ matrix defined as

$$L_f(p_1, \ldots, p_n) = \left[ \frac{f(p_i) - f(p_j)}{p_i - p_j} \right].$$

When $i = j$ the quotient in (1.1) is interpreted as $f'(p_i)$. Such a matrix is called a Loewner matrix associated with $f$.

For the function $f(t) = t^r$ where $r > 0$, we use the symbol $L_r$ for a Loewner matrix associated with this function. Thus

$$L_r = \left[ \frac{p_i^r - p_j^r}{p_i - p_j} \right].$$

The function $f$ is said to be operator monotone on $[0, \infty)$ if for two positive semidefinite matrices $A$ and $B$ (of any size $n$) the inequality $A \geq B$ implies $f(A) \geq f(B)$.

Karl Löwner (later Charles Loewner) in [9] showed that $f$ is operator monotone if and only if for all $n$, and all $p_1, \ldots, p_n$, the Loewner matrices $L_f(p_1, \ldots, p_n)$ are p.s.d. and that the function $f(t) = t^r$ is operator monotone if and only if $0 < r \leq 1$. Consequently, if $0 < r \leq 1$, then the matrix (1.2) is p.s.d., and therefore all its eigenvalues are non-negative.

Recall the notion of operator convexity: Assume that $f$ is a $C^2$ function from $(0, \infty)$ into itself, $f(0) = 0$ and $f'(0) = 0$. We say that $f$ is operator convex if

$$f((1-t)A + tB) \leq (1-t)f(A) + tf(B), \quad 0 \leq t \leq 1,$$
for all p.s.d. matrices $A$ and $B$ (of any size $n$).

Let $H^n$ be the subspace of $\mathbb{C}^n$ consisting of all $x = (x_1, \ldots, x_n)$ for which $\sum_{i=1}^{n} x_i = 0$.

An $n \times n$ Hermitian matrix $A$ is said to be conditionally positive definite (c.p.d. for short) or almost positive if

$$\langle x, Ax \rangle \geq 0 \text{ for all } x \in H^n,$$

and conditionally negative definite (c.n.d. for short) if $-A$ is c.p.d. We refer the reader to [1, 4, 8] for properties of these matrices.

We proved:

**Theorem 1.1.** Let $f$ be an operator convex function. Then all Loewner matrices associated with $f$ are conditionally negative definite.

**Theorem 1.2.** Let $f(t) = tg(t)$ where $g$ is an operator convex function. Then all Loewner matrices associated with $f$ are conditionally positive definite.

**Theorem 1.3.** Let $L_r$ be the $n \times n$ Loewner matrix (1.2) associated with distinct points $p_1, \ldots, p_n$. Then

(i) $L_r$ is conditionally negative definite for $1 \leq r \leq 2$, and conditionally positive definite for $2 \leq r \leq 3$.

(ii) $L_r$ is nonsingular for $1 < r < 2$ and for $2 < r < 3$.

(iii) As a consequence, for $1 < r < 2$ the matrix $L_r$ has one positive and $n - 1$ negative eigenvalues, and for $2 < r < 3$ it has one negative and $n - 1$ positive eigenvalues.

Here is the converse of Theorems 1.1 and 1.2:

**Theorem 1.4.** Let $f$ be a $C^2$ function from $(0, \infty)$ into itself with $f(0) = f'(0) = 0$. Suppose all Loewner matrices $L_f$ are conditionally negative definite. Then $f$ is operator convex.

**Theorem 1.5.** Let $f$ be a $C^3$ function from $(0, \infty)$ into itself with $f(0) = f'(0) = f''(0) = 0$. Suppose all Loewner matrices $L_f$ are conditionally positive definite. Then there exists an operator convex function $g$ such that $f(t) = tg(t)$.

**Remark.** Theorems 1.1, 1.2, 1.4 and 1.5 together say the following. Let $f$ be a $C^3$ function from $(0, \infty)$ into itself with $f(0) = 0$. Let $g(t) = tf(t), h(t) = t^2f(t)$. Then the following three conditions are equivalent.
(i) All Loewner matrices $L_f$ are p.s.d.
(ii) All Loewner matrices $L_g$ are c.n.d.
(iii) All Loewner matrices $L_h$ are c.p.d.

2 Generalisations by Hiai-Sano

We already review characterizations in [3] for operator convexity of nonnegative functions on $[0, \infty)$ in terms of the conditional negative or positive definiteness of the Loewner matrices. Uchiyama [10] extended, by a rather different method, results in such a way that the assumption $f \geq 0$ is removed and the boundary condition $f(0) = f'(0) = 0$ is relaxed. Note that the conditional positive definiteness of the Loewner matrices and the matrix/operator monotony were related in [7] and [4, Chapter XV] for a real function on a general open interval.

We proved:

**Theorem 2.1.** Let $f$ be a real $C^1$ function on $(0, \infty)$. For each $n \in \mathbb{N}$ consider the following conditions:

(a)$_n$ $f$ is $n$-convex on $(0, \infty)$;
(b)$_n$ $\liminf_{t \to \infty} f(t)/t > -\infty$ and $L_f(t_1, \ldots, t_n)$ is c.n.d. for all $t_1, \ldots, t_n \in (0, \infty)$;
(c)$_n$ $\limsup_{t \searrow 0} tf(t) \geq 0$ and $L_{tf(t)}(t_1, \ldots, t_n)$ is c.p.d. for all $t_1, \ldots, t_n \in (0, \infty)$.

Then for every $n \in \mathbb{N}$ the following implications hold:

$$(a)_{2n+1} \Rightarrow (b)_n, \quad (b)_{4n+1} \Rightarrow (a)_n, \quad (a)_{n+1} \Rightarrow (c)_n, \quad (c)_{2n+1} \Rightarrow (a)_n.$$ 

**Corollary 2.2.** Let $f$ be a real $C^1$ function on $(0, \infty)$. Then the following conditions are equivalent:

(a) $f$ is operator convex on $(0, \infty)$;
(b) $\liminf_{t \to \infty} f(t)/t > -\infty$ and $L_f(t_1, \ldots, t_n)$ is c.n.d. for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in (0, \infty)$;
(c) $\limsup_{t \searrow 0} tf(t) \geq 0$ and $L_{tf(t)}(t_1, \ldots, t_n)$ is c.p.d. for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in (0, \infty)$.

Moreover, if the above conditions are satisfied, then $\lim_{t \to \infty} f(t)/t$ and $\lim_{t \searrow 0} tf(t)$ exist in $(-\infty, \infty]$ and $[0, \infty)$, respectively.

**Theorem 2.3.** Let $f$ be a real $C^1$ function on $(0, \infty)$. For each $n \in \mathbb{N}$ consider the following conditions:
(a) \[ f \text{ is } n\text{-monotone on } (0, \infty); \]

(b) \[ \limsup_{t \to \infty} f(t)/t < +\infty, \limsup_{t \to \infty} f(t) > -\infty, \text{ and } L_f(t_1, \ldots, t_n) \text{ is c.p.d. for all } t_1, \ldots, t_n \in (0, \infty); \]

(c) \[ \liminf_{t \searrow 0} tf(t) \leq 0, \limsup_{t \to \infty} f(t) > -\infty, \text{ and } L_{tf(t)}(t_1, \ldots, t_n) \text{ is c.p.d. for all } t_1, \ldots, t_n \in (0, \infty); \]

(d) \[ \liminf_{t \searrow 0} tf(t) \leq 0, \limsup_{t \searrow 0} t^2f(t) \geq 0, \text{ and } L_{t^2f(t)}(t_1, \ldots, t_n) \text{ is c.p.d. for all } t_1, \ldots, t_n \in (0, \infty). \]

Then for every \( n \in \mathbb{N} \) the following implications hold:

\[ (a)_n' \Rightarrow (b)_n', \quad (b)_{4n+1}' \Rightarrow (a)_{n}', \quad (a)_{2n+2}' \Rightarrow (c)_n', \quad (c)_{2n+1}' \Rightarrow (a)_{n}', \quad (a)_{n}' \Rightarrow (d)_{n}', \quad (c)_{2n+1}' \Rightarrow (d)_n', \quad (d)_{2n+1}' \Rightarrow (c)_n'. \]

**Corollary 2.4.** Let \( f \) be a real \( C^1 \) function on \((0, \infty)\). Then the following conditions are equivalent:

(a)' \[ f \text{ is operator monotone on } (0, \infty); \]

(b)' \[ \limsup_{t \to \infty} f(t)/t < +\infty, \limsup_{t \to \infty} f(t) > -\infty, \text{ and } L_f(t_1, \ldots, t_n) \text{ is c.p.d. for all } n \in \mathbb{N} \text{ and all } t_1, \ldots, t_n \in (0, \infty); \]

(c)' \[ \liminf_{t \searrow 0} tf(t) \leq 0, \limsup_{t \to \infty} f(t) > -\infty, \text{ and } L_{tf(t)}(t_1, \ldots, t_n) \text{ is c.p.d. for all } n \in \mathbb{N} \text{ and all } t_1, \ldots, t_n \in (0, \infty); \]

(d)' \[ \liminf_{t \searrow 0} tf(t) \leq 0, \limsup_{t \searrow 0} t^2f(t) \geq 0, \text{ and } L_{t^2f(t)}(t_1, \ldots, t_n) \text{ is c.p.d. for all } n \in \mathbb{N} \text{ and all } t_1, \ldots, t_n \in (0, \infty). \]

Moreover, if the above conditions are satisfied, then \( \lim_{t \to \infty} f(t)/t, \lim_{t \to \infty} f(t), \) and \( \lim_{t \searrow 0} tf(t) \) exist in \([0, \infty), (-\infty, \infty], \) and \((-\infty, 0], \) respectively, and \( \lim_{t \searrow 0} t^\alpha f(t) = 0 \) for any \( \alpha > 1. \)

**Proposition 2.5.** Consider the power functions \( t^\alpha \) on \((0, \infty)\), where \( \alpha \in \mathbb{R} \). Then:

1. \( t^\alpha \) is 2-monotone if and only if \( 0 \leq \alpha \leq 1 \), or equivalently, \( t^\alpha \) is operator monotone. Moreover, \( -t^\alpha \) is 2-monotone if and only if \(-1 \leq \alpha \leq 0. \)

2. \( t^\alpha \) is 2-convex if and only if either \(-1 \leq \alpha \leq 0 \) or \( 1 \leq \alpha \leq 2 \), or equivalently, \( t^\alpha \) is operator convex.

3. \( L_{t^\alpha}(t_1, t_2) \) is c.p.d. for all \( t_1, t_2 \in (0, \infty) \) if and only if either \( 0 \leq \alpha \leq 1 \) or \( \alpha \geq 2. \)

4. \( L_{t^\alpha}(t_1, t_2) \) is c.n.d. for all \( t_1, t_2 \in (0, \infty) \) if and only if either \( \alpha \leq 0 \) or \( 1 \leq \alpha \leq 2. \)
(5) $L_{t^\alpha}(t_1, t_2, t_3)$ is c.p.d. for all $t_1, t_2, t_3 \in (0, \infty)$ if and only if either $0 \leq \alpha \leq 1$ or $2 \leq \alpha \leq 3$.

(6) $L_{t^\alpha}(t_1, t_2, t_3)$ is c.n.d. for all $t_1, t_2, t_3 \in (0, \infty)$ if and only if either $-1 \leq \alpha \leq 0$ or $1 \leq \alpha \leq 2$.

**Theorem 2.6.** Let $f$ be a real $C^1$ function on $(a, b)$ where $-\infty < a < b < \infty$. For each $n \in \mathbb{N}$ consider the following conditions:

$(\alpha)_n$ $f$ is $n$-monotone on $(a, b)$;

$(\beta)_n$ $\limsup_{t \searrow b} (b-t)f(t) < +\infty$, $\limsup_{t \nearrow b} f(t) > -\infty$, and $L_{(b-t)^2 f(t)}(t_1, \ldots, t_n)$ is c.p.d. for all $t_1, \ldots, t_n \in (a, b)$;

$(\gamma)_n$ $\liminf_{t \searrow a} (t-a)f(t) \leq 0$, $\limsup_{t \nearrow b} f(t) > -\infty$, and $L_{(t-a)(b-t)f(t)}(t_1, \ldots, t_n)$ is c.n.d. for all $t_1, \ldots, t_n \in (a, b)$;

$(\delta)_n$ $\liminf_{t \searrow a} (t-a)f(t) \leq 0$, $\limsup_{t \nearrow a} (t-a)^2 f(t) \geq 0$, and $L_{(t-a)^2 f(t)}(t_1, \ldots, t_n)$ is c.p.d. for all $t_1, \ldots, t_n \in (a, b)$.

Then for every $n \in \mathbb{N}$ the following implications hold:

$(\alpha)_n \implies (\beta)_n$ if $n \geq 2$, $(\beta)_{4n+1} \implies (\alpha)_n$, $(\alpha)_{2n+2} \implies (\gamma)_n$, $(\gamma)_{2n+1} \implies (\alpha)_n$,

$(\alpha)_n \implies (\delta)_n$ if $n \geq 2$, $(\gamma)_{2n+1} \implies (\delta)_n$, $(\delta)_{2n+1} \implies (\gamma)_n$.

**Corollary 2.7.** Let $f$ be a real $C^1$ function on $(a, b)$ where $-\infty < a < b < \infty$. Then the following conditions are equivalent:

$(\alpha)$ $f$ is operator monotone on $(a, b)$;

$(\beta)$ $\limsup_{t \searrow b} (b-t)f(t) < +\infty$, $\limsup_{t \nearrow b} f(t) > -\infty$, and $L_{(b-t)^2 f(t)}(t_1, \ldots, t_n)$ is c.p.d. for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in (a, b)$;

$(\gamma)$ $\liminf_{t \searrow a} (t-a)f(t) \leq 0$, $\limsup_{t \nearrow b} f(t) > -\infty$, and $L_{(t-a)(b-t)f(t)}(t_1, \ldots, t_n)$ is c.n.d. for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in (a, b)$;

$(\delta)$ $\liminf_{t \searrow a} (t-a)f(t) \leq 0$, $\limsup_{t \nearrow a} (t-a)^2 f(t) \geq 0$, and $L_{(t-a)^2 f(t)}(t_1, \ldots, t_n)$ is c.p.d. for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in (a, b)$.

**References**


