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(Noncommutative Structure in Operator Theory and its Application)

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On estimations for parametrized operator means
作用素平均族の評価式について

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For a nonnegative operator monotone function $f$ on $[0, \infty)$, Kubo and Ando [2] introduced an operator mean for positive operators $m_f$:

$$A m_f B = A^{\frac{1}{2}} f \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$$

where the semi-continuity $\lim_{\varepsilon \to 0} (A + \varepsilon I) m_f (B + \varepsilon I) \downarrow A m_f B$ assures that we may assume operators are invertible. Recently Kittaneh-Manasrah [1] gave a refined Young inequality, which is immediately extended to an inequality among operator means in the sense of by Furuichi-Lin [3]: Let $t$ be a weight $t \in [0, 1]$, then

$$A \nabla_t B - A \#_t B \geq \min \{ 2t, 2(1-t) \} (A \nabla B - A \# B)$$

where $A \nabla_t B = (1-t)A + tB$; the arithmetic mean and $A \#_t B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}$; the geometric one (for convenience’ sake, we omit $t$ if $t = \frac{1}{2}$).

In this talk, we generalize it for parametrized operator means: For $-1 \leq r \leq 1$, it is known that the functions $f_{r,t}(x) = (1-t + tx^r)^{\frac{1}{r}}$ are operator monotone, so they define operator means

$$A m_{r,t} B = A^{\frac{1}{2}} \left( (1-t)I + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{r} \right)^{\frac{1}{r}} A^{\frac{1}{2}}.$$ 

If $r = 0$, $A \#_t B$ is the limit $\lim_{r \to 0} A m_{r,t} B$. Then we have

\begin{center}
\textbf{Theorem.} For $0 \leq s \leq r \leq 1$,

$$A m_{r,t} B - A m_{s,t} B \geq \min \{ (2t)^{\frac{1}{r}}, (2(1-t))^{\frac{1}{r}} \} (A m_r B - A m_s B).$$
\end{center}

We have only to show the numerical inequality:

\begin{equation}
(0) \quad f_{r,t}(x) - f_{s,t}(x) \geq \min \{ (2t)^{\frac{1}{r}}, (2(1-t))^{\frac{1}{r}} \} \left( f_{r,\frac{1}{2}}(x) - f_{s,\frac{1}{2}}(x) \right).
\end{equation}

It is easy to show the above inequality if $r = 1/n$, but I have not yet a simple proof for the general case. I have shown it by the following properties:

\begin{center}
\textbf{Lemma.} The following properties hold:
\end{center}
(0) $f$ is operator monotone.

(1) $F_{t,x}(r) = (1 - t + tx^r)^{1/r}$ is monotone-increasing for $r$.

(2) $\ell_z(r) = \frac{z-1}{r}$ is monotone-increasing for $r$.

(3) If $0 < x \leq 1$, then $H(r) = g(r)^{1/r-1} = (1 - t + tx^r)^{1/r-1}$ is monotone-increasing for $r$.

(4) Let $J_{r,x}(t) = (1 - t + tx^r)^{1/r}$. Then, $rt J_{r,x}'(t) = J_{r,x}(t) - J_{r,x}(t)^{1-r}$.
\hspace{1cm} Moreover, if $0 < x \leq 1$, then $J_{r,x}'(t) \geq J_{s,x}'(t)$.

\textbf{Proof.} (0) Considering the analytic continuation of $f$ to the upper half plane $\text{Im} \ z > 0$, we have
\begin{align*}
0 < \text{Arg} \ (1 - t + tz^r) < \text{Arg} \ z,
\end{align*}
so that
\begin{align*}
0 < \text{Arg} \ f_{r,t}(z) = \text{Arg} \ (1 - t + tz^r)^{1/r} < \text{Arg} \ z.
\end{align*}
It follows that $\text{Im} \ f_{r,t}(z) > 0$, which shows $f$ is operator monotone.

(1) Let $g(r) = 1 - t + tx^r$, $y = F_{t,x}(r) = g(r)^{1/r}$. Then log $y = \log g(r)/r$. By the convexity of $\eta(x) = x \log x$, we have
\begin{align*}
F_{t,x}'(r) = y' &= \frac{yg'(r)}{rg(r)} - \frac{y \log g(r)}{r^2} = \frac{y^{1-r}g'(r)}{r^2} - \frac{y \log g(r)}{r^2} \\
&= \frac{y^{1-r}}{r^2} (t x^r \log x^r - g(r) \log g(r)) = \frac{y^{1-r}}{r^2} (t \eta(x^r) - \eta(g(r))) \\
&\geq \frac{y^{1-r}}{r^2} (t \eta(x^r) - (1-t) \eta(1) - t \eta(x^r)) = 0,
\end{align*}
which shows $F_{t,x}(r)$ is monotone increasing.

(2) By the Klein inequality $\log y \geq 1 - 1/y$,
\begin{align*}
\ell_x'(r) &= \frac{rx^r \log x - (x^r - 1)}{r^2} = \frac{x^r \log x^r - (x^r - 1)}{r^2} \geq \frac{x^r - 1 - (x^r - 1)}{r^2} = 0.
\end{align*}
(3) Suppose $0 < x \leq 1$ and $1 - t + tx^r \leq 1$. By $1 - r < 1 - s$ and (1), we have
\begin{align*}
H(r) = (1 - t + tx^r)^{(1-r)/r} \geq (1 - t + tx^s)^{(1-r)/s} \geq (1 - t + tx^s)^{(1-s)/s} = H(s).
\end{align*}
(4) The former inequality follows from
\begin{align*}
rt J_{r,x}'(t) &= t(1 - t + tx^r)^{1-r} (x^r - 1) = (1 - t + tx^r)^{1-r} (1 - t + tx^r - 1) \\
&= (1 - t + tx^r)^{1-1} - (1 - t + tx^r)^{1-r} = J_{r,x}(t) - J_{r,x}(t)^{1-r}.
\end{align*}
Since
\[ J_{r,x}'(t) = (1 - t + tx^r)^{1/r - 1} \frac{x^r - 1}{r}, \]

It follows from (2) and (3) that \( J_{r,x}'(t) \geq J_{s,x}'(t) \).

\[ \square \]

**Proof of theorem.** Suppose \( 0 < x \leq 1 \). For \( t \leq 1/2 \), put \( K(t) = \frac{J_{r,x}(t) - J_{s,x}(t)}{(2t)^{1/r}} \). It follows from Lemma that

\[ K'(t) = \frac{2}{r(2t)^{1/r+1}} \left( (1-t)J_{r,x}'(t) - J_{s,x}'(t) \right) - \frac{2}{r(2t)^{1/r}} (J_{r,x}(t) - J_{s,x}(t)) \]

which shows \( K \) is monotone decreasing and attains the minimum \( J_{r,x}(1/2) - J_{s,x}(1/2) \) at \( t = 1/2 \).

Next suppose \( t > 1/2 \). Putting \( L(t) = \frac{J_{r,x}(t) - J_{s,x}(t)}{(2(1-t))^{1/r}} \), we have by Lemma that

\[ L'(t) = \frac{2}{r(2(1-t))^{1/r+1}} \left( (1-t)J_{r,x}'(t) - J_{s,x}'(t) \right) - \frac{2}{r(2(1-t))^{1/r}} (J_{r,x}(t) - J_{s,x}(t)) \]

Thus \( L \) is monotone decreasing and attains the maximum at \( t = 1/2 \). Therefore

\[ J_{r,x}(t) - J_{s,x}(t) \geq (2 \min \{1-t, t\})^{\frac{1}{r}} (J_{r,x}(1/2) - J_{s,x}(1/2)), \]

that is,

\[ (1 - t + tx^r)^{1/r} - (1 - t + tx^s)^{1/s} \geq (2 \min \{1-t, t\})^{\frac{1}{r}} \left( \left( \frac{1 + x^r}{2} \right)^{1/r} - \left( \frac{1 + x^s}{2} \right)^{1/s} \right) \]

holds for \( 0 < x \leq 1 \). By the homogeneity of \( x \), it also holds for \( x > 1 \).

\[ \square \]

If the following conjecture holds, we immediately have a simple proof for (0):

**Conjecture.** For \( 0 < s \leq r \leq 1 \),

\[ \left\{ \begin{array}{ll} \frac{(1-t+tx^r)^{\frac{1}{r}} - (1-t+tx^s)^{\frac{1}{s}}}{1-(2t)^{\frac{1}{r}}} & \geq \frac{(1-t+tx^s)^{\frac{1}{s}} - ((1-t)(1+x^r))^\frac{1}{s}}{1-(2t)^{\frac{1}{s}}} \quad (t < \frac{1}{2}) \\ \frac{(1-t+tx^r)^{\frac{1}{r}} - ((1-t)(1+x^r))^\frac{1}{r}}{1-(2(1-t))^\frac{1}{r}} & \geq \frac{(1-t+tx^s)^{\frac{1}{s}} - ((1-t)(1+x^r))^\frac{1}{s}}{1-(2(1-t))^\frac{1}{s}} \quad (t > \frac{1}{2}) \end{array} \right. \]
In fact, it is also shown easily for $r = \frac{1}{n} > s$: For the case $t < 1/2$, the Jensen inequality for the power function for $ns < 1$ implies

$$\text{LHS} = \left( \frac{(1 - t + tx^\frac{1}{n})^{n} - \left(2t\left(\frac{1+x^\frac{1}{n}}{2}\right)^{n}\right)}{1 - (2t)^{n}} \right)^{n}$$

$$= \left( \frac{\sum_{k=0}^{n-1}(2t)^{k}(1 - t + tx^\frac{1}{n})^{n-k-1}\left(\frac{1+x^\frac{1}{n}}{2}\right)^{k}}{\sum_{k=0}^{n-1}(2t)^{k}} \right)^{ns\cdot\frac{1}{s}}$$

$$\geq \left( \frac{\sum_{k=0}^{n-1}(2t)^{k}(1 - t + tx^{s})^{n-k-1}\left(\frac{1+x^{s}}{2}\right)^{k}}{\sum_{k=0}^{n-1}(2t)^{k}} \right)^{\frac{1}{s}}$$

$$= \left( \frac{(1 - t + tx^{s})^{n} - (t(1+x^{s}))^{n}}{1 - (2t)^{n}} \right)^{\frac{1}{s}} = \text{RHS}.$$

参考文献

