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PERTURBATIONS OF POLAROID TYPE OPERATORS ON BANACH SPACES AND APPLICATIONS

PIETRO AIENA AND ELVIS APONTE

ABSTRACT. A bounded linear operator T defined on a Banach space is said to be polaroid if every isolated point of the spectrum is a pole of the resolvent. The "polaroid" condition is related to the conditions of being left or right polaroid. In these paper we explore these conditions, and the condition of being a-polaroid, under perturbations. Moreover, we present a general framework which allows to us to obtain, and also to extend, recent results concerning Weyl type theorems (generalized or not) for $T+K$, where K is algebraic.

1. INTRODUCTION

In [6] it has been proved that if T is polaroid, or left polaroid, or a-polaroid then some of the Weyl type theorems, in their classical form or in their generalized form, are equivalent. For this reason it has some interest to consider the problem of preserving the polaroid conditions from T to $T+K$ in the case where K is a suitable operator commuting with T. In this talk we shall discuss the case where K is an algebraic commuting perturbations, i.e. there exists a nontrivial polynomial h such that $h(K)=0$. It is well known that important examples of algebraic operators are given by the operators K for which K^n is a finite-dimensional operator for some $n \in \mathbb{N}$. The polaroid conditions, together with the single-valued extension property (SVEP), ensure that the several versions of Weyl type theorems hold (and are equivalent!) for many classes of operators. Since the SVEP is transferred from T to $T+K$, K algebraic and commuting with T, then our results allows to us to obtain that Weyl type theorems (generalized or not) hold for $T+K$.

This note is a free-style paraphrase of a presentation of the results contained in [5], held in Kyoto, 27-29 October 2010. The first author thanks the organizer Masatoshi Fujii for his kind invitation. He also thanks Muneo Chō for his generous hospitality, in the week before the conference, at Kanagawa University, Yokohama.

2. POLAROID TYPE OPERATORS

We begin by fixing the terminology used in this paper. Let $L(X)$ be the algebra of all bounded linear operators acting on an infinite dimensional complex Banach space X and if $T \in L(X)$ let be $\alpha(T) := \dim \ker T$ and $\beta(T)$ the codimension of the range $T(X)$. Recall that the operator $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and the range $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$, if $\beta(T) < \infty$. If either T is upper

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or lower semi-Fredholm then T is said to be a *semi-Fredholm operator*, while if T is both upper and lower semi-Fredholm then T is said to be a *Fredholm operator.* If T is semi-Fredholm then the *index* of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. An operator $T \in L(X)$ is said to be a *Weyl operator*, $T \in W(X)$, if T is a Fredholm operator having index 0. The classes of *upper semi-Weyl's* and *lower semi-Weyl's* operators are defined, respectively:

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\},$$
$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_-(X)$. The *Weyl spectrum* and the *upper semi-Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}.$$

and

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}.$$

The *ascent* of an operator $T \in L(X)$ is defined as the smallest non-negative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest non-negative integer $q := q(T)$ such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3]. Moreover, if $\lambda \in \mathbb{C}$ the condition $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ is equivalent to saying that λ is a pole of the resolvent. In this case λ is an eigenvalue of T and an isolated point of the spectrum $\sigma(T)$, see [30, Prop. 50.2]. A bounded operator $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder, lower semi-Browder*) if T is Fredholm and $p(T) = q(T) < \infty$ (resp. T is upper semi-Fredholm and $p(T) < \infty$, T is lower semi-Fredholm and $q(T) < \infty$). Denote by $B(X)$, $B_+(X)$ and $B_-(X)$ the classes of Browder operators, upper semi-Browder operators and lower semi-Browder operators, respectively. Clearly, $B(X) \subseteq W(X)$, $B_+(X) \subseteq W_+(X)$ and $B_-(X) \subseteq W_-(X)$. Let

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\}$$

denote the *Browder spectrum* and $\sigma_{\mathrm{ub}}(T)$ denote the *upper semi-Browder spectrum* of T, defined as

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\}.$$

then $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ and $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$.

The concept of Drazin invertibility [26] has been introduced in a more abstract setting than operator theory [26]. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) if and only if $p(T) = q(T) < \infty$ and this is equivalent to saying that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent, see [32, Corollary 2.2] and [31, Prop. A]. Drazin invertibility for bounded operators suggests the following definitions.

Definition 2.1. *$T \in L(X)$ is said to be* left Drazin invertible *if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be* right Drazin invertible *if $q := q(T) < \infty$ and $T^q(X)$ is closed.*

Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$ then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$, see [30, Prop. 50.2]. Note that every left or right Drazin invertible operator is quasi-Fredholm, see [19] for definition and details.

The *left Drazin spectrum* is then defined as

$$\sigma_{\mathrm{ld}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

the *right Drazin spectrum* is defined as

$$\sigma_{\mathrm{rd}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\},$$

and the *Drazin spectrum* is defined as

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

Obviously, $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{ld}}(T) \cup \sigma_{\mathrm{rd}}(T)$.

3. Left and right polaroid operators

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective with closed range. The classical *approximate point spectrum* is defined by

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

while the *surjectivity spectrum* is defined as

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}.$$

It is well known that $\sigma_{\mathrm{a}}(T^*) = \sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{s}}(T^*) = \sigma_{\mathrm{a}}(T)$.

Definition 3.1. *Let $T \in L(X)$, X a Banach space. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{a}}(T)$ then λ is said to be a* left pole *of the resolvent of T. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a* right pole *of the resolvent of T.*

Clearly, λ is a pole of T if and only if λ is both a left and a right pole of T. In fact, if λ is a pole of T then $0 < p := p(\lambda I - T) = q(\lambda I - T) < \infty$ and $T^p(X) = T^{p+1}(X)$ coincides with the kernel of the spectral projection associated with the spectral set $\{\lambda\}$, so $\lambda I - T$ is both left and right Drazin invertible. Moreover, the condition $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ entails that $\lambda \in \sigma_{\mathrm{a}}(T)$ as well as $\lambda \in \sigma_{\mathrm{s}}(T)$.

Definition 3.2. *Let $T \in L(X)$. Then*

(i) *T is said to be* left polaroid *if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a left pole of the resolvent of T, while $T \in L(X)$ is said to be* right polaroid *if every isolated point of $\sigma_{\mathrm{s}}(T)$ is a right pole of the resolvent of T.*

(ii) *T is said to be* polaroid *if every isolated point of $\sigma(T)$ is a pole of the resolvent of T.*

(iii) *T is said to be* a-polaroid *if every $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T.*

The concept of left and right polaroid are dual each other:

Theorem 3.3. [6, Theorem 2.8] *If $T \in L(X)$ then the following equivalences hold:*

(i) *T is left polaroid if and only if T' is right polaroid.*

(ii) *T is right polaroid if and only if T' is left polaroid.*

(iii) *T is polaroid if and only if T' is polaroid.*

The following property has relevant role in local spectral theory, see the recent monographs by Laursen and Neumann [33] and [1].

Definition 3.4. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have* the single valued extension property *at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$.*
An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, $T \in L(X)$ has SVEP at every isolated point of the spectrum.
We also have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually, if T' denotes the dual of T,

$$q(\lambda I - T) < \infty \Rightarrow T' \text{ has SVEP at } \lambda, \tag{2}$$

see [1, Theorem 3.8]. Furthermore, from definition of localized SVEP it easily seen that

$$\sigma_a(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually,

$$\sigma_s(T) \text{ does not cluster at } \lambda \Rightarrow T' \text{ has SVEP at } \lambda. \tag{4}$$

The *quasi-nilpotent part* of $T \in L(X)$ is defined as the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$. Moreover, T is quasi-nilpotent if and only if $H_0(\lambda I - T) = X$, see Theorem 1.68 of [1]. Note that $H_0(T)$ generally is not closed and ([1, Theorem 2.31]

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{5}$$

The *analytical core* of T is defined $K(T) := \{x \in X :$ there exist $c > 0$ and a sequence $(x_n)_{n\geq 1} \subseteq X$ such that $Tx_1 = x, Tx_{n+1} = x_n$ for all $n \in \mathbb{N}$, and $||x_n|| \leq c^n||x||$for all $n \in \mathbb{N}\}$. Note that $T(K(T)) = K(T)$, and $K(T)$ is contained in the *hyper-range* of T defined by $T^\infty(X) := \bigcap_{n=0}^\infty T^n(X)$, see [1, Chapter 1] for details.

Remark 3.5. If $\lambda I - T$ is semi-Fredholm, or also quasi-Fredholm, then the implications above are equivalences, see [1] or [3]

In [6, Theorem 2.6] it has been observed that if T is both left and right polaroid then T is polaroid. The following theorem shows that this is true if T is either left or right polaroid.

Theorem 3.6. *If $T \in L(X)$ the following implications hold:*

$$T \text{ a-polaroid} \Rightarrow T \text{ left polaroid} \Rightarrow T \text{ polaroid}$$

Furthermore, if T is right polaroid then T is polaroid.

Proof. The first implication is clear, since a pole is always a left pole. Assume that T is left polaroid and let $\lambda \in \operatorname{iso}\sigma(T)$. It is known that the boundary of the spectrum is contained in $\sigma_{\mathrm{a}}(T)$, in particular every isolated point of $\sigma(T)$, thus $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and hence λ is a left pole of the resolvent of T. By [16, Theorem 2.4] then there exists a exists a natural $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$. Now, since λ is isolated in $\sigma(T)$, by [1, Theorem 3.74] the following decomposition holds,

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^\nu \oplus K(\lambda I - T).$$

Therefore,

$$(\lambda I - T)^\nu(X) = (\lambda I - T)^\nu(K(\lambda I - T)) = K(\lambda I - T).$$

So

$$X = \ker(\lambda I - T)^\nu \oplus (\lambda I - T)^\nu(X),$$

which implies, by [1, Theorem 3.6], that $p(\lambda I - T) = q(\lambda I - T) \leq \nu$, from which we conclude that λ is a pole of the resolvent for every isolated point of $\sigma(T)$, i.e. T is polaroid.

To show the last assertion suppose that T is right polaroid. By Theorem 3.3 then T' is left polaroid and hence, by the first part, T' is polaroid, or equivalently T is polaroid. ∎

In [6] it has been observed that if T' has SVEP(respectively, T has SVEP) then the polaroid type conditions for T (respectively, for T') are equivalent. We give now a more precise result.

Theorem 3.7. ([5]) *Let $T \in L(X)$. Then we have*

(i) *If T' has SVEP then the properties of being polaroid, a-polaroid and left polaroid for T are all equivalent.*

(ii) *If T has SVEP then the properties of being polaroid, a-polaroid and left polaroid for T' are all equivalent.*

4. Perturbations of polaroid type operators

In this section we consider the permanence of the polaroid conditions under perturbations. First we need the following result:

Lemma 4.1. [13] *If $T \in L(X)$ and N is a nilpotent operator commuting with T then $H_0(T + N) = H_0(T)$.*

The polaroid and a-polaroid condition is preserved by commuting nilpotent perturbations:

Theorem 4.2. ([5]) *Suppose that $T \in L(X)$ and let N be a nilpotent operator which commutes with T. Then we have*

(i) *$T + N$ is polaroid if and only if T is polaroid*

(ii) *$T + N$ is a-polaroid if and only if T is a-polaroid.*

If T is left polaroid then T is polaroid, so $T + N$ is polaroid by [13, Theorem 2.10]. The next result shows that assuming SVEP the also $T + N$ is left polaroid

Corollary 4.3. *Suppose that $T \in L(X)$ and let N be a nilpotent operator which commutes with T.*

(i) *If T' has SVEP and T is left polaroid then $T + N$ is left polaroid.*

(ii) *If T has SVEP and T is right polaroid then $T + N$ is right polaroid.*

Proof. (i) Suppose that T is left polaroid. Then, by Theorem 3.7, T is a-polaroid and hence $T + N$ is a-polaroid by Theorem 4.2. Consequently, $T + N$ is left polaroid.

(ii) If T is right polaroid then T' is left polaroid and hence, again by Theorem 3.7, T' is a-polaroid. Since N' is also nilpotent, by Theorem 4.2 then $T' + N'$ is a-polaroid and hence left polaroid. By Theorem 3.3 it then follows that $T + N$ is right polaroid. ∎

It is not known to the authors if the results of Corollary 4.3 hold without assuming SVEP. The answer is positive for Hilbert space operators:

Theorem 4.4. ([5]) *Suppose that $T \in L(H)$, H a Hilbert space, and let N be a nilpotent operator which commutes with T. Then T is left polaroid (respectively, right polaroid) if and only if $T + N$ is left polaroid (respectively, right polaroid).*

Recall that a bounded operator $T \in L(X)$ is said to be *algebraic* if there exists a non-constant polynomial h such that $h(T) = 0$. Trivially, every nilpotent operator is algebraic and it is well-known that every finite-dimensional operator is algebraic. It is also known that every algebraic operator has a finite spectrum.

In the sequel we consider the perturbation $T + K$ of a polaroid type theorem whenever K is algebraic. In the sequel the part of an operator T means the restriction of T to a closed T-invariant subspace.

Definition 4.5. *An operator $T \in L(X)$ is said to be* hereditarily polaroid *if every part of T is polaroid.*

Every hereditarily polaroid operator has SVEP, see [27, Theorem 2.8]. By using Theorem 4.2 we obtain our main result:

Theorem 4.6. ([5]) *Suppose that $T \in L(X)$ and $K \in L(X)$ is an algebraic operator which commutes with T.*

(i) *If T is hereditarily polaroid operator then $T + K$ is polaroid while $T' + K'$ is a-polaroid.*

(i) *If T' is hereditarily polaroid operator then $T' + K'$ is polaroid while $T + K$ is a-polaroid.*

The next simple example shows that the result of Corollary 4.3, as well as the result of Theorem 4.2, cannot be extended to quasi-nilpotent operators Q commuting with T.

Example 4.7. Let $Q \in L(\ell^2(\mathbb{N}))$ is defined by

$$Q(x_1, x_2, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}),$$

Then Q is quasi-nilpotent and if $e_n : (0, \dots, 1, 0,$ where 1 is the n-th term and all others are 0, then $e_{n+1} \in \ker Q^{n+1}$ while $e_{n+1} \notin \ker Q^n$, so that $p(Q) = \infty$. If we take $T = 0$, the null operator, then T is both left and a-polaroid, while $T + Q = Q$ is is not left polaroid, as well as not a-polaroid.

However, the following theorem shows that $T + Q$ is polaroid if a very special case. Recall first that if $\alpha(T) > \infty$ then $\alpha(T^n) < \infty$ for all $n \in \mathbb{N}$.

Theorem 4.8. ([5]) *Suppose that $Q \in L(X)$ is a quasi-nilpotent operator which commutes with $T \in L(X)$ and suppose that all eigenvalues of T have finite multiplicity.*

(i) *If T is polaroid operator then $T + Q$ is polaroid .*

(i) *If T is left polaroid operator then $T + Q$ is left polaroid .*

(iii) *If T is a-polaroid operator then $T + Q$ is a-polaroid.*

The argument of the proof of part (i) of Theorem 4.3 works also if we assume that every isolated point of $\sigma(T)$ is a finite rank pole (in this case T is said to be *finitely polaroid*).

5. Weyl type theorems

In this section we give a general framework for Weyl type theorem for $T + K$, where K is algebraic and commute with T. First we need to give some preliminary defintions. If $T \in L(X)$ set

$$E(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\},$$

and

$$E^a(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T)\}.$$

Evidently, $E^0(T) \subseteq E(T) \subseteq E^a(T)$ for every $T \in L(X)$. Define

$$\pi_{00}(T) : \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$\pi^a_{00}(T) : \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Let $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, i.e. $p_{00}(T)$: is the set of all poles of the resolvent of T.

Definition 5.1. *A bounded operator $T \in L(X)$ is said to satisfy* Weyl's theorem, *in symbol* (W), *if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. T is said to satisfy* a-Weyl's theorem, *in symbol* (aW), *if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T)$. T is said to satisfy property* (w), *if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$.*

Recall that $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, while $T \in L(X)$ is said to satisfy *a-Browder's theorem* if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. Weyl's theorem for T entails Browder's theorem for T, while a-Weyl's theorem entails a-Browder's theorem. Either a-Weyl's theorem or property (w) entails Weyl's theorem. Property (w) and a-Weyl's theorem are independent, see [15].

The concept of semi-Fredholm operators has been generalized by Berkani ([19], [24]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space

$T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm* (resp. *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([24]). This enables one to define the index of a semi B-Fredholm as ind T = ind $T_{[n]}$. A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl, lower semi B-Weyl*) if for some integer $n \geq 0$ $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl). In an obvious way all the classes of operators generate spectra, for instance the *B-Weyl spectrum* $\sigma_{\mathrm{bw}}(T)$ and the *upper B-Weyl spectrum* $\sigma_{\mathrm{ubw}}(T)$. Analogously, a bounded operator $T \in L(X)$ is said to be *B-Browder* (respectively, (respectively, *upper semi B-Browder, lower semi B-Browder*) if for some integer $n \geq 0$ $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Browder, lower semi-Browder). The *B-Browder spectrum* is denoted by $\sigma_{\mathrm{bb}}(T)$, the *upper semi B-Browder spectrum* by $\sigma_{\mathrm{ubb}}(T)$. Note that $\sigma_{\mathrm{ubb}}(T)$ coincides with the left Drazin spectrum $\sigma_{\mathrm{ld}}(T)$ ([9]).

Remark 5.2. The converse of the implications (1)-(5) hold also whenever $\lambda I - T$ is semi B-Fredholm, see [3], in particular left or right Drazin invertible.

The generalized versions of Weyl type theorems are defined as follows:

Definition 5.3. *A bounded operator $T \in L(X)$ is said to satisfy* generalized Weyl's theorem, *in symbol,* (gW), *if $\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = E(T)$. $T \in L(X)$ is said to satisfies* generalized a-Weyl's theorem, *in symbol,* (gaW), *if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = E^a(T)$. $T \in L(X)$ is said to satisfy* generalized property (w), *in symbol,* (gw), *if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = E(T)$.*

In the following diagrams we resume the relationships between all Weyl type theorems:

$$(gw) \quad \Rightarrow (w) \Rightarrow (\mathrm{W})$$

$$(\mathrm{gaW}) \quad \Rightarrow (\mathrm{aW}) \Rightarrow (\mathrm{W}),$$

see [18, Theorem 2.3], [15] and [23]. Generalized property (w) and generalized a-Weyl's theorem are also independent, see [18]. Furthermore,

$$(gw) \quad \Rightarrow (\mathrm{gW}) \Rightarrow (\mathrm{W})$$

$$(\mathrm{gaW}) \quad \Rightarrow (\mathrm{gW}) \Rightarrow (\mathrm{W})$$

see [18] and [23]. The converse of all these implications in general does not hold. Furthermore, by [2, Theorem 3.1],

$$(W) \text{ holds for } T \Leftrightarrow \text{Browder's theorem holds for } T \text{ and } p_{00}(T) = \pi_{00}(T).$$

Under the polaroid conditions we have a very clear situation:

Theorem 5.4. *Let $T \in L(X)$. Then we have:*

(i) *If T is polaroid then (W) and (gW) for T are equivalent.*

(ii) *If T is left-polaroid then (aW) and (gaW) are equivalent for T, while (W) and (gW) are equivalent for T.*

(iii) *If T is a-polaroid then (aW), (gaW), (w) and (gw) are equivalent for T, while (W) and (gW) are equivalent for T.*

Proof. The equivalence in (i) of (W) and (gW) and the equivalence in (ii) of (W) and (gW) have been proved in [6, Theorem 3.7]. The equivalence of (W) and (gW) for T, if T is left polaroid, follows from (i) and from Theorem 3.6. The equivalence in (iii) is [6, Corollary 3.8]. ∎

Theorem 5.5. [10, Theorem 2.3] *Let $T \in L(X)$ be polaroid and suppose that either T or T' has SVEP. Then both T and T' satisfy Weyl's theorem.*

For a bounded operator $T \in L(X)$, define $\Pi^a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. It is clear that $\Pi^a_{00}(T)$ is the set of all left poles of the resolvent.

Theorem 5.6. *Let $T \in L(X)$ be left polaroid and suppose that either T or T' has SVEP. Then T satisfies generalized a-Weyl's theorem.*

Proof. T satisfies a-Browder's theorem and the left polaroid condition entails that $\Pi^a(T) = E^a(T)$. By [14, Theorem 2.18] then (gaW) holds for T. ∎

Theorem 5.7. [10] *Let $T \in L(X)$ be polaroid. Then we have:*

(i) *if T' has SVEP then (gaW) and (gw) hold for T.*

(ii) *If T has SVEP then (gaW) and (gw) hold for T'.*

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$.

Theorem 5.8. *Suppose that $T \in L(X)$ has SVEP and let $f \in \mathcal{H}_{nc}(\sigma(T))$.*

(i) *If T is polaroid then $f(T)$ satisfies (gW).*

(ii) *If T is left polaroid then $f(T)$ satisfies (gaW).*

(iii) *If T is a-polaroid then $f(T)$ satisfies both (gaW) and (gw).*

Proof. (i) $f(T)$ is polaroid by [6, Lemma 3.11] and by [1, Theorem 2.40] has SVEP. Combining Theorem 5.5 and Theorem 5.4 we then conclude that $f(T)$ satisfies (gW) .

(ii) $f(T)$ is left polaroid by [6, Lemma 3.11] and has SVEP. Combining Theorem 5.6 and Theorem 5.4 it then follows that $f(T)$ satisfies (gaW).

(iii) By part (ii) $f(T)$ satisfies (gaW), since it is also left polaroid. $f(T)$ is a-polaroid by [6, Lemma 3.11] and has SVEP. By Theorem 5.4 then $f(T)$ satisfies also (gw). ∎

The next two examples show that the assumption of being polaroid in part (i) of Theorem 5.8 is not sufficient to ensure property (gaW), or (gw).

Example 5.9. Denote by $R \in L(\ell^2(\mathbb{N}))$ the canonical right shift and let Q denote the quasi-nilpotent operator defined as

$$Q(x_1, x_2, \dots) := (0, \frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } x = (x_1, x_2, \dots) \in \ell^2(\mathbb{N}).$$

Let $T := R \oplus Q$. Then T has SVEP, since both R and Q have SVEP, and is polaroid, since $\sigma(T) = D(0,1)$, where $D(0,1)$ is the closed unit disc of $\mathbb{C}$ centered at 0 and radius 1, has no isolated points. We also have $\sigma_{\rm a}(T) = \Gamma \cup \{0\}$, where Γ denotes the unit circle of $\mathbb{C}$. Hence, $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm a}(T) = \Gamma \cup \{0\}$. Now, by Remark 3.5 for every $\lambda \notin \sigma_{\rm uw}(T)$ the SVEP of T at λ implies that $\lambda \notin \operatorname{acc}\sigma_{\rm a}(T) = \Gamma$, thus $\Gamma \subseteq \sigma_{\rm uw}(T)$. Clearly, $p(T) = p(R) + p(Q) = \infty$, so $0 \in \sigma_{\rm ub}(T) = \sigma_{\rm uw}(T)$, where the last equality holds since T satisfies a-Browder's theorem. Therefore, $\sigma_{\rm uw}(T) = \Gamma \cup \{0\}$, hence $\sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = \emptyset$. But $\pi^a_{00}(T) = \{0\}$, so a-Weyl's theorem does not hold for T.

It is easily seen that property (gw) holds for T. Indeed, $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm uw}(T) = \Gamma \cup \{0\}$, and repeating the same argument used above (just use Remark 5.2, instead of Remark 3.5, and generalized a-Browder's theorem for T) we easily obtain $\sigma_{\rm ubw}(T) = \Gamma \cup \{0\}$. Clearly, $E(T) = \emptyset$ and hence $E(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ubw}(T)$.

Example 5.10. Take $0 < \varepsilon < 1$ and define $S \in L(\ell^2(\mathbb{N}))$ by

$$S(x_1, x_2, \dots) := (\varepsilon x_1, 0, x_2, x_3, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then $\sigma(S^*) = D(0,1)$, so S^* is polaroid and $\sigma_{\rm a}(S^*) = \Gamma \cup \{0\}$, see [5], which implies the SVEP for S^*. Moreover, $\sigma_{\rm uw}(S^*) = \Gamma$, and $\pi_{00}(S^*) = \emptyset$, so property (w) (and hence (gw)) does not hold for S^*. Note that $\pi^a_{00}(S^*) = \{\varepsilon\}$, so a-Weyl's theorem holds for S^*.

Also the assumption of being left polaroid in part (ii) of Theorem 5.8 is not sufficient to ensure property (gw):

Example 5.11. Denote by T the hyponormal operator given by the direct sum of the 1-dimensional zero operator U and the unilateral right shift R on $\ell^2(\mathbb{N})$. Evidently, T has SVEP and $\operatorname{iso}\sigma_{\rm a}(T) = \{0\}$ since $\sigma_{\rm a}(T) = \Gamma \cup \{0\}$. Clearly, $T \in \Phi_+(X)$, and hence $T^2 \in \Phi_+(X)$, so $T^2(X)$ is closed, and since $p(T) = p(U) = 1$it then follows that 0 is a left pole of T, i.e. T is left polaroid. We show that T does not satisfy (w) (and hence (gw)). We know that $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm a}(T) = \Gamma \cup \{0\}$ and repeating the same argument of Example 5.9 we have $\Gamma \subseteq \sigma_{\rm uw}(T) \subseteq \Gamma \cup \{0\}$. Since $T \in B_+(X) \subseteq W_+(X)$ it then follows that $0 \notin \sigma_{\rm uw}(T)$, so $\sigma_{\rm uw}(T) = \Gamma$, and hence

$$\sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = \{0\} \neq \pi_{00}(T) = \emptyset,$$

thus T does not satisfy (w) (and hence (gw).

Theorem 5.12. *Suppose $K \in L(X)$ an algebraic operator commuting with $T \in L(X)$ and let $f \in \mathcal{H}_{nc}(\sigma(T+K))$. Then we have*

(i) *If $T \in L(X)$ is hereditarily polaroid then $f(T+K)$ satisfies (gW), while $f(T'+K')$ satisfies every Weyl type theorem (generalized or not).*

(ii) *If $T' \in L(X)$ is hereditarily polaroid then $f(T'+K')$ satisfies (gW), while $f(T+K)$ satisfies every Weyl type theorem (generalized or not).*

Proof. (i) $T+K$ is polaroid and has SVEP. Then $f(T+K)$ is polaroid. We also know that T has SVEP and hence, by [13, Theorem 2.14], $T+K$ has SVEP, from which it follows that $f(T+K)$ has SVEP. From Theorem 5.8 we then conclude that $f(T+K)$ satisfies (gW). The second assertion easily follows from Theorem 4.6: since K' is algebraic then $T'+K'$, and hence $f(T'+K')$, is a-polaroid and has SVEP. By Theorem 5.6 then (gaW) holds for $f(T'+K')$, or equivalently, by Theorem 5.4, (gw) holds for $f(T'+K')$.

(ii) The proof is analogous. ∎

Part of statement (i) of Theorem 5.12 has been proved by Duggal [27, Theorem 3.6] by using different methods.

Remark 5.13. In the case of Hilbert space operators, in Theorem 5.7 and Theorem 6 the assertions holds if T' is replaced by the Hilbert adjoint T^*. Furthermore, the assumption that T is hereditarily polaroid in Theorem 6 may be replaced by the assumption that T is *polynomially* hereditarily polaroid, i.e. there exists a non-trivial polynomial h such that $h(T)$ is hereditarily polaroid (actually, T is polynomially hereditarily polaroid if and only if T is hereditarily polaroid, see [27, Example 2.5]).

The class of hereditarily polaroid operators is rather large. It contains the $H(p)$-operators introduced by Oudghiri in [36], where $T \in L(X)$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{6}$$

From the implication (5) we see that every operator T which belongs to the class $H(p)$ has SVEP. Moreover, every $H(p)$ operator T is polaroid. Furthermore, if T is $H(p)$ then the every part of T is $H(p)$ [36, Lemma 3.2], so T is hereditarily polaroid. Property $H(p)$ is satisfied by every generalized scalar operator (see [33] for details), and in particular for p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces, see [36]. Therefore, algebraically p-hyponormal or algebraically M-hyponormal operators are $H(p)$.

Corollary 5.14. *Suppose that $T \in L(X)$ is generalized scalar and $K \in L(X)$ is an algebraic operator which commutes with T. Then all Weyl type theorems, generalized or not, hold for $T+K$ and $T'+K'$.*

Proof. Observe that for every generalized scalar operator T both T and T' have SVEP. The assertion for $T'+K'$ is clear by Theorem 5.12. By Theorem 4.6 $T+K$ is polaroid, by [13, Theorem 2.14] $T'+K'$ has SVEP and hence, by Theorem 3.7, $T+K$ is a-polaroid. The assertion for $T+K$ then follows by part (iii) of Theorem 5.8. ∎

Another important class of hereditarily polaroid operators is given by *paranormal* operators on Hilbert spaces, defined as the operators for which

$$\|Tx\|^2 \le \|T^2x\|\|x\| \quad \text{for all } x \in H.$$

In fact, these operators have SVEP, are polaroid and obviously their restrictions to a part are still paranormal, see [12]. Weyl's theorem for $T+K$, in the case that T is $H(p)$ has been proved by Oudghiri [36], while Weyl's theorem for $T+K$ in the case that T is paranormal has been proved in [12]. Therefore, Theorem 5.12 extends and subsumes both results. Theorem 5.12 also extends the results of [13, Theorem 2.15 and Theorem 2.16], since every algebraically paranormal operator is polaroid and has SVEP. Other examples of hereditarily polaroid operators are given by the *completely hereditarily normaloid* operators on Banach spaces. In particular, all (p,k)-*quasihyponormal* operators on Hilbert spaces are hereditarily polaroid, see for details [27]. Also the *algebraically quasi-class A operators* on a Hilbert space considered in [29], are hereditarily polaroid. In fact, every part of an algebraically quasi-class A operator T is algebraically quasi-class A and every algebraically quasi-class A operator is polaroid [29, Lemma 2.3]. Other classes of polaroid operators may be find in [4].

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Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo (Italy), E-mail paiena@unipa.it

Departamento de Matemáticas, Facultád de Ciencias UCLA, Barquisimeto (Venezuela), E-mail elvisedo22@gmail.com