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A CORRESPONDENCE OF CANONICAL BASES IN THE $q$-DEFORMED HIGHER LEVEL FOCK SPACES

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ABSTRACT. The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. The $q$-decomposition matrix is a transition matrix from the standard basis to the canonical basis defined by Uglov in the $q$-deformed Fock space. In this paper, we show that parts of $q$-decomposition matrices of level $\ell$ coincide with that of level $\ell - 1$ under certain conditions of multi charge.

1. Introduction

The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMMO91]. For a multi charge $s = (s_1, \ldots, s_\ell) \in \mathbb{Z}_+^\ell$, the $q$-deformed Fock space $F_q[s]$ of level $\ell$ is the $\mathbb{Q}(q)$-vector space whose basis are indexed by $\ell$-tuples of Young diagrams, i.e. $\{|\lambda; s\rangle | \lambda \in \Pi^\ell\}$, where $\Pi$ is the set of Young diagrams.

The canonical bases $\{G^+(\lambda; s) | \lambda \in \Pi^\ell\}$ and $\{G^-(\lambda; s) | \lambda \in \Pi^\ell\}$ are bases of the Fock space $F_q[s]$ that are invariant under a certain involution $\varepsilon$ [Ug100]. Define matrices $\Delta^+(q) = (\Delta_{\lambda\mu}^+(q))_{\lambda, \mu}$ and $\Delta^-(q) = (\Delta_{\lambda\mu}^-(q))_{\lambda, \mu}$ by

$$G^+(\lambda; s) = \sum_{\mu} \Delta^+_{\lambda\mu}(q) |\mu; s\rangle, \quad G^-(\lambda; s) = \sum_{\mu} \Delta^-_{\lambda\mu}(q) |\mu; s\rangle.$$ 

We call $\Delta^+_{\lambda\mu}(q)$ and $\Delta^-_{\lambda\mu}(q)$ $q$-decomposition numbers. These $q$-decomposition matrices plays an important role in representation theory. However it is difficult to compute $q$-decomposition matrices.

In the case of $\ell = 1$, Varagnolo-Vasserot [VV99] proved that $\Delta^+(q)$ coincides with the decomposition matrix of $\nu$-Schur algebra. For $\ell \geq 2$, Yvonne [Yvo07] conjectured that the matrix $\Delta^+(q)$ coincides with the $q$-analogue of the decomposition matrix of cyclotomic Schur algebras at a primitive $n$-th root of unity under a suitable condition of multi charge. Rouquier [Rou08, Theorem 6.8, §6.5] conjectured that, for arbitrary multi charge, the multiplicities of simple modules in standard modules in the category $\mathcal{O}$ of rational Cherednik algebras are equal to the corresponding coefficients $\Delta^+_\lambda\mu(q)$.

We say that the $j$-th component $s_j$ of the multi charge is sufficiently large for $|\lambda; s\rangle$ if $s_j - s_i \geq 1$ for $i = 1, 2, \ldots, \ell$, and that $s_j$ is sufficiently small for $|\lambda; s\rangle$ if $s_i - s_j \geq 1$ for $i = 1, 2, \ldots, \ell$ (see Definition 3.1). If $s_j$ is sufficiently large for $|\lambda; s\rangle$ and $|\lambda; s\rangle > |\mu; s\rangle$, then the $j$-th components of $\lambda$ and $\mu$ are both the empty Young diagram $\emptyset$ (Lemma 3.2). On the other hand, if $s_j$ is sufficiently small for $|\lambda; s\rangle$ and $|\lambda; s\rangle > |\mu; s\rangle$, then $\mu^{(j)} = \emptyset$ implies $\lambda^{(j)} = \emptyset$. (Lemma 3.3).

Our main results are as follows.

**Theorem A.** (Theorem 3.4) [Iij]

Let $\varepsilon \in \{+, -\}$. If $s_j$ is sufficiently large for $|\lambda; s\rangle$, then

$$\Delta^\varepsilon_{\lambda\mu; s}(q) = \Delta^\varepsilon_{\lambda\mu; s}(q).$$
K. IJIMA

where \( \lambda \) (resp. \( \mu, s \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)), \( \Delta^e_{\lambda, \mu, s}(q) \) is the \( q \)-decomposition number of level \( \ell \) and \( \Delta^e_{\lambda, \mu, s}(q) \) is the \( q \)-decomposition number of level \( \ell - 1 \).

**Theorem B.** (Theorem 3.5) [Iij]

Let \( \epsilon \in \{+,-\} \). If \( s \) is sufficiently small for \( |\mu; s| \) and \( \mu^{(0)} = 0 \), then

\[
\Delta^e_{\lambda, \mu, s}(q) = \Delta^e_{\lambda, \mu, s}(q),
\]

where \( \lambda \) (resp. \( \mu, s \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

This paper is organized as follows. In Section 2, we review the \( q \)-deformed Fock spaces of higher levels and its canonical bases. In Section 3, we state the main results.

**Acknowledgments.** I am deeply grateful to Hyohe Miyachi and Soichi Okada for their advice.

**Notations.** For a positive integer \( N \), a partition of \( N \) is a non-increasing sequence of non-negative integers summing to \( N \). We write \( |\lambda| = N \) if \( \lambda \) is a partition of \( N \). The length \( l(\lambda) \) of \( \lambda \) is the number of non-zero components of \( \lambda \). And we use the same notation \( \lambda \) to represent the Young diagram corresponding to \( \lambda \). For an \( \ell \)-tuple \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(t)}) \) of Young diagrams, we put \( |\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(t)}| \).

2. The \( q \)-deformed Fock spaces of higher levels

2.1. \( q \)-wedge products and straightening rules. Let \( n, \ell, s \) be integers such that \( n \geq 2 \) and \( \ell \geq 1 \). We define \( P(s) \) and \( P^{++}(s) \) as follows;

\[
\begin{align*}
(1) & \quad P(s) = \{ k = (k_1, k_2, \cdots) \in \mathbb{Z}^\infty \mid k_r = s - r + 1 \text{ for any sufficiently large } r \}, \\
(2) & \quad P^{++}(s) = \{ k = (k_1, k_2, \cdots) \in P(s) \mid k_1 > k_2 > \cdots \}.
\end{align*}
\]

Let \( \Lambda^s \) be the \( \mathbb{Q}(q) \) vector space spanned by the \( q \)-wedge products

\[
(3) \quad u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots, \quad (k \in P(s))
\]

subject to certain commutation relations, so-called straightening rules. Note that the straightening rules depend on \( n \) and \( \ell \). [Ugl00, Proposition 3.16].

**Example 2.1.** (i) For every \( k_1 \in \mathbb{Z}, u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1} \). Therefore \( u_{k_1} \wedge u_{k_1} = 0 \).

(ii) Let \( n = 2, \ell = 2, k_1 = -2, \) and \( k_2 = 4 \). Then

\[
u_{-2} \wedge u_4 = q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0.
\]

(iii) Let \( n = 2, \ell = 2, k_1 = -1, k_2 = -2 \) and \( k_3 = 4 \). Then

\[
u_{-1} \wedge u_{-2} \wedge u_4 = u_{-1} \wedge (u_{-2} \wedge u_4) = u_{-1} \wedge (q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0)
\]

By applying the straightening rules, every \( q \)-wedge product \( u_k \) is expressed as a linear combination of so-called ordered \( q \)-wedge products, namely \( q \)-wedge products \( u_k \) with \( k \in P^{++}(s) \). The ordered \( q \)-wedge products \( \{ u_k \mid k \in P^{++}(s) \} \) form a basis of \( \Lambda^s \) called the standard basis.
2.2. Abacus. It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with $N$ runners labeled $1, 2, \cdots, N$ from left to right. The positions on the $i$-th runner are labeled by the integers having residue $i$ modulo $N$.

\[
\begin{array}{cccccc}
& & & & & \\
& -N & -N + 1 & -N + 2 & \cdots & 0 \\
1 & 2 & 3 & \cdots & N - 1 & N \\
N & N + 1 & N + 2 & \cdots & 2N - 1 & 2N \\
& & & & & \\
\end{array}
\]

Each $k \in P^{+}(s)$ (or the corresponding $q$-wedge product $u_{k}$) can be represented by a bead-configuration on the abacus with $N\ell$ runners and beads put on the positions $k_{1}, k_{2}, \cdots$. We call this configuration the abacus presentation of $u_{k}$.

Example 2.2. If $n = 2$, $\ell = 3$, $s = 0$, and $k = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots)$, then the abacus presentation of $u_{k}$ is

\[
\begin{array}{c|c|c}
 d = 1 & d = 2 & d = 3 \\
\hline
 \vdots & \vdots & \vdots \\
 1 \otimes 1 & 1 \otimes 1 & 1 \otimes 1 \\
 1 \otimes 0 & -9 & -8 \\
 -5 & -3 & -2 \\
 1 & 2 & 3 \\
 \vdots & \vdots & \vdots \\
 c = 1 & c = 2 & c = 1 \\
\hline
\end{array}
\]

We use another labeling of runners and positions. Given an integer $k$, let $c, d$ and $m$ be the unique integers satisfying

\[
k = c + n(d - 1) - n\ell m , \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell.
\]

Then, in the abacus presentation, the position $k$ is on the $c + n(d - 1)$-th runner (see the previous example). Relabeling the position $k$ by $c - nm$, we have $\ell$ abaci with $n$ runners.

Example 2.3. In the previous example, relabeling the position $k$ by $c - nm$, we have

\[
\begin{array}{c|c|c}
 d = 1 & d = 2 & d = 3 \\
\hline
 \vdots & \vdots & \vdots \\
 -5 & -2 & -5 \\
 -3 & 0 & -3 \\
 1 & 2 & 1 \\
 \vdots & \vdots & \vdots \\
 c = 1 & c = 1 & c = 1 \\
\hline
\end{array}
\]
We assign to each of \( \ell \) abacus presentations with \( n \) runners a \( q \)-wedge product of level 1. In fact, straightening rules in each "sector" are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see Example 2.5 below)

We introduce some notation.

**Definition 2.4.** For an integer \( k \), let \( c, d \) and \( m \) be the unique integers satisfying (4), and write

\[
(5) \quad u_k = u_{c-nm}^{(d)}.
\]

Also we write \( u_{c_1-nm_1}^{(d_1)} > u_{c_2-nm_2}^{(d_2)} \) if \( k_1 > k_2 \), where \( k_i = c_i + n(d_i - 1) - n\ell m_i \), \( (i = 1, 2) \).

We regard \( u_{c-nm}^{(d)} \) as \( u_{c-nm} \) in the case of \( \ell = 1 \).

**Example 2.5.** If \( n = 2, \ell = 3 \), then we have

\[
(6) \quad u_{-10} \land u_1 = -q^{-1} u_1 \land u_{-10} + (q^{-2} - 1) u_{-4} \land u_{-5},
\]

that is,

\[
u_{-2} \land u_1 = -q^{-1} u_1 \land u_{-2} + (q^{-2} - 1) u_0 \land u_{-1}.
\]

On the other hand, in the case of \( n = 2, \ell = 1 \),

\[
u_{-2} \land u_1 = -q^{-1} u_1 \land u_{-2} + (q^{-2} - 1) u_0 \land u_{-1}.
\]

### 2.3. \( \ell \)-tuples of Young diagrams.

Another indexation of the ordered \( q \)-wedge products is given by the set of pairs \( (\lambda, s) \) of \( \ell \)-tuples of Young diagrams \( \lambda = (\lambda^{(1)}, \cdots, \lambda^{(\ell)}) \) and integer sequences \( s = (s_1, \cdots, s_\ell) \) summing up to \( s \). Let \( k = (k_1, k_2, \cdots) \in P^{+}(s) \), and write

\[
k_r = c_r + n(d_r - 1) - n\ell m_r, \quad 1 \leq c_r \leq n, \quad 1 \leq d_r \leq \ell, \quad m_r \in \mathbb{Z}.
\]

For \( d \in \{1, 2, \cdots, \ell\} \), let \( k_1^{(d)}, k_2^{(d)}, \cdots \) be integers such that

\[
\beta^{(d)} = \{c_r - nm_r \mid d_r = d\} = \{k_1^{(d)}, k_2^{(d)}, \cdots \} \quad \text{and} \quad k_1^{(d)} > k_2^{(d)} > \cdots
\]

Then we associate to the sequence \( (k_1^{(d)}, k_2^{(d)}, \cdots) \) an integer \( s_d \) and a partition \( \lambda^{(d)} \) by

\[
k_r^{(d)} = s_d - r + 1 \quad \text{for sufficiently large } r \quad \text{and} \quad \lambda_r^{(d)} = k_r^{(d)} - s_d + r - 1 \quad \text{for } r \geq 1.
\]

In this correspondence, we also write

\[
(7) \quad u_k = |\lambda; s\rangle \quad (k \in P^{+}(s)).
\]

**Example 2.6.** If \( n = 2, \ell = 3, s = 0, \) and \( k = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots) \), then

\[
(8) \quad k_1 = 6 = 2 + 2(3 - 1) - 6 \cdot 0, \quad k_2 = 3 = 1 + 2(2 - 1) - 6 \cdot 0, \quad k_3 = 2 + 2(1 - 1) - 6 \cdot 0, \quad \cdots \quad \text{and so on.}
\]

Hence,

\[
\beta^{(1)} = \{2, 1, 0, -1, -2, \cdots\}, \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \cdots\}, \quad \beta^{(3)} = \{2, -3, -4, -5, \cdots\}.
\]

Thus, \( s = (2, 0, -2) \) and \( \lambda = (\emptyset, (1, 1), (4)) \).

Note that we can read off \( s = (2, 0, -2) \) and \( \lambda = (\emptyset, (1, 1), (4)) \) from the abacus presentation. (see Example 2.3)
2.4. The q-deformed Fock spaces of higher levels.

Definition 2.7. For $s \in \mathbb{Z}^{\ell}$, we define the q-deformed Fock space $F_{q}[s]$ of level $\ell$ to be the subspace of $\Lambda^{s}$ spanned by $|\lambda; s\rangle$ ($\lambda \in \Pi^{\ell}$):

(7) $F_{q}[s] = \bigoplus_{\lambda \in \Pi^{\ell}} \mathbb{Q}(q)|\lambda; s\rangle$.

We call $s$ a multi charge.

2.5. The bar involution.

Definition 2.8. The involution $\overline{\cdot}$ of $\Lambda^{s}$ is the $\mathbb{Q}$-vector space automorphism such that $\overline{q} = q^{-1}$ and

(8) $\overline{u_{k}} = \overline{u_{k_{1}} \wedge \cdots \wedge u_{k_{r}}} \wedge u_{k_{r+1}} \wedge \cdots = (-q)^{\kappa(c_{1}, \cdots, c_{r})}q^{-\kappa(c_{1}, \cdots, c_{r})}(u_{k_{r}} \wedge \cdots \wedge u_{k_{1}} \wedge u_{k_{r+1}} \wedge \cdots)$,

where $c_{i}, d_{i}$ are defined by $k_{i}$ as in (4), $r$ is an integer satisfying $k_{r} = s - r + 1$. And $\kappa(a_{1}, \cdots, a_{r})$ is defined by

$\kappa(a_{1}, \cdots, a_{r}) = \#\{(i, j) | i < j, a_{i} = a_{j}\}$.

Remarks (i) The involution is well defined, i.e. it doesn’t depend on $r$ [Ugl00].

(ii) The involution comes from the bar involution of affine Hecke algebra $H_{r}$. (see [Ugl00] for more detail.)

(iii) The involution preserves the q-deformed Fock space $F_{q}[s]$ of higher level.

2.6. The dominance order. We define a partial ordering $|\lambda; s\rangle \geq |\mu; s\rangle$. For $|\lambda; s\rangle$ and $|\mu; s\rangle$, we define multi-sets $\overline{\lambda}$ and $\overline{\mu}$ as

$\overline{\lambda} = \{\lambda_{a}^{(d)} + s_{d} | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(J^{l^{(d)}}))\}$,

$\overline{\mu} = \{\mu_{a}^{(d)} + s_{d} | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$.

We denote by $\overline{\lambda}_{1}, \overline{\lambda}_{2}, \cdots$ (resp. $\overline{\mu}_{1}, \overline{\mu}_{2}, \cdots$) the sequence obtained by rearranging the elements in the multi-set $\lambda$ (resp. $\mu$) in decreasing order.

Definition 2.9. Let $|\lambda; s\rangle = u_{k_{1}} \wedge u_{k_{2}} \wedge \cdots$ and $|\mu; s\rangle = u_{g_{1}} \wedge u_{g_{2}} \wedge \cdots$. We define $|\lambda; s\rangle \geq |\mu; s\rangle$ if $|\lambda| = |\mu|$ and

(9) $\begin{cases} (a) & \overline{\lambda} \neq \overline{\mu}, \quad \sum_{j=1}^{r}\lambda_{j} \geq \sum_{j=1}^{r}\mu_{j} (\text{for all } r = 1, 2, 3, \cdots) , \text{ or} \\ (b) & \overline{\lambda} = \overline{\mu}, \quad \sum_{j=1}^{r}\lambda_{j} \geq \sum_{j=1}^{r}\mu_{j} (\text{for all } r = 1, 2, 3, \cdots). \end{cases}$

Remark. The order in Definition 2.9 is different from the order in [Ugl00] (see Example 2.10 below). However, the unitriangularity in (11) holds for both of them.

Example 2.10. Let $n = \ell = 2$, $s = (1, -1)$, $\lambda = ((1,1), \emptyset)$, and $\mu = (\emptyset, (2))$. Then, $|\lambda; s\rangle = u_{2} \wedge u_{1} \wedge u_{-1} \wedge u_{-3} \wedge \cdots$ and $|\mu; s\rangle = u_{3} \wedge u_{1} \wedge u_{-2} \wedge u_{-3} \wedge \cdots$. In Uglov’s order, $|\mu; s\rangle$ is greater than $|\lambda; s\rangle$. However, $|\lambda; s\rangle > |\mu; s\rangle$ under our order since $\{\lambda_{1}, \lambda_{2}, \lambda_{3}\} = \{2, 2, -1\}$ and $\{\mu_{1}, \mu_{2}, \mu_{3}\} = \{1, 1, 1\}$.
We define a matrix $(a_{\lambda,\mu}(q))_{\lambda,\mu}$ by

\[(10) \quad \overline{\langle \lambda; s \rangle} = \sum_{\mu} a_{\lambda,\mu}(q) \langle \mu; s \rangle.\]

Then the matrix $(a_{\lambda,\mu}(q))_{\lambda,\mu}$ is unitriangular with respect to $\geq$, that is

\[(11) \begin{cases} & (a) \text{ if } a_{\lambda,\mu}(q) \neq 0, \text{ then } \langle \lambda; s \rangle \geq \langle \mu; s \rangle, \\ & (b) \quad a_{\lambda,\lambda}(q) = 1. \end{cases}\]

Thus, by the standard argument, the unitriangularity implies the following theorem.

**Theorem 2.11.** [Ugl00] There exist unique bases \(\{G^+(\lambda; s)|\lambda \in \Pi^f\}\) and \(\{G^-(\lambda; s)|\lambda \in \Pi^f\}\) of \(F_q[s]\) such that

\[(i) \quad \overline{G^+(\lambda; s)} = G^+(\lambda; s), \quad \overline{G^-(\lambda; s)} = G^-(\lambda; s)\]

\[(ii) \quad G^+(\lambda; s) \equiv |\lambda; s\rangle \mod qL^+, \quad G^-(\lambda; s) \equiv |\lambda; s\rangle \mod q^{-1}L^-\]

where

\[L^+ = \bigoplus_{\lambda \in \Pi^f} \mathbb{Q}[q]|\lambda; s\rangle, \quad L^- = \bigoplus_{\lambda \in \Pi^f} \mathbb{Q}[q^{-1}]|\lambda; s\rangle.\]

**Definition 2.12.** Define matrices \(\Delta^+(q) = (\Delta_{\lambda,\mu}^+(q))_{\lambda,\mu}\) and \(\Delta^-(q) = (\Delta_{\lambda,\mu}^-(q))_{\lambda,\mu}\) by

\[(12) \quad G^+(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^+(q) |\mu; s\rangle, \quad G^-(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^-(q) |\mu; s\rangle.\]

The entries \(\Delta_{\lambda,\mu}^\pm(q)\) are called \(q\)-decomposition numbers. Note that \(q\)-decomposition numbers \(\Delta^\pm(q)\) depend on \(n, \ell\) and \(s\). The matrices \(\Delta^+(q)\) and \(\Delta^-(q)\) are also unitriangular with respect to \(\geq\).

It is known [Ugl00, Theorem 3.26] that the entries of \(\Delta^-(q)\) are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type \(A\), and that they are polynomials in \(p = -q\) with non-negative integer coefficients (see [KT02]).

### 3. A Comparison of \(q\)-Decomposition Numbers

#### 3.1. Sufficiently large and sufficiently small.

**Definition 3.1.** Let \(s = (s_1, s_2, \cdots, s_\ell) \in \mathbb{Z}^\ell\) be a multi charge and \(1 \leq j \leq \ell\).

(i). We say that the \(j\)-th component \(s_j\) of the multi charge \(s\) is **sufficiently large** for \(|\lambda; s\rangle \in F_q[s]\) if

\[(13) \quad s_j - s_i \geq \lambda_i^{(0)} \quad \text{for all } i = 1, 2, \cdots, \ell.\]

More generally, we say that \(s_j\) is sufficiently large for a \(q\)-wedge \(u_k\) if

\[(14) \quad s_j \geq c_r - nm_r \quad \text{for all } r = 1, 2, \cdots, \]

where \(k_r = c_r + n(d_r - 1) - nm_r, (r = 1, 2, \cdots), 1 \leq c \leq n\) and \(1 \leq d \leq \ell\) (see §2).

(ii). We say that \(s_j\) is **sufficiently small** for \(|\lambda; s\rangle\) if

\[(15) \quad s_i - s_j \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \quad \text{for all } i \neq j.\]
Note that the definition of sufficiently small depends only on the size of \( \lambda \) and the multi charge \( s \). When we fix the multi charge \( s \), we say that \( s_j \) is sufficiently small for \( N \) if

\[ s_i - s_j \geq N \quad \text{for all} \quad i \neq j. \]

**Remark.** If \( |\lambda; s| \) is 0-dominant in the sense of [Ugl00], that is

\[ s_i - s_{i+1} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(f)}| \quad \text{for all} \quad i = 1, 2, \ldots, f-1, \]

then \( s_1 \) is sufficiently large for \( |\lambda; s| \) and \( s_f \) is sufficiently small for \( |\lambda; s| \).

**Lemma 3.2.** If \( s_j \) is sufficiently large for \( |\lambda; s| \) and \( |\lambda; s| \geq |\mu; s| \), then

(i) \( \lambda^{(j)} = \emptyset \),

(ii) \( s_j \) is also sufficiently large for \( |\mu; s| \). In particular, \( \lambda^{(j)} = \emptyset \).

**Proof.** It is clear that \( \lambda^{(j)} = \emptyset \) by the definition.

Note that

\[ s_j \text{ is sufficiently large for } |\lambda; s| \iff s_j - s_i \geq \lambda^{(i)}_j \quad \text{for all } i = 1, 2, \ldots, f \]

\[ \iff s_j \geq \max(\lambda^{(1)}_1 + s_1, \ldots, \lambda^{(f)}_{f-1} + s_f) = \lambda_1. \]

If \( |\lambda; s| \geq |\mu; s| \), then \( \lambda_1 \geq \mu_1 \) and so \( s_j \geq \mu_1 \). It means that \( s_j \) is sufficiently large for \( |\mu; s| \). \( \square \)

**Lemma 3.3.** Suppose that \( s_j \) is sufficiently small for \( |\lambda; s| \). If \( |\lambda; s| \geq |\mu; s| \) and \( \mu^{(j)} = \emptyset \), then \( \lambda^{(j)} = \emptyset \).

**Proof.** Suppose that \( l(\lambda^{(j)}) \geq 1 \). Then \( s_j \) is the minimal integer in the set \( \{ \lambda^{(d)}_a + s_d | 1 \leq d \leq f, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)})) \} \) because \( \mu^{(j)} = \emptyset \) and \( s_j \) is the minimal integer in \( s \). On the other hand, the minimal integer in the set \( \{ \lambda^{(d)}_a + s_d | 1 \leq d \leq f, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)})) \} \) is greater than \( s_j \) because \( s_j \) is sufficiently small for \( |\lambda; s| \). Therefore \( |\lambda; s| \not\geq |\mu; s| \). This is a contradiction. \( \square \)

### 3.2. Main Results

Now, we are ready to state our main theorems.

**Theorem 3.4 ([Iij]).** Let \( \varepsilon \in \{+, -\} \). If \( s_j \) is sufficiently large for \( |\lambda; s| \), then

\[ \Delta^\varepsilon_{\lambda, \mu; s}(q) = \Delta^\varepsilon_{\lambda, \mu; s}(q), \]

where \( \check{\lambda} \) (resp. \( \check{\mu}, \check{s} \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

**Theorem 3.5 ([Iij]).** Let \( \varepsilon \in \{+, -\} \). If \( s_j \) is sufficiently small for \( |\mu; s| \) and \( \mu^{(j)} = \emptyset \), then

\[ \Delta^\varepsilon_{\lambda, \mu; s}(q) = \Delta^\varepsilon_{\lambda, \mu; s}(q), \]

where \( \check{\lambda} \) (resp. \( \check{\mu}, \check{s} \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

**Example 3.6.** (i) If \( n = f = 4, s = (3, -3) \) and \( \lambda = (0, (6)), \mu = (0, (5, 1)) \), then \( s_1 \) is sufficiently large for \( |\lambda; s| \). Hence

\[ \Delta^+_{\lambda, \mu; s}(q) = \Delta^-_{\lambda, \mu; s}(q) = \Delta^-_{(0), (5, 1)(-3)}(q) = -q^{-1}. \]

(ii) If \( n = f = 4, s = (3, -3) \) and \( \lambda = ((6), 0), \mu = ((5, 1), 0) \), then \( s_2 \) is sufficiently small for \( |\mu; s| \). Hence
\[ \Delta_{\check{\lambda},\check{\mu};\check{s}}^{-}(q) = \Delta_{(6),(5,1);(-3)}^{-}(q) = -q^{-1}. \]

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