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Extremal structure of the set of absolute norms [1]

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Abstract. Recently, we have a series of papars about geometrical properties of absolute normalized norms on $\mathbb{R}^2$ (or on $\mathbb{C}^2$). In this note we describe the results about the extremal structure of the set of absolute normalized norms on $\mathbb{R}^2$.

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(|x|,|y|)\| = \|(x,y)\|$ for all $x, y \in \mathbb{R}$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are basic examples:

$$\|(x,y)\|_p = \begin{cases} (|x|^p + |y|^p)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \max\{|x|,|y|\}, & \text{if} \quad p = \infty. \end{cases}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{R}^2$, and let Ψ_2 be the set of all (continuous) convex functions on the unit interval $[0,1]$ with $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ for $t \in [0,1]$. It is well-known that AN_2 and Ψ_2 are in a one-to-one correspondence with $\psi(t) = \|(1-t,\ t)\|$ for $t \in [0,1]$ and

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right), & \text{if} \quad (x,y) \neq (0,0), \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

For $1 \leq p \leq \infty$, let ψ_p be the corresponding convex function with $\|\cdot\|_p$. Namely,

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \max\{1-t, t\}, & \text{if} \quad p = \infty. \end{cases}$$

Recently, geometrical properties of absolute normalized norms have been studied by several authors. For example, Saito, Kato and Takahashi in [9] calculated and

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estimated the von Neumann-Jordan constant for absolute normalized norms on $\mathbb{C}^2$ by considering Ψ_2. Mitani and Saito [7] calculated the James constant for absolute normalized norms on $\mathbb{R}^2$.

In this note we consider the extremal structure of the set AN_2 of absolute normalized norms on $\mathbb{R}^2$. Note here that the set AN_2 has the convex structure in the sense that $\|\cdot\|, \|\cdot\|' \in AN_2, 0 \le \lambda \le 1 \Rightarrow (1-\lambda)\|\cdot\| + \lambda\|\cdot\|' \in AN_2$. Moreover, the correspondence $\psi \to \|\cdot\|_\psi$ preserves the operation to take a convex combination. Namely, it holds that $(1-\lambda)\|\cdot\|_\psi + \lambda\|\cdot\|_{\psi'} = \|\cdot\|_{(1-\lambda)\psi+\lambda\psi'}$. So, $\psi, \psi' \in \Psi_2, 0 \le \lambda \le 1 \Rightarrow (1-\lambda)\psi + \lambda\psi' \in \Psi_2$.

Definition 1 *We call a norm $\|\cdot\| \in AN_2$ an extreme point of AN_2 if*

$$\|\cdot\| = \frac{1}{2}(\|\cdot\|' + \|\cdot\|''), \ \|\cdot\|', \|\cdot\|'' \in AN_2 \Rightarrow \|\cdot\|' = \|\cdot\|''.$$

Also we call a function $\psi \in \Psi_2$ an extreme point of Ψ_2 if

$$\psi = \frac{1}{2}(\psi' + \psi''), \ \psi', \psi'' \in \Psi_2 \Rightarrow \psi' = \psi''.$$

Example 1 Let

$$\psi(t) = \begin{cases} -\frac{2}{3}t + 1 & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ \frac{2}{3}t + \frac{1}{3} & \text{if} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

Then $\psi \in \Psi_2$. Put

$$\varphi(t) = 2\psi(t) - \psi_\infty(t)$$

It is clear that

$$\varphi(t) = \begin{cases} -\frac{1}{3}t + 1 & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ \frac{1}{3}t + \frac{2}{3} & \text{if} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

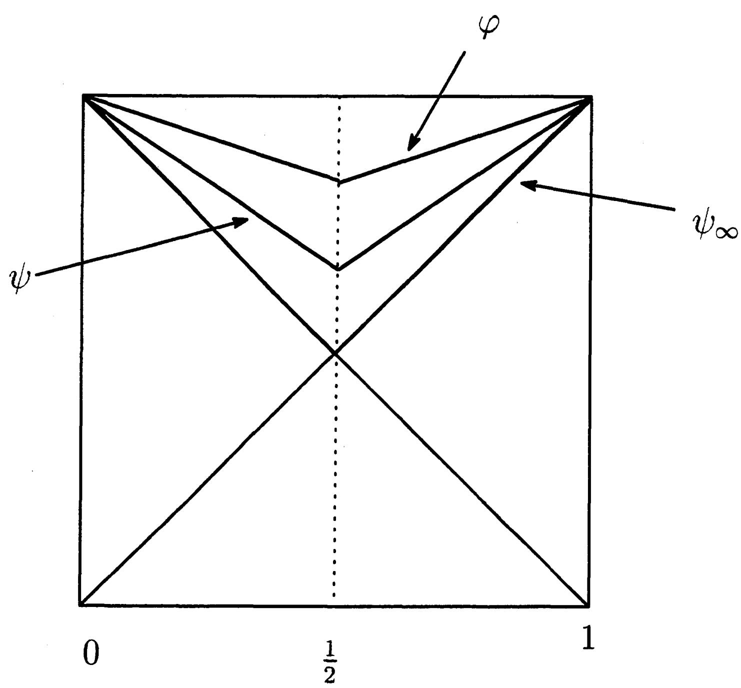
Then φ is convex on $[0,1]$. Hence $\varphi \in \Psi_2$. Note that

$$\|(x,y)\|_\psi = \max\left\{|x| + \frac{|y|}{3}, \frac{|x|}{3} + |y|\right\}$$

and

$$\|(x,y)\|_\varphi = \max\left\{|x| + \frac{2}{3}|y|, \frac{2}{3}|x| + |y|\right\}.$$

Hence $\psi = \frac{1}{2}(\varphi + \psi_\infty)$ and $\varphi \neq \psi_\infty$. Thus ψ is not an extreme point of Ψ_2 ($\|\cdot\|_\psi$ is not an extreme point of AN_2).



It is clear that ψ_1 (or ψ_∞) is an extreme point of Ψ_2. Let us consider the family of extreme points of AN_2. For $0 \le \alpha \le \frac{1}{2} < \beta \le 1$, we define

$$\psi_{\alpha,\beta}(t) = \begin{cases} 1-t & (0 \le t \le \alpha) \\ \dfrac{\alpha+\beta-1}{\beta-\alpha}t + \dfrac{\beta-2\alpha\beta}{\beta-\alpha} & (\alpha \le t \le \beta) \\ t & (\beta \le t \le 1). \end{cases}$$

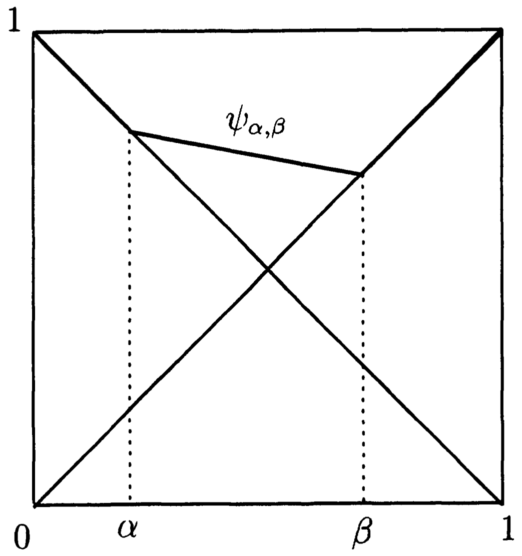
For $0 \le \alpha < \frac{1}{2} = \beta$ we put $\psi_{\alpha,\beta} = \psi_\infty$. Then $\psi_{\alpha,\beta} \in \Psi_2$ for all α, β. The corresponding norm is
$\|(x_1, x_2)\|_{\psi_{\alpha,\beta}}$

$$= \begin{cases} |x_1| & (|x_2| \le \frac{\alpha}{1-\alpha}|x_1|) \\ \frac{\beta(1-2\alpha)}{\beta-\alpha}|x_1| + \frac{(2\beta-1)(1-\alpha)}{\beta-\alpha}|x_2| & (\frac{\alpha}{1-\alpha}|x_1| \le |x_2|, \frac{1-\beta}{\beta}|x_2| \le |x_1|) \\ |x_2| & (\frac{1-\beta}{\beta}|x_2| \le 1). \end{cases}$$

We put $E = \{\psi_{\alpha,\beta} \in \Psi_2 : 0 \le \alpha \le \frac{1}{2} \le \beta \le 1\}$.

Then we have the following.

Theorem 1 ([5], cf. [3]) *The following are equivalent:*
(i) $\|\cdot\|_\psi$ *is an extreme point of* AN_2.
(ii) ψ *is an extreme point of* Ψ_2.
(iii) $\psi \in E$.



As applications we calculate the von Neumann-Jordan constant and the James constant of $(\mathbb{R}^2, \|\cdot\|)$ when $\|\cdot\|$ is a extreme point of AN_2. The von Neumann-Jordan constant of X was introduced by Clarkson as the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

for all $x, y \in X$ with $(x, y) \neq (0,0)$. For any Banach space X, we have $1 \leq C_{\mathrm{NJ}}(X) \leq 2$. (ii) X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$. (iii) If $1 \leq p \leq \infty$ and dim $L_p \geq 2$, then $C_{\mathrm{NJ}}(L_p) = 2^{2/\min\{p,q\}-1}$, where $1/p + 1/q = 1$.

Saito, Kato and Takahashi in [9] calculated the constant $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_\psi))$, as follows.

Proposition 1 ([9]) *Let $\psi \in \Psi_2$.*

(i) *If $\psi \geq \psi_2$, then*

$$C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0 \leq t \leq 1} \frac{\psi(t)^2}{\psi_2(t)^2}$$

(ii) *If $\psi \leq \psi_2$, then*

$$C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_\psi)) = \max_{0 \leq t \leq 1} \frac{\psi_2(t)^2}{\psi(t)^2}$$

(iii) *If ψ is symmetric with respect to $t = 1/2$, and $M_1 = \max\{\frac{\psi(t)}{\psi_2(t)} : 0 \leq t \leq 1\}$ or $M_2 = \max\{\frac{\psi_2(t)}{\psi(t)} : 0 \leq t \leq 1\}$ is taken at $t = 1/2$, then*

$$C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_\psi)) = M_1^2 M_2^2.$$

We consider a function $\psi \in E$ such that ψ is symmetric with respect to $t = 1/2$, that is, $\psi_{1-\beta,\beta} \in E$. Then $\psi_{1-\beta,\beta} \leq \psi_2$ if and only if $1/2 \leq \beta \leq 1/\sqrt{2}$. Applying Proposition 1 (iii) we have the following.

Theorem 2 **([5])** *Let* $1/2 \leq \beta \leq 1$. *Then*

$$C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_{\psi_{1-\beta,\beta}})) = \begin{cases} \frac{\beta^2+(1-\beta)^2}{\beta^2}, & \text{if } 1/2 \leq \beta \leq 1/\sqrt{2}, \\ 2(\beta^2 + (1-\beta)^2), & \text{if } 1/\sqrt{2} \leq \beta \leq 1. \end{cases}$$

We consider a function $\psi_{\alpha,\beta} \in E$ with $\psi_{\alpha,\beta} \leq \psi_2$. Since $\psi_2/\psi_{\alpha,\beta}$ takes its maximum at $t = \alpha$ (resp. $t = \beta$) if $\alpha + \beta \geq 1$ (resp. $\alpha + \beta \leq 1$), we have by Proposition 1,

Theorem 3 **([5])** *If* $\psi_{\alpha,\beta} \leq \psi_2$, *then*

$$C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \begin{cases} \frac{\alpha^2+(1-\alpha)^2}{(1-\alpha)^2}, & \text{if } \alpha + \beta \geq 1, \\ \frac{\beta^2+(1-\beta)^2}{\beta^2}, & \text{if } \alpha + \beta \leq 1. \end{cases}$$

The James constant $J(X)$ of a Banach space X is defined by

$$J(X) = \sup\left\{ \min\{\|x+y\|, \|x-y\|\} : x, y \in X,\ \|x\| = \|y\| = 1\right\}.$$

It is known that (i) $J(X) < 2$ if and only is X is uniformly non-square, that is, there is a $\delta > 0$ such that

$$\|(x-y)/2\| > 1-\delta,\ \|x\| = \|y\| = 1 \Rightarrow \|(x+y)/2\| \leq 1-\delta.$$

(ii) For all Banach space X, $\sqrt{2} \leq J(X) \leq 2$. (iii) If X is a Hilbert space, then $J(X) = \sqrt{2}$. (iv) Let $1 \leq p \leq \infty$, $1/p + 1/q = 1$, then $J(L_p) = \max\{2^{1/p}, 2^{1/q}\}$.

Mitani and Saito [7] the James constant of $(\mathbb{R}^2, \|\cdot\|_\psi)$ when ψ is symmetric with respect to $t = 1/2$, that is, $\psi(1-t) = \psi(t)$ for $t \in [0,1]$.

Theorem 4 **([7])** *Let* $\psi \in \Psi_2$. *If* ψ *is symmetric with respect to* $t = 1/2$, *then*

$$J((\mathbb{R}^2,\ \|\cdot\|_\psi)) = \max_{0 \leq t \leq 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right).$$

We calculate $J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}}))$ for any α, β with $0 \leq \alpha \leq 1/2 \leq \beta \leq 1$. Let $\alpha = 1 - \beta$. Then $\psi_{\alpha,\beta}$ is symmetric with respect to $t = 1/2$.

Theorem 5 ([7]) *For* $\beta \in [1/2, 1]$,

$$J((\mathbb{R}^2, \|\cdot\|\psi_{1-\beta,\beta})) = \begin{cases} 1/\beta, & \text{if } \beta \in [1/2, 1/\sqrt{2}] \\ 2\beta, & \text{if } \beta \in [1/\sqrt{2}, 1]. \end{cases}$$

Let $\alpha \neq 1-\beta$. We define $x(\theta) = (\cos\theta, \sin\theta)/\|(\cos\theta, \sin\theta)\|_\psi$ for $0 \leq \theta \leq 2\pi$. Clearly, we have $\|x(\theta)\|_\psi = 1$. Then,

Lemma 1 ([1]) *Let* $\theta_0 < \theta_1 < \theta_2 < \theta_3 (\leq \theta_0 + \pi)$. *Then*

(i) $\|x(\theta_1) - x(\theta_2)\|_\psi \leq \|x(\theta_0) - x(\theta_3)\|_\psi$

(ii) $\|x(\theta_1) + x(\theta_2)\|_\psi \geq \|x(\theta_0) + x(\theta_3)\|_\psi$.

Using this lemma, we obtain following.

Theorem 6 *Let* $0 \leq \alpha < 1/2 < \beta < 1$ *and* $\alpha < 1-\beta$.
(i) *If* $\psi_{\alpha,\beta}(1/2) \leq \frac{1}{2(1-\alpha)}$, *then*

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = \frac{1}{\psi_{\alpha,\beta}(1/2)}.$$

(ii) *If* $\frac{1}{2(1-\alpha)} \leq \psi_{\alpha,\beta}(1/2) \leq c(\alpha,\beta)$, *then*

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = 1 + \frac{1}{2\psi_{\alpha,\beta}(1/2) + \frac{2\beta-1}{\beta-\alpha}}.$$

(iii) *If* $\psi_{\alpha,\beta}(1/2) \geq c(\alpha,\beta)$, *then*

$$J((\mathbb{R}^2, \|\cdot\|_{\psi_{\alpha,\beta}})) = 2\psi_{\alpha,\beta}(1/2),$$

where

$$c(\alpha,\beta) = \frac{1}{4}\left(1 - \frac{2\beta-1}{\beta-\alpha} + \sqrt{(1 + \frac{2\beta-1}{\beta-\alpha})^2 + 4}\right).$$

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