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Invariant subspaces and other animals

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An "invariant subspace" T for a linear operator T on a vector space is, precisely, a linear subspace Y for which

0.1 $$T(Y) \subseteq Y \subseteq X \ .$$

The point of an invariant subspace is the *restriction* operator

0.2 $$T_Y : Y \to Y \ ,$$

where of course $T_Y(y) = Ty$ for each $y \in Y$. The relationship between T and its restriction T_Y involves also the induced *quotient*

0.3 $$T'_Y : X/Y \to X/Y \ ,$$

where $T'_Y(x + Y) = (Tx) + Y$ for each $x \in X$. Now the "three space property" of invertibility says that if any two of the three operators T, T_Y and T'_Y is invertible then so is the third. Recalling that invertibility is the same as one one and onto, this follows from the six implications ([1] Theorems 3.11.1, 3.11.2)

0.4 $$T_Y, T'_Y \textit{ one one} \Longrightarrow T \textit{ one one} \Longrightarrow T_Y \textit{ one one} \ ;$$

0.5 $$T_Y, T'_Y \textit{ onto} \Longrightarrow T \textit{ onto} \Longrightarrow T_Y \textit{ onto} \ ;$$

0.6 $$T \textit{ one one, } T_Y \textit{ onto} \Longrightarrow T'_Y \textit{ one one} \ ;$$

0.7 $$T \textit{ onto, } T'_Y \textit{ one one} \Longrightarrow T_Y \textit{ onto} \ .$$

All this remains valid for bounded operators on Banach spaces, when of course we only consider closed invariant subspaces. In terms of the *spectrum*

$$\sigma(T) = \{\lambda \in \mathbf{C} : T - \lambda I \textit{ not invertible}\} \ ,$$

the spectrum of each of the operators T, T_Y and T'_Y is contained in the union of the other two. Equivalently

0.8 $$\sigma(T) \subseteq \sigma(T_Y) \cup \sigma(T'_Y) \subseteq \sigma(T) \cup (\sigma(T_Y) \cap \sigma(T'_Y)) \ .$$

This leads to a new kind of invariant subspace ([3] (2.3)):

1. Definition *An invariant subspace $Y \subseteq X$ is called spectrally invariant for T if*

1.1 $$\sigma(T_Y) \cap \sigma(T'_Y) = \emptyset ,$$

in which case also

1.2 $$\sigma(T) = \sigma(T_Y) \cup \sigma(T'_Y) .$$

Of course (0.2) is a consequence of (1.1) and (0.8). For bounded operators on Banach spaces, spectrally invariant subspaces are both *reducing* and *hyperinvariant*: there is a projection $P = P^2 \in B(X)$ for which

1.3 $$ST = TS \Longrightarrow SP = PS$$

with

1.4 $$Y = P(X) .$$

Naturally the projection comes from the splitting of the spectrum via functional calculus ([1] Definition 9.7.1):

1.5 $$P = f(T) \equiv \frac{1}{2\pi i} \oint_{\sigma(T)} (zI - T)^{-1} dz$$

with the function f given by the characteristic function of the restriction spectrum,

1.6 $$f = \chi_K \text{ where } K = \sigma(T_Y) .$$

Since both the range $P(X)$ and its complement $P^{-1}(0)$ are invariant under T it is clear that $P(X)$ is a reducing subspace for T; since by (1.5) the range of P is invariant under everything which commutes with T it is also hyperinvariant. It is also clear that the restriction and the quotient of T with respect to $P(X)$ are the same as with respect to Y: with a little more work it turns out that Y and $P(X)$ are the same.

Intermediate between the invariant and the hyperinvariant are two further kinds of invariant subspace ([3] Definition 1):

2. Definition *The invariant subspace $Y \subseteq X$ is called holomorphically invariant for T if*

2.1 $$f \in \text{Holo}(\sigma(T)) \Longrightarrow f(T)Y \subseteq Y \ ,$$

and comm square invariant for T if

2.2 $$S \in \text{comm}^2(T) \Longrightarrow SY \subseteq Y \ .$$

Evidently

$$\text{spectrally invariant} \Longrightarrow \text{hyperinvariant} \Longrightarrow \text{comm square invariant}$$

$$\Longrightarrow \text{holomorphically invariant} \Longrightarrow \text{invariant} \ ;$$

we claim that none of these implications is reversible. Our counterexamples will all be built from the *forward* and the *backward shifts* u and v, and the *standard weight* w, where for each $x = (x_1, x_2, x_3, \ldots) \in E = \ell_p$ with $p = 2$ and each $n \in \mathbf{N}$

2.3 $$(ux)_1 = 0, (ux)_{n+1} = x_n \ ; \ (vx)_n = x_{n+1} \ ; \ (wx)_n = (1/n)x_n \ .$$

The spectrum σ, the onto spectrum τ^{right} and the eigenvalues π^{left} are given by

2.4 $$\tau^{right}(v) = \partial\mathbf{D} \subseteq \mathbf{D} = \sigma(v) = \sigma(u) = \tau^{right}(u) \ ,$$

2.5 $$\sigma(w) = \mathbf{O} \cup \mathbf{N}^{-1} \ ; \ \sigma(wu) = \mathbf{O} \equiv \{0\}$$

and

2.6 $$\pi^{left}(u) = \emptyset \ ; \ \pi^{left}(v) = \text{int } \mathbf{D} \ ,$$

where $\mathbf{D} = \{|z| \leq 1\} \subseteq \mathbf{C}$ is the closed unit disc. The eigenvalues of the backward shift v all have one dimensional eigenspaces:

$$|\lambda| < 1 \Longrightarrow 1 - \lambda u \textit{ invertible and } v - \lambda = v(1 - \lambda u) \ ,$$

giving

2.7 $$v^{-1}(0) = (1 - uv)(E) = \mathbf{C}\delta_1 = \{(\lambda, 0, 0, \ldots) : \lambda \in \mathbf{C}\}$$

and

2.8 $$(v - \lambda)^{-1}(0) = (1 - \lambda u)^{-1} v^{-1}(0) = (1 - \lambda u)^{-1}(1 - uv)(E) \ .$$

In fact our examples are on the direct sum $X = E \oplus E$ of two copies of $E = \ell_p = \ell_2$, and appear is *operator matrices.*

Not every invariant subspace is holomorphically invariant ([3] Example 1):

3. Example *With*

3.1
$$U = \begin{pmatrix} u & 1-uv \\ 0 & v \end{pmatrix} \ , \ V = \begin{pmatrix} v & 0 \\ 1-uv & u \end{pmatrix} \ , \ P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

3.2
$$Y = P(X) \subseteq X \ ,$$

3.3
$$U(Y) \subseteq Y$$

but not

3.4
$$U^{-1}Y = V(Y) \subseteq Y \ .$$

Not every comm square invariant subspace is hyperinvariant ([3] Example 2):

4. Example *With*

4.1
$$\mathbf{u} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \ , \ \mathbf{v} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \ ,$$

4.2
$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ ,$$

we have

4.3
$$\mathbf{u}P - P\mathbf{u} = \mathbf{v}P - P\mathbf{v} = O$$

and

4.4
$$\mathbf{u}Q - Q\mathbf{u} = \mathbf{v}Q - Q\mathbf{v} = O \ ,$$

but

4.5
$$PQ \neq QP \ ,$$

so that

4.6
$$P \in \mathrm{comm}(\mathbf{v}) \setminus \mathrm{comm}^2(\mathbf{v})$$

and

4.7
$$P \in \mathrm{comm}(\mathbf{u}) \setminus \mathrm{comm}^2(\mathbf{u})$$

and

4.8
$$Y = P(X) \ \textit{invariant under} \ \mathbf{v} \ , \ \mathbf{u} \ , P \ \textit{but not} \ Q \ .$$

Not every holomorphically invariant subspace is comm square invariant ([3] Example 3). This is the most delicate of our examples: we need in particular to see that not everything in the double commutant need be a holomorphic function [2]:

5. Example *With*

5.1
$$T = \begin{pmatrix} u & 0 \\ 0 & 1-u \end{pmatrix} , \; S = \begin{pmatrix} v & 0 \\ 0 & 1-v \end{pmatrix} , \; P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

we have

5.2
$$P \in \mathrm{comm}^2(S) \setminus \mathrm{Holo}(S)$$

and

5.3
$$P \in \mathrm{comm}^2(T) \setminus \mathrm{Holo}(T) .$$

Also

$$W = (S - \lambda I)^{-1}(0) = \begin{pmatrix} (v-\lambda)^{-1}(0) \\ (1-v-\lambda)^{-1}(0) \end{pmatrix}$$

$$= \begin{pmatrix} (1-\lambda u)^{-1} & 0 \\ 0 & (1-(1-\lambda)u)^{-1} \end{pmatrix} \begin{pmatrix} (1-uv)E \\ (1-uv)E \end{pmatrix}$$

is (hyper)invariant under S, and

5.4
$$Y = \begin{pmatrix} (1-\lambda u)^{-1} & 0 \\ 0 & (1-(1-\lambda)u)^{-1} \end{pmatrix} Y' \text{ where } Y' = \begin{pmatrix} 1-uv \\ 1-uv \end{pmatrix} E$$

is (holomorphically) invariant under S but not invariant under P.

Indeed since the diagonal elements of S do not have disjoint spectrum, P cannot ([2] Theorem 1) be a holomorphic function of S:

$$\sigma(v) \cap \sigma(v-1) \neq \emptyset \Longrightarrow P \notin \mathrm{Holo}(S) .$$

On the other hand

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \mathrm{comm}(S) \Longrightarrow m(1-v) - vm = (1-v)n - nv = 0$$

$$\Longrightarrow m = n = 0 \Longleftrightarrow \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \mathrm{comm}(P)$$

and there is ([2] Theorem 2) implication

$$x = vx + xv \Longrightarrow (1-v)x(1-u^n v^n) = 0 \; (n \in \mathbf{N})$$

$$\Longrightarrow x = xuv = xu^2v^2 = xu^3v^3 = \ldots \Longrightarrow x = 0 .$$

Not every hyperinvariant reducing subspace is spectrally invariant ([3] Example 4):

6. Example *With*

6.1 $$R = \begin{pmatrix} v & 0 \\ 0 & wu \end{pmatrix} ,$$

the null space $R^{-1}(0)$ *is hyperinvariant and reducing for* R*, but not spectrally invariant.*

Alternatively

6.2 $$W = (S - \lambda I)^{-1}(0)$$

is hyperinvariant and reducing for S but not spectrally invariant.

Neither hyperinvariance nor reducing implies the other ([3] Example 5):

7. Example *The subspace*

$$P(X) = E \oplus O$$

is comm square invariant and reducing but not hyperinvariant for $\mathbf{u}$ *and for* $\mathbf{v}$*, and is hyperinvariant but not reducing for* Q*.*

Alternatively, on ℓ_∞ the closure of the range of w is hyperinvariant but ([1] Theorem 5.10.2) uncomplemented.

We remark that each of the operators $\mathbf{u}$ and $\mathbf{v}$ satisfies the condition (1.2) but not the disjointness (1.1).

References

[1] R.E.Harte, *Invertibility and singularity*, Dekker 1988

[2] R.E. Harte, *Block diagonalization in Banach algebras*, Proc. Amer. Math. Soc. **129** (2000) 181-190

[3] S.V. Djordković, R.E. Harte and D.A. Larson, *Partially hyperinvariant subspaces*, Operators and Matrices (to appear).