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ASYMMETRIC VARIATION OF CHOI INEQUALITY FOR POSITIVE LINEAR MAP

TAKAYUKI FURUTA

Introduction

Let $\Phi$ be a unital positive linear map between two matrix algebras $\mathcal{A}$ and $\mathcal{M}$.

Kadison inequality [10] states that for $A \in \mathcal{A}^{sa}$ (the self adjoint elements in $\mathcal{A}$)

$$\Phi(A)^2 \leq \Phi(A^2).$$

It is known in [e.g.,[1]] that $\Phi(A)^r \leq \Phi(A^r)$ holds for $A > 0$ and $r \in [-1, 0]$ and $r \in [1, 2]$, and more generally

$$f(\Phi(A)) \leq \Phi(f(A))$$

for operator convex function $f$, and $A \in \mathcal{A}^{sa}$ with spectra of $A$ in the domain of $f$. We cite nice references [2] and [12] to this subject. Choi [4] shows that for $A \in \mathcal{A}^{+}$ (the positive cone of $\mathcal{A}$);

(C1) $\Phi(A^p) \leq \Phi(A)^p$ for $0 \leq p \leq 1$.  
(C2) $\Phi(A)^p \leq \Phi(A^p)$ for $1 \leq p \leq 2$.

The study of positive linear maps is of central importance in several parts of matrix analysis and functional analysis.

J-C. Bourin and E. Ricard show very interesting asymmetric extension of Kadison inequality as follows by using quite ingenious method.

**Theorem A** (Bourin-Ricard [3]). Let $A \in \mathcal{A}^{+}$ and $0 \leq p \leq q$. Then

$$|\Phi(A^p)\Phi(A^q)| \leq \Phi(A^{p+q}).$$
§1. A result interpolating Theorem A and Choi inequality (C2)

Löwner-Heinz inequality asserts that If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$. As an extension of Löwner-Heinz inequality, we state the following result to give proofs of our results.

**Theorem B.**
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) \[(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}\]

and

(ii) \[(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}\]

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

![Figure 1](image)

The original proof of Theorem B is shown in [6], an elementary one-page proof is in [7] and alternative ones are in [5],[9] and [8]. It is shown in [11] that the conditions $p$, $q$ and $r$ in **Figure 1** are best possible.

**Theorem 1.1.** Let $A \in \mathcal{A}^+$, (i) $0 \leq p \leq q$ and (ii) \[\frac{q}{q+p} \leq r \leq \frac{2q}{q+p}\]. Then

\[(1.0) \quad |\Phi(A^p)^r \Phi(A^q)^r| \leq \Phi(A^{(p+q)r}).\]

**Proof.**
Put $X = \Phi(A^p)^{\frac{r}{q}}$ and $Y = \Phi(A^q)$. Then $X \geq Y \geq 0$ by Choi (C1). Put $\alpha = 2r \geq 0$ and $\beta = \frac{2qr}{p} \geq 0$. Then $(1+\beta)2 \geq \alpha + \beta$ holds by (i) and (ii), so that (ii) of Theorem B ensures

\[(1.1) \quad \Phi(A^q)^{\frac{p+q}{q}} \geq \left(\Phi(A^q)^{\frac{\alpha}{2}} \Phi(A^p)^{\frac{\beta}{2}} \Phi(A^q)^{\frac{\alpha}{2}}\right)^{\frac{1}{2}}\]

and (1.1) yields

\[(1.2) \quad \Phi(A^q)^{\frac{(p+q)r}{q}} \geq \left(\Phi(A^q)^r \Phi(A^p)^{2r} \Phi(A^q)^r\right)^{\frac{1}{2}}\]

and

\[(1.3) \quad \Phi(A^{(p+q)r}) \geq \Phi(A^q)^{\frac{(p+q)r}{q}}\]

by Choi (C2) and (ii).
so that we have the desired result (1.0) by (1.2) and (1.3)

\[ \Phi(A^{(p+q)r}) \geq \left( \Phi(A^q)^r \Phi(A^p)^{2r} \Phi(A^q)^r \right)^{\frac{1}{2}} = |\Phi(A^p)^r \Phi(A^q)^r|. \]

\[ \square \]

**Remark 1.** Theorem 1.1 implies Theorem A by putting \( r = 1 \) and also Theorem 1.1 implies Choi inequality (C2) by putting \( p = 0 \).

| \( r = 1 \) | \( p = 0 \) |
| Theorem A | Choi inequality (C2) |

Theorem 1.1 can be extended to the class of positive, sub-unital linear maps. The result also holds in the general setting of positive linear maps between unital \( C^* \)-algebra.

**Corollary 1.2.** Let \( A \in A^+ \) and \( 0 \leq p \leq q \). Then

\[ |\Phi(A^p)^{\frac{1}{r}} \Phi(A^q)^{\frac{1}{2}}| \leq \Phi(A^q) \]

and

\[ |\Phi(A^p)^{\frac{2q}{q+p}} \Phi(A^q)^{\frac{2q}{q+p}}| \leq \Phi(A^{2q}). \]

**Proof.** Put \( r = \frac{q}{q+p} \) and \( r = \frac{2q}{q+p} \) in Theorem 1.1 respectively.

§2. **Asymmetric variations of \( \Phi(A)^{-1} \leq \Phi(A^{-1}) \) paralleled to Theorem 1.1**

Let \( A \in A^{++} \) be defined by \( A \in A^+ \) and \( A \) is invertible and let \( \Phi \) be strictly positive and unital. By the almost similar way to Theorem 1.1, we show the following result.

| \( \frac{q}{q+p} \leq r \leq \frac{2q}{q+p} \) |
| Theorem 2.1. Let \( A \in A^{++} \), (i) \( 0 \leq p \leq q \) and (ii) \( \frac{q}{q+p} \leq r \leq \frac{2q}{q+p} \). Then

\[ |\Phi(A^{-p})^{-r} \Phi(A^q)^r| \leq \Phi(A^{(p+q)r}). \]

**Proof.** Since \( f(t) = t^s \) is operator convex for \( s \in [-1,0] \), \( \Phi(A)^s \leq \Phi(A^s) \) holds for \( A > 0 \) and \( s \in [-1,0] \) as stated in Introduction. Put \( X = \Phi(A^q)^{\frac{2q}{p}} \) and \( Y = \Phi(A^{-p})^{-1} \). Then \( X \geq Y > 0 \). Put \( \alpha = 2r \geq 0 \) and \( \beta = \frac{2qr}{p} \geq 0 \). Then \( (1 + \beta)2 \geq \alpha + \beta \) by (i) and (ii). so that (ii) of Theorem B ensures
$$\Phi(A^q)^{\frac{(p+q)r}{q}} \geq \left( \Phi(A^q)^r \Phi(A^{-p})^{-2r} \Phi(A^q)^r \right)^{\frac{1}{2}}$$

and (2.1) yields

$$\Phi(A^q)^{\frac{(p+q)r}{q}} \geq \left( \Phi(A^q)^r \Phi(A^{-p})^{-2r} \Phi(A^q)^r \right)^{\frac{1}{2}}$$

and

$$\Phi(A^{(p+q)r}) \geq \Phi(A^q)^{\frac{(p+q)r}{q}}$$

by Choi (C2) and (ii) so that we have the desired (2.0) by (2.2) and (2.3)

$$\Phi(A^{(p+q)r}) \geq \left( \Phi(A^q)^r \Phi(A^{-p})^{-2r} \Phi(A^q)^r \right)^{\frac{1}{2}} = |\Phi(A^{-p})^{-r} \Phi(A^q)^r| \square$$

**Corollary 2.2.** Let $A \in \mathcal{A}^{++}$ and $0 \leq p \leq q$. Then

$$|\Phi(A^{-p})^\frac{-2q}{q+p} \Phi(A^q)^\frac{q}{q+p}| \leq \Phi(A^{q})$$

$$|\Phi(A^{-p})^\frac{-2q}{q+p} \Phi(A^q)^\frac{q}{q+p}| \leq \Phi(A^{2q})$$

**Proof.** Put $r = \frac{q}{q+p}$ and $r = \frac{2q}{q+p}$ in Theorem 2.1 respectively. $\square$

**Remark 2.** Theorem 2.1 interpolating Choi inequality (C2) by potting $p = 0$ and $|\Phi(A^{-p})^{-1} \Phi(A^q)| \leq \Phi(A^{p+q})$ for $0 \leq p \leq q$ by putting $r = 1$.

**Theorem 2.1**

$$r = 1 \searrow \searrow p = 0$$

$$|\Phi(A^{-p})^{-1} \Phi(A^q)| \leq \Phi(A^{p+q}).$$

Choi inequality (C2)

The complete form of this talk has been published in the following paper:

REFERENCES

[6] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101 (1987), 85-88.

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