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Orbits of operators and operator semigroups

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1. Introduction

Denote by $B(X)$ the set of all bounded linear operators acting on a Banach space X. For simplicity we assume that all Banach spaces are complex unless stated explicitly otherwise. However, the notions make sense also in real Banach spaces and most of the results remain true (with slight modifications) in the real case.

Let X be a Banach space and let T be a bounded linear operator on X. By an orbit of T we mean a sequence of the form $(T^n x)_{n=0}^{\infty}$, where $x \in X$ is a fixed vector.

By a weak orbit of T we mean a sequence of the form $(\langle T^n x, x^* \rangle)_{n=0}^{\infty}$, where $x \in X$ and x^* are fixed vectors.

Orbits of operators appear frequently in operator theory. They are closely connected with the famous invariant subspace/subset problem, they play a central role in the linear dynamics and appear also in other branches of operator theory, for example in the local spectral theory or in the theory of operator semigroup.

Typically, the behaviour of an orbit $(T^n x)$ depends essentially on the choice of the initial vector x. This can be illustrated by the following simple example:

Example 1.1. Let S be the backward shift on a Hilbert space H, i.e., $Se_0 = 0$ and $Se_i = e_{i-1}\quad (i \geq 1)$, where $\{e_i : i = 0, 1, 2, \dots\}$ is an orthonormal basis in H. Let $T = 2S$. Then:

(i) there is a dense subset of points $x \in H$ such that $\|T^n x\| \to 0$;
(ii) there is a dense subset of points $x \in H$ such that $\|T^n x\| \to \infty$;
(iii) there is a residual subset (= complement of a set of the first category) of points $x \in H$ such that the set $\{T^n x : n = 0, 1, \dots\}$ is dense in H.

The first statement is very simple — any finite linear combination of the basis vectors e_n satisfies property (i). The second and third statements are nontrivial and follow from general results of the theory of orbits.

The aim of this note is to give a survey of results concerning the behaviour of orbits. In the next section we discuss the connections of orbits with the invariant subspace/subset problem. In Section 3 we give a brief survey of results concerning vectors with very irregular orbits (type (iii) of Example 1.1). In Section 4 we study vectors with large orbits - type (ii). Weak orbits will be discussed in Section 5.

2. Invariant subspace/subset problem

Let $T \in B(X)$. A non-empty subset $M \subset X$ is called invariant for T if $TM \subset M$. The set M is non-trivial if $\{0\} \neq M \neq X$ (the trivial subsets $\{0\}$ and X are

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always invariant for any operator $T \in B(X)$). The invariant subspace problem may be formulated as follows:

Problem 2.1. Let T be an operator on a Hilbert space H of dimension ≥ 2. Does there exist a non-trivial closed subspace invariant for T?

It is easy to see that the problem has sense only for separable infinite-dimensional spaces. Indeed, if H is non-separable and $x \in H$ any non-zero vector, then the vectors $x, Tx, T^2x, \dots$ span a non-trivial closed subspace invariant for T.

If $\dim H < \infty$, then T has at least one eigenvalue and the corresponding eigenvector generates an invariant subspace of dimension 1. Note that the existence of eigenvalues is equivalent to the fundamental theorem of algebra that each non-constant complex polynomial has a root. Thus the invariant subspace problem is non-trivial even for finite-dimensional spaces.

Examples of Banach space operators without non-trivial closed invariant subspaces were given by Enflo [E], Beuzamy [Be1] and Read [R1]. Read [R2] also gave an example of an operator T (acting on ℓ^1) with a stronger property that T has no non-trivial closed invariant subset.

It is not known whether such an operator exists on a Hilbert space. The following invariant subset problem may be easier than Problem 2.1.

Problem 2.2. (invariant subset problem) Let T be an operator on a Hilbert space H. Does there exist a non-trivial closed subset invariant for T?

Both Problems 2.1 and 2.2 are also open for operators on reflexive Banach spaces. More generally, the following problem is open:

Problem 2.3. Let T be an operator on a Banach space X. Does T^* have a non-trivial closed invariant subset/subspace?

The existence of non-trivial invariant subspaces/subsets is closely connected with the behaviour of orbits. It is easy to see that an operator $T \in B(X)$ has no non-trivial closed invariant subspace if and only if all orbits corresponding to non-zero vectors span all the space X (i.e., each non-zero vector is cyclic).

Similarly, $T \in B(X)$ has no non-trivial closed invariant subset if and only if all orbits corresponding to non-zero vectors are dense, i.e., all non-zero vectors are hypercyclic.

Thus orbits provide the basic information about the structure of an operator.

The simplest way of constructing nontrivial closed invariant subsets/subspaces is to construct a vector whose orbit has a special properties. For example, if a vector x satisfies $\|T^n x\| \to \infty$ then the orbit $\{T^n x : n = 0, 1, \dots\}$ is a nontrivial closed invariant subset. Similarly, if x is a nonzero vector with $\|T^n x\| \to 0$ then $\{T^n x : n = 0, 1, \dots\} \cup \{0\}$ is a nontrivial closed invariant subset. So vectors of type (i) and (ii) of Example 1.1 provide nontrivial closed invariant subsets.

Construction of more structured invariant subsets (for example subspaces) requires usually to consider weak orbits. The basic idea of the Scott Brown technique is to construct a weak orbit with the property $\langle T^n x, x^* \rangle = 0 \quad (n \geq 1)$ and $\langle x, x^* \rangle = 1$.

Then the subspace generated by the vectors $T^n x, n = 1, 2, \ldots$ is a nontrivial closed invariant subspace.

Similarly, if we manage to construct a weak orbit with the property $\operatorname{Re}\langle T^n x, x^* \rangle \geq 0$ for all $n \geq 0$, then the set $\{\sum \alpha_n x_n : \alpha_n \geq 0\}^-$ forms a nontrivial closed invariant cone, see Section 5.

Note that it is very simple to find an operator such that almost all (up to a set of the first category) vectors are hypercyclic, but it is very difficult to find an operator such that all non-zero vectors are hypercyclic.

3. Hypercyclic vectors

Let $T \in B(X)$. As mentioned in the previous section, a vector $x \in X$ is called hypercyclic for T if the set $\{T^n x : n = 0, 1, \ldots\}$ is dense in X. An operator $T \in B(X)$ is called hypercyclic if there exists at least one vector $x \in X$ hypercyclic for T.

Hypercyclic vectors have been studied intensely in the last years. In this section we give a brief survey of results concerning hypercyclic vectors. For more information see the monograph [BM2].

The first example of a hypercyclic vector was given by Rolewicz [Ro]. Although hypercyclic vectors seem at first glance to be rather strange and exceptional, they are quite common and in some sense it is a typical property of a vector.

The notion has sense only in separable Banach spaces. Clearly, in non-separable Banach spaces there are no hypercyclic operators.

It is easy to find an operator that has no hypercyclic vectors. For example, if $\|T\| \leq 1$, then all orbits are bounded, and therefore not dense. On the other hand, if T is hypercyclic, then almost all vectors are hypercyclic for T.

Theorem 3.1. *Let $T \in B(X)$ be a hypercyclic operator. Then the set of all vectors $x \in X$ that are hypercyclic for T is a dense G_δ set, and hence residual in X.*

Indeed, let $x \in X$ be a vector hypercyclic for T. For any $n \in \mathbb{N}$, the vector $T^n x$ is also hypercyclic for T and therefore the set of all vectors hypercyclic for T is dense in X.

Note that the space X is separable. Let (U_j) be a countable base of open sets in X. A vector $u \in X$ is hypercyclic for T if and only if it belongs to the set $\bigcap_{j=1}^\infty \left(\bigcup_{n=0}^\infty T^{-n} U_j \right)$, which is a G_δ set.

Lemma 3.2. *Let $T \in B(X)$ be a hypercyclic operator. Then the point spectrum $\sigma_p(T^*)$ is empty.*

Indeed, suppose on the contrary that $\lambda \in \mathbb{C}$ belongs to the point spectrum of T^*. Let $x^* \in X^*$ be a corresponding eigenvector, i.e., $x^* \neq 0$ and $T^* x^* = \lambda x^*$.

Let $x \in X$ be a vector hypercyclic for T. Then the set $\{\langle T^n x, x^* \rangle : n = 0, 1, \ldots\}$ is dense in $\mathbb{C}$. We have

$$\{\langle T^n x, x^* \rangle : n = 0, 1, \ldots\} = \{\langle x, T^{*n} x^* \rangle : n = 0, 1, \ldots\}$$

$$\{\lambda^n \langle x, x^* \rangle : n = 0, 1, \ldots\}.$$

The last set is bounded if either $|\lambda| \le 1$ or $\langle x, x^* \rangle = 0$. If $|\lambda| > 1$ and $\langle x, x^* \rangle \ne 0$, then $|\lambda^n \langle x, x^* \rangle| \to \infty$. So the set $\{\lambda^n \langle x, x^* \rangle : n = 0, 1, \dots\}$ cannot be dense in $\mathbb{C}$, which is a contradiction.

Corollary 3.3. *Let* $\dim X < \infty$. *Then there are no hypercyclic operators acting in* X.

Theorem 3.4. *Let* $T \in B(X)$ *be a hypercyclic operator. Then there exists a dense linear manifold* $M \subset X$ *such that each non-zero vector* $x \in M$ *is hypercyclic for* T.

Indeed, let $x \in X$ be a vector hypercyclic for T. Let

$$M = \{q(T)x : q \text{ a polynomial}\}.$$

Clearly M is a dense linear manifold since it contains the orbit $\{T^n x : n = 0, 1, \dots\}$. We show that $q(T)x$ is hypercyclic for T for each non-zero polynomial q.

Write $q(z) = \beta(z - \alpha_1) \cdots (z - \alpha_n)$, where $n \ge 0$, $\alpha_1, \dots, \alpha_n$ are the roots of q and $\beta \ne 0$. Since $\sigma_p(T^*) = \emptyset$, the operators $T - \alpha_i$ have dense ranges. Hence $q(T)$ has also dense range. We have

$$\{T^n q(T)x : n = 0, 1, \dots\} = q(T)\{T^n x : n = 0, 1, \dots\},$$

which is dense in X, since $\{T^n x : n = 0, 1, \dots\}$ is dense in X.

The next theorem provides a criterion for hypercyclicity of an operator, see [K]. Denote by $B_X = \{x \in X : \|x\| \le 1\}$ the closed unit ball in a Banach space X.

Theorem 3.5. *Let* X *be a separable Banach space. Let* $T \in B(X)$. *Suppose that there exists an increasing sequence of positive integers* (n_k) *such that the following two conditions are satisfied:*

(i) there exists a dense subset $X_0 \subset X$ *such that* $\lim_{k\to\infty} T^{n_k} x = 0 \quad (x \in X_0)$;

(ii) $\overline{\bigcup_k T^{n_k} B_X} = X$.

Then T *is hypercyclic.*

The criterion provided by Theorem 3.5 is usually easy to apply. For example, it implies easily the hypercyclicity of the operator $T = 2S$, where S is the backward shift, see Example 1.1.

In the same way it is possible to obtain the hypercyclicity of any weighted backward shift with weights w_i which satisfy $\sup_n (w_1 \cdots w_n) = \infty$.

It was a longstanding open problem whether there are hypercyclic operators that do not satisfy the conditions of Theorem 3.5. The problem has several equivalent formulations. The simplest formulation is whether there exists a hypercyclic operator $T \in B(X)$ such that $T \oplus T \in B(X \oplus X)$ is not hypercyclic. The problem was solved recently by [DR], see also [BM1], where such an operator was constructed in any space $\ell^p \quad (1 \le p < \infty)$ or c_0.

The next result [GS] shows that an operator acting on a separable Banach space is hypercyclic if and only if it is topologically transitive.

Theorem 3.6. *Let X be a separable Banach space and let $T \in B(X)$. Then T is hypercyclic if and only if, for all non-empty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$.*

Indeed, suppose that T is hypercyclic and let U, V be non-empty open subsets of X. Since the set of all hypercyclic vectors is dense, there is an $x \in U$ hypercyclic for T. Therefore there is an $n \in \mathbb{N}$ such that $T^n x \in V$, and so $T^n U \cap V \neq \emptyset$.

Conversely, suppose that $T^n U \cap V \neq \emptyset$ for all non-empty open subsets U, V of X. Let (U_j) be a countable basis of open subsets of X. For each j let $M_j = \bigcup_{n \in \mathbb{N}} T^{-n} U_j$. Clearly, M_j is open. We show that it is also dense.

Let $y \in X$ and $\varepsilon > 0$. By assumption, there are $n \in \mathbb{N}$ and $x \in X$, $\|x - y\| < \varepsilon$ such that $T^n x \in U_j$. Thus $x \in M_j$ and M_j is dense.

By the Baire category theorem, $\bigcap_j M_j$ is non-empty and clearly each vector in $\bigcap_j M_j$ is hypercyclic for T.

For $T \in B(X)$ and $x \in X$ write $\operatorname{Orb}(T, x) = \{T^n x : n = 0, 1, \ldots\}$. For $y \in X$ and $\varepsilon > 0$ denote by $B(y, \varepsilon) = \{u \in X : \|u - y\| < \varepsilon\}$ the open ball with center y and radius ε.

The next theorem shows that for hypercyclicity of an operator it is sufficient to have a vector whose orbit is a d-net for some $d > 0$.

Theorem 3.7. [F] *Let $T \in B(X)$, $d > 0$ and let $x \in X$ satisfy that for each $y \in X$ there is an $n \in \mathbb{N}$ with $\|T^n x - y\| < d$. Then T is hypercyclic.*

The following deep result show that if an orbit is somewhere dense then it is everywhere dense.

Theorem 3.8. [BF] *Let $T \in B(X)$ and $x \in X$. Suppose that $\overline{\operatorname{Orb}(T, x)}$ has non-empty interior. Then x is hypercyclic for T.*

Theorem 3.8 implies easily the following interesting result [A].

Corollary 3.9. *Let $T \in B(X)$ be a hypercyclic operator. Then for every positive n, the operator T^n is also hypercyclic. Moreover, T and T^n share the same collection of hypercyclic vectors.*

Indeed, let x be hypercyclic for T. We have $\operatorname{Orb}(T, x) = \bigcup_{j=0}^{n-1} \operatorname{Orb}(T^n, T^j x)$, and so $X = \overline{\operatorname{Orb}(T, x)} = \bigcup_{j=0}^{n-1} \overline{\operatorname{Orb}(T^n, T^j x)}$. By the Baire category theorem, there is a k, $0 \leq k \leq n - 1$ such that $\overline{\operatorname{Orb}(T^n, T^k x)}$ has non-empty interior. By Theorem 3.8, $T^k x$ is hypercyclic for T^n. Since T has dense range, the set $T^{n-k} \operatorname{Orb}(T^n, T^k x) = \operatorname{Orb}(T^n, T^n x)$ is also dense. So $T^n x$ is hypercyclic for T^n, and so x is also hypercyclic for T^n.

Let $T \in B(X)$ and let $x \in X$ be a hypercyclic vector for T. Then x is also hypercyclic for $-T$. Indeed,

$$\operatorname{Orb}(-T, x) \supset \{(-T)^{2n} x : n = 0, 1, \ldots\} = \operatorname{Orb}(T^2, x),$$

which is dense by Corollary 3.9.

Similarly, one can show that x is hypercyclic for each operator λT, where $\lambda = e^{2\pi it}$ with t rational, $0 \le t < 1$. The next result shows that in fact x is hypercyclic for each operator λT with $|\lambda| = 1$.

It is easy to show that in general x is not hypercyclic for λT if $|\lambda| \ne 1$.

Theorem 3.10. [LM] *Let $T \in B(X)$ be a hypercyclic operator and let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then λT is also hypercyclic. Moreover, T and λT share the same collection of hypercyclic vectors.*

Many results for orbits can be formulated also for strongly continuous semigroup of operators. A strongly continuous semigroup of operators on a Banach space X is a collection $\{T(t) : t \ge 0\} \subset B(X)$ such that $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t, s \ge 0$ and the mapping $t \mapsto T(t)$ is continuous in the strong operator topology.

Let $\mathcal{T} = (T(T)_{t \ge 0}$ be a strongly continuous semigroup on a Banach space X. A vector $x \in X$ is called hypercyclic for the semigroup $\mathcal{T}$ if the set $\{T(t)x : t \ge 0\}$ is dense in X.

The previous result has a generalization for semigroups.

Theorem 3.11. [CMP] *Let $\mathcal{T} = (T(T)_{t \ge 0}$ be a strongly continuous semigroup on a Banach space X. Let $x \in X$ be a vector hypercyclic for the semigroup $\mathcal{T}$. Then x is hypercyclic for each operator $T(t_0)$ $(t_0 > 0)$.*

Equivalently, if $\{T(t)x : t \ge 0\}^- = X$ then $\{T(nt_0)x : n = 0, 1, \ldots\}^- = X$ for each $t_0 > 0$.

4. Large orbits

In this section we study orbits that are "large" in some sense (e.g., of type (ii)). As mentioned above, orbits satisfying $\|T^n x\| \to \infty$ provide a simple example of a non-trivial closed invariant subset.

It is an easy consequence of the Banach-Steinhaus theorem that an operator $T \in B(X)$ has unbounded orbits if and only if $\sup \|T^n\| = \infty$.

More precisely, it is possible to prove the following stronger result:

Theorem 4.1. *Let $T \in B(X)$, let $(a_n)_{n \ge 0}$ be a sequence of positive numbers such that $a_n \to 0$. Then the set of all $x \in X$ with the property that*

$$\|T^n x\| \ge a_n \|T^n\| \qquad \text{for infinitely many powers } n$$

is residual.

Indeed, for $k \in \mathbb{N}$ set

$$M_k = \{x \in X : \text{ there exists } n \ge k \text{ such that } \|T^n x\| > a_n \|T^n\|\}.$$

Clearly, M_k is an open set. We prove that M_k is dense. Let $x \in X$ and $\varepsilon > 0$. Choose $n \ge k$ such that $a_n \varepsilon^{-1} < 1$. There exists $z \in X$ of norm 1 such that $\|T^n z\| > a_n \varepsilon^{-1} \|T^n\|$. Then

$$2a_n \|T^n\| < \|T^n(2\varepsilon z)\| \le \|T^n(x + \varepsilon z)\| + \|T^n(x - \varepsilon z)\|,$$

and so either $\|T^n(x+\varepsilon z)\| > a_n\|T^n\|$ or $\|T^n(x-\varepsilon z)\| > a_n\|T^n\|$. Thus either $x+\varepsilon z \in M_k$ or $x-\varepsilon z \in M_k$, and so dist $\{x, M_k\} \le \varepsilon$. Since x and ε were arbitrary, the set M_k is dense.

By the Baire category theorem, the intersection $\bigcap_{k=1}^{\infty} M_k$ is a dense G_δ-set, hence it is residual. Clearly, each $x \in \bigcap_{k=1}^{\infty} M_k$ satisfies $\|T^n x\| > a_n\|T^n\|$ for infinitely many powers n.

Definition 4.2. Let $T \in B(X)$ and $x \in X$. The local spectral radius of T at x is defined by $r_x(T) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$.

Recall that the spectral radius $r(T)$ of T satisfies $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$.

It is easy to see that $r_x(T) \le r(T)$ for each $x \in X$.

However, if we set $a_n = n^{-1}$ in the previous result, we obtain

$$r_x(T) = \limsup_{n\to\infty} \|T^n x\|^{1/n} \ge \limsup_{n\to\infty} \left(\frac{\|T^n\|}{n}\right)^{1/n} = r(T)$$

for each x satisfying $\|T^n x\| \ge n^{-1}\|T^n\|$ for infinitely many n.

Corollary 4.3. *Let $T \in B(X)$. Then the set $\{x \in X : r_x(T) = r(T)\}$ is residual.*

In fact, a much stronger result is also true: there are points $x \in X$ such that all powers $T^n x$ are "large" in the norm.

Theorem 4.4. [M1], see also [Be2] *Let $T \in B(X)$, let $(a_j)_{j=0}^{\infty}$ be a sequence of positive numbers satisfying $\lim_{j\to\infty} a_j = 0$. Then:*

(i) for each $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| \le \sup\{a_j : j = 0, 1, \dots\} + \varepsilon$ and $\|T^j x\| \ge a_j r(T^j)$ for all $j \ge 0$;

(ii) there is a dense subset L of X with the following property: for each $y \in L$ we have $\|T^j y\| \ge a_j r(T^j)$ for all n sufficiently large.

Corollary 4.5. *The set $\{x \in X : \limsup_{n\to\infty} \|T^n x\|^{1/n} = r(T)\}$ is residual for each $T \in B(X)$. The set $\{x \in X : \liminf_{n\to\infty} \|T^n x\|^{1/n} = r(T)\}$ is dense.*

In particular, there is a dense subset of points $x \in X$ with the property that the limit $\lim_{n\to\infty} \|T^n x\|^{1/n}$ exists and is equal to $r(T)$.

Corollary 4.6. *Let $T \in B(X)$. Then*

$$\sup_{\substack{x\in X\\ \|x\|=1}} \inf_{n\ge 1} \|T^n x\|^{1/n} = \inf_{n\ge 1} \sup_{\substack{x\in X\\ \|x\|=1}} \|T^n x\|^{1/n} = r(T).$$

In general it is not possible to replace the word "dense" in Corollary 4.5 by "residual".

Example 4.7 Let H be a separable Hilbert space with an orthonormal basis $\{e_j : j = 0, 1, \dots\}$ and let S be the backward shift, $Se_j = e_{j-1}$ $(j \ge 1)$, $Se_0 = 0$. Then $r(S) = 1$ and the set $\{x \in H : \liminf_{n\to\infty} \|S^n x\|^{1/n} = 0\}$ is residual.

In particular, the set $\{x \in H :\ \text{the limit} \lim_{n\to\infty} \|S^n x\|^{1/n} \text{ exists}\}$ is of the first category (but it is always dense by Corollary 9).

Indeed, for $k \in \mathbb{N}$ let

$$M_k = \{x \in X :\ \text{there exists } n \geq k \text{ such that } \|S^n x\| < k^{-n}\}.$$

Clearly, M_k is an open subset of X. Further, M_k is dense in X. To see this, let $x \in X$ and $\varepsilon > 0$. Let $x = \sum_{j=0}^{\infty} \alpha_j e_j$ and choose $n \geq k$ such that $\sum_{j=n}^{\infty} |\alpha_j|^2 < \varepsilon^2$. Set $y = \sum_{j=0}^{n-1} \alpha_j e_j$. Then $\|y - x\| < \varepsilon$ and $S^n y = 0$. Thus $y \in M_k$ and M_k is a dense open subset of X.

By the Baire category theorem, the intersection $M = \bigcap_{k=0}^{\infty} M_k$ is a dense G_δ-subset of X, hence it is residual.

Let $x \in M$. For each $k \in \mathbb{N}$ there is an $n_k \geq k$ such that $\|S^{n_k} x\| < k^{-n_k}$, and so $\liminf_{n\to\infty} \|S^n x\|^{1/n} = 0$.

Since the set $\{x \in H : \limsup_{n\to\infty} \|S^n x\|^{1/n} = r(S) = 1\}$ is also residual, we see that the set $\{x \in H :\ \text{the limit } \lim_{n\to\infty} \|S^n x\|^{1/n} \text{ exists}\}$ is of the first category.

Remark 4.8. If $r(T) = 1$ and $a_n > 0$, $a_n \to 0$, then Theorem 4.4 says that there exists x such that $\|T^n x\| \geq a_n$ for all n. This is the best possible result since the previous example $S \in B(H)$ satisfies $S^n x \to 0$ for all $x \in H$. By Theorem 4.4, there are orbits converging to 0 arbitrarily slowly.

Theorem 4.4 implies that there is always a dense subset of points x satisfying $\sum_j \frac{\|T^j x\|}{r(T^j)} = \infty$. In fact, the set of all points with this property is even residual.

Theorem 4.9. *Let $T \in B(X)$, $r(T) \neq 0$ and let $0 < p < \infty$. Then the set*

$$\left\{x \in X : \sum_{j=0}^{\infty} \left(\frac{\|T^j x\|}{r(T^j)}\right)^p = \infty\right\}$$

is residual.

In Theorem 4.1 we proved an estimate of $\|T^n x\|$ in terms of the norm $\|T^n\|$. It is also possible to construct points $x \in X$ with $\|T^n x\| \geq a_n \cdot \|T^n\|$ for all n; in this case, however, there is a restriction on the sequence (a_n). The results are based on the following interesting plank theorems, see [B1] and [B2].

Theorem 4.10. *Let X be a real or complex Banach space, $a_n \geq 0$, $\sum a_n < 1$, $f_n \in X^*$, $\|f_n\| = 1$ and let $y \in X$. Then there exists $x \in X$ such that $\|x - y\| = 1$ and*

$$|\langle x, f_n\rangle| \geq a_n$$

for all n.

Theorem 4.11. *Let X be a complex Hilbert space, $a_n \geq 0$, $\sum a_n \leq 1$, $f_n \in X$, $\|f_n\| = 1$. Then there exists $x \in X$ such that $\|x\| = 1$ and*

$$|\langle x, f_n\rangle| \geq a_n$$

for all n.

It is interesting to note that there is a difference between the real and complex Hilbert spaces. Real Hilbert spaces are not better than general Banach spaces.

The plank theorem implies easily the existence of large orbits of operators, see [MV].

Theorem 4.12. *Let X, Y be Banach spaces, let $(T_n) \subset B(X,Y)$ be a sequence of operators. Let (a_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Then there exists $x \in X$ such that $\|T_n x\| \geq a_n \|T_n\|$ for all n.*

Moreover, it is possible to choose such an x in each ball in X of radius greater than $\sum_{n=1}^{\infty} a_n$.

Corollary 4.13. *Let $T \in B(X)$ satisfy $\sum_{n=1}^{\infty} \|T^n\|^{-1} < \infty$. Then there exists a dense subset of points $x \in X$ such that $\|T^n x\| \to \infty$.*

Better results can be obtained for Hilbert space operators.

Theorem 4.14. *Let H, K be Hilbert spaces and let $T_n \in B(H,K)$ be a sequence of operators. Let a_n be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Let $\varepsilon > 0$. Then:*

(i) there exists $x \in H$ such that $\|x\| \leq \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} + \varepsilon$ and $\|T_n x\| \geq a_n \|T_n\|$ for all n;

(ii) there is a dense subset of vectors $x \in H$ such that $\|T_n x\| \geq a_n \|T_n\|$ for all n sufficiently large.

Corollary 4.15. *Let $T \in B(H)$ be a Hilbert space operator and $\sum_{n=1}^{\infty} \|T^n\|^{-2} < \infty$. Then there exists a dense subset of points $x \in H$ such that $\|T^n x\| \to \infty$.*

Corollary 4.16. *Let $T \in B(X)$ satisfy $\sum_{n=1}^{\infty} \|T^n\|^{-1} < \infty$. Then T has a non-trivial closed invariant subset.*

If X is a Hilbert space, then it is sufficient to assume that $\sum_{n=1}^{\infty} \|T^n\|^{-2} < \infty$.

Corollary 4.17. *Let $T \in B(X)$ satisfy $r(T) \neq 1$. Then T has a non-trivial closed invariant subset.*

Indeed, if $r(T) > 1$, then there exists an $x \in X$ with $\|T^n x\| \to \infty$. If $r(T) < 1$, then $\|T^n x\| \to 0$ for each $x \in X$. In both cases there are non-trivial closed invariant subsets.

The previous results are in some sense the best possible.

Example 4.18. [MV] There exists a Banach space X and an operator $T \in B(X)$ such that $\|T^n\| = n+1$ for all n, but there is no vector $x \in X$ with $\|T^n x\| \to \infty$.

There exists a Hilbert space operator T such that $\|T^n\| = \sqrt{n+1}$ for all n and there in no vector x with $\|T^n x\| \to \infty$.

It is also possible to consider orbits that are large in the sense of $\sum \frac{\|T^n x\|}{\|T^n\|}$.

Theorem 4.19. *Let $T \in B(X)$ be a non-nilpotent operator and $0 < p < 1$. Then the set*

$$\left\{x \in X : \sum_{j=0}^{\infty}\left(\frac{\|T^j x\|}{\|T^j\|}\right)^p = \infty\right\}$$

is residual.

The previous result is not true for $p = 1$.

Example 4.20. There are a Banach space X and a non-nilpotent operator $T \in B(X)$ such that $\sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} < \infty$ for all $x \in X$.

Many results from this section can be reformulated also for strongly continuous semigroup of operators.

Theorem 4.21. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X, let $f \in L^\infty$, $f \geq 0$, $\lim_{t\to\infty} f(t) = 0$.*

Then there exists $x \in X$ such that

$$\|T(t)x\| \geq f(t)e^{\omega_0 t} \qquad (t \geq 0)$$

where ω_0 is the growth bound of the semigroup $\mathcal{T}$ (note that $r(T(t)) = e^{\omega_0 t}$ for all $t \geq 0$).

Theorem 4.22. *Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X, let $f \in L^1$, $f(t) \searrow 0 \quad (t \to \infty)$.*

Then there exists $x \in X$ such that

$$\|T(t)x\| \geq f(t)\|T(t)\| \qquad (t \geq 0)$$

If X is a Hilbert space then it is possible to take $f \in L^2$.

Weak orbits

Some results for orbits can be generalized also for weak orbits.

The following result can be obtained by applying the plank theorem twice.

Theorem 5.1. [MV] *Let $T \in B(X)$, $a_n \geq 0$, $\sum \sqrt{a_n} < \infty$. Then there exist $x \in X$ and $x^* \in X^*$ such that*

$$|\langle T^n x, x^*\rangle| \geq a_n \|T^n\| \qquad (n \in \mathbb{N})$$

(if X is a Hilbert space, then it is sufficient to require that $\sum a_n < \infty$)

Generalizations of results based on the spectral theory (for example Theorem 4.4) are more complicated. Usually it is necessary to assume that T is power bounded or that $T^n \to 0$ in some sense.

Theorem 5.2. *Let H be a Hilbert space, $T \in B(X)$, $T^n \to 0$ in the weak operator topology, let $r(T) = 1$. Let (a_n) be any sequence of nonnegative numbers such that $a_n \to 0$. Then there exist $x, y \in H$ such that*

$$|\langle T^n x, y\rangle| \geq a_n \qquad (n \in \mathbb{N})$$

A better result can be obtained if we assume that $1 \in \sigma(T)$.

Theorem 5.3. *Let H be a Hilbert space, $T \in B(X)$, $T^n \to 0$ in the weak operator topology, $1 \in \sigma(T)$, let (a_n) be a sequence of nonnegative numbers, $a_n \to 0$. Then there exists $x \in H$ such that*

$$Re\ \langle T^n x, x\rangle \geq a_n \qquad (n \in \mathbb{N})$$

Let X be a Banach space. A subset $C \subset X$ is called a cone if $C + C \subset C$ and $tC \subset C$ for each $t \geq 0$.

Corollary 5.4. *Let H be a Hilbert space, let $T \in B(H)$ be power bounded, $1 \in \sigma(T)$. Then T has a nontrivial closed invariant cone.*

Similar results can be proved also for Banach space operators, see [M2].

Theorem 5.5. *Let X be a Banach space, $c_0 \not\subset X$, $T \in B(X)$, $T^n \to 0$ in the strong operator topology, $1 \in \sigma(T)$. Then there exist $x \in X, x^* \in X^*$ such that*

$$Re\ \langle T^n x, x^*\rangle \geq 0 \qquad (n \in \mathbb{N})$$

Corollary 5.6. *Let X be a reflexive Banach space, $T \in B(X)$ power bounded, $1 \in \sigma(T)$. Then T has a nontrivial closed invariant cone.*

It is also possible to prove some results concerning weak orbits for semigroups of operators.

Theorem 5.7. *Let H be a Hilbert space, $\mathcal{T} = (T(t))_{t\geq 0}$ a strongly continuous semigroup on H, $\mathcal{T}$ weakly stable (i.e., $T(t) \to 0$ in the weak operator topology), $\omega_0 = 0$. Let $f \in L^\infty$, $f(t) \searrow 0 \quad (t \to \infty)$. Then there exist $x, y \in H$ such that*

$$|\langle T(t)x, y\rangle| \geq f(t) \qquad (t \geq 0)$$

Moreover, one can take $\|x\| \leq \sup\{f(s) : s \geq 0\} + \varepsilon$.

The upper density of a subset $A \subset \mathbb{N}$ is defined by

$$\overline{\text{Dens}} = \limsup_{n\to\infty} \frac{1}{n} \text{card}\,\{a \in A : a \leq n\}.$$

Theorem 5.8. *Let $T \in B(X)$, $r(T) \geq 1$, $a_n \searrow 0$. Then there exist $x \in X$, $x^* \in X^*$ and a subset $A \subset \mathbb{N}$ with $\overline{\text{Dens}}\, A = 1$ such that*

$$Re\ \langle T^n x, x^* \rangle \geq a_n$$

for all $n \in A$.

The only known application of the plank theorem for weak orbits of operator semigroup is the following result.

Theorem 5.9. *Let H be a Hilbert space, $\mathcal{T} = (T(t))_{t \geq 0}$ uniformly continuous semigroup and $\varepsilon > 0$ Then there exist $x, y \in H$ such that*

$$|\langle T(t)x, y \rangle| \geq \frac{1}{(t+1)^{2+\varepsilon}} \|T(t)\|$$

for all $t \geq 0$.

Problem 5.10. Is it sufficient to assume in Theorem 5.9 that $\mathcal{T}$ is strongly continuous?

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