A CORRESPONDENCE OF CANONICAL BASES IN THE $q$-DEFORMED HIGHER LEVEL FOCK SPACES

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ABSTRACT. The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. The $q$-decomposition matrix is a transition matrix from the standard basis to the canonical basis defined by Uglov in the $q$-deformed Fock space. In this paper, we show that parts of $q$-decomposition matrices of level $\ell$ coincide with that of level $\ell - 1$ under certain conditions of multi charge.

1. Introduction

The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMMO91]. For a multi charge $s = (s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell$, the $q$-deformed Fock space $F_q[s]$ of level $\ell$ is the $Q(q)$-vector space whose basis are indexed by $\ell$-tuples of Young diagrams, i.e., $\{|\lambda; s| \lambda \in \Pi^\ell\}$, where $\Pi$ is the set of Young diagrams.

The canonical bases $\{G^+ (\lambda; s) | \lambda \in \Pi^\ell\}$ and $\{G^- (\lambda; s) | \lambda \in \Pi^\ell\}$ are bases of the Fock space $F_q[s]$ that are invariant under a certain involution $^\pi$ [Ug100]. Define matrices $\Delta^+ (q) = (\Delta^+_{\lambda, \mu}(q))_{\lambda, \mu}$ and $\Delta^- (q) = (\Delta^-_{\lambda, \mu}(q))_{\lambda, \mu}$ by

$$G^+ (\lambda; s) = \sum_{\mu} \Delta^+_{\lambda, \mu}(q) |\mu; s\rangle \quad , \quad G^- (\lambda; s) = \sum_{\mu} \Delta^-_{\lambda, \mu}(q) |\mu; s\rangle.$$

We call $\Delta^+_{\lambda, \mu}(q)$ and $\Delta^-_{\lambda, \mu}(q)$ $q$-decomposition numbers. These $q$-decomposition matrices play an important role in representation theory. However, it is difficult to compute $q$-decomposition matrices.

In the case of $\ell = 1$, Varagnolo-Vasserot [VV99] proved that $\Delta^+ (q)$ coincides with the decomposition matrix of $q$-Schur algebra. For $\ell \geq 2$, Yvonne [Yvo07] conjectured that the matrix $\Delta^+ (q)$ coincides with the $q$-analogue of the decomposition matrix of cyclotomic Schur algebras at a primitive $n$-th root of unity under a suitable condition of multi charge. Rouquier [Rou08, Theorem 6.8, \S 6.5] conjectured that, for arbitrary multi charge, the multiplicities of simple modules in standard modules in the category $O$ of rational Cherednik algebras are equal to the corresponding coefficients $\Delta^+_{\lambda, \mu}(q)$.

We say that the $j$-th component $s_j$ of the multi charge is sufficiently large for $|\lambda; s|$ if $s_j - s_i \geq \lambda_i^{(j)}$ for any $i = 1, 2, \ldots, \ell$, and that $s_j$ is sufficiently small for $|\lambda; s|$ if $s_i - s_j \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}|$ for any $i = 1, 2, \ldots, \ell$ (see Definition 3.1). If $s_j$ is sufficiently large for $|\lambda; s|$ and $|\lambda; s| > |\mu; s|$, then the $j$-th components of $\lambda$ and $\mu$ are both the empty Young diagram $\emptyset$ (Lemma 3.2). On the other hand, if $s_j$ is sufficiently small for $|\lambda; s|$ and $|\lambda; s| > |\mu; s|$, then $\mu^{(j)} = \emptyset$ implies $\lambda^{(j)} = \emptyset$ (Lemma 3.3).

Our main results are as follows.

Theorem A. (Theorem 3.4) [Iij]

Let $\epsilon \in \{+, -\}$. If $s_j$ is sufficiently large for $|\lambda; s|$, then

$$\Delta_{\lambda, \mu; 3}^{\epsilon}(q) = \Delta_{\lambda, \mu; 3}^{\epsilon}(q).$$
where \( \bar{\lambda} \) (resp. \( \bar{\mu}, \bar{s} \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)), \( \Delta^{e}_{\bar{\lambda} \bar{\mu} \bar{s}}(q) \) is the \( q \)-decomposition number of level \( \ell \) and \( \Delta^{e}_{\bar{\lambda} \bar{\mu} \bar{s}}(q) \) is the \( q \)-decomposition number of level \( \ell - 1 \).

**Theorem B.** (Theorem 3.5) [Iij]

Let \( \epsilon \in \{+, -\} \). If \( s_{j} \) is sufficiently small for \( |\mu; s| \) and \( \mu^{(0)} = 0 \), then

\[
\Delta^{e}_{\lambda \mu s}(q) = \Delta^{e}_{\bar{\lambda} \bar{\mu} \bar{s}}(q),
\]

where \( \bar{\lambda} \) (resp. \( \bar{\mu}, \bar{s} \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

This paper is organized as follows. In Section 2, we review the \( q \)-deformed Fock spaces of higher levels and its canonical bases. In Section 3, we state the main results.

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**Notations.** For a positive integer \( N \), a partition of \( N \) is a non-increasing sequence of non-negative integers summing to \( N \). We write \( |\lambda| = N \) if \( \lambda \) is a partition of \( N \). The length \( \ell(\lambda) \) of \( \lambda \) is the number of non-zero components of \( \lambda \). And we use the same notation \( \lambda \) to represent the Young diagram corresponding to \( \lambda \). For an \( \ell \)-tuple \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(\ell)}) \) of Young diagrams, we put \( |\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(\ell)}| \).

2. **The \( q \)-deformed Fock spaces of higher levels**

2.1. **\( q \)-wedge products and straightening rules.** Let \( n, \ell, s \) be integers such that \( n \geq 2 \) and \( \ell \geq 1 \). We define \( P(s) \) and \( P^{++}(s) \) as follows;

\[
\begin{align*}
(1) & \quad P(s) = \{ k = (k_1, k_2, \cdots) \in \mathbb{Z}^{\infty} \mid k_r = s - r + 1 \text{ for any sufficiently large } r \} , \\
(2) & \quad P^{++}(s) = \{ k = (k_1, k_2, \cdots) \in P(s) \mid k_1 > k_2 > \cdots \} .
\end{align*}
\]

Let \( \Lambda^{s} \) be the \( \mathbb{Q}(q) \) vector space spanned by the \( q \)-wedge products

\[
uk = u_{k_1} \wedge u_{k_2} \wedge \cdots , \quad (k \in P(s))
\]

subject to certain commutation relations, so-called straightening rules. Note that the straightening rules depend on \( n \) and \( \ell \). [Ugl00, Proposition 3.16].

**Example 2.1.** (i) For every \( k_1 \in \mathbb{Z}, u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1} \). Therefore \( u_{k_1} \wedge u_{k_1} = 0 \).

(ii) Let \( n = 2, \ell = 2, k_1 = -2, \) and \( k_2 = 4 \). Then

\[
\begin{align*}
\wedge u_{-2}, q u_{4} \wedge u_{-2} + (q^2 - 1) u_{2} \wedge u_{0}.
\end{align*}
\]

(iii) Let \( n = 2, \ell = 2, k_1 = -1, k_2 = -2 \) and \( k_3 = 4 \). Then

\[
\begin{align*}
\wedge u_{-1}, q u_{4} \wedge u_{-2} + (q^2 - 1) u_{2} \wedge u_{0}
\end{align*}
\]

By applying the straightening rules, every \( q \)-wedge product \( u_k \) is expressed as a linear combination of so-called ordered \( q \)-wedge products, namely \( q \)-wedge products \( u_k \) with \( k \in P^{++}(s) \). The ordered \( q \)-wedge products \( \{u_k \mid k \in P^{++}(s)\} \) form a basis of \( \Lambda^{s} \) called the standard basis.
2.2. **Abacus.** It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with $N$ runners labeled $1, 2, \cdots N$ from left to right. The positions on the $i$-th runner are labeled by the integers having residue $i$ modulo $N$.

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-N+1 & -N+2 & \cdots & -1 & 0 \\
1 & 2 & \cdots & N-1 & N \\
N+1 & N+2 & \cdots & 2N-1 & 2N \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

Each $k \in P^{++}(s)$ (or the corresponding $q$-wedge product $u_k$) can be represented by a bead-configuration on the abacus with $n\ell$ runners and beads put on the positions $k_1, k_2, \cdots$. We call this configuration the **abacus presentation** of $u_k$.

**Example 2.2.** If $n = 2$, $\ell = 3$, $s = 0$, and $k = (6, 3, 2, -2, -4, -5, -7, -8, -9, \cdots)$, then the abacus presentation of $u_k$ is

\[
\begin{array}{cccc}
d = 1 & d = 2 & d = 3 \\
\cdots & \cdots & \cdots & \cdots \\
1 & 0 & 13 & 12 & \cdots m = 3 \\
1 & 0 & -9 & -8 & -7 & -6 & \cdots m = 2 \\
-5 & 4 & -3 & -2 & -1 & 0 & \cdots m = 1 \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots m = 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

We use another labeling of runners and positions. Given an integer $k$, let $c, d$ and $m$ be the unique integers satisfying

\begin{equation}
k = c + n(d - 1) - n\ell m, \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell.
\end{equation}

Then, in the abacus presentation, the position $k$ is on the $c + n(d - 1)$-th runner (see the previous example). Relabeling the position $k$ by $c - nm$, we have $\ell$ abaci with $n$ runners.

**Example 2.3.** In the previous example, relabeling the position $k$ by $c - nm$, we have

\[
\begin{array}{cccc}
d = 1 & d = 2 & d = 3 \\
\cdots & \cdots & \cdots & \cdots \\
-5 & -4 & -5 & -4 & \cdots m = 3 \\
-3 & -2 & -3 & -2 & -3 & -2 & \cdots m = 2 \\
-1 & 0 & -1 & 0 & -1 & 0 & \cdots m = 1 \\
1 & 2 & 1 & 2 & \cdots m = 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
c = 1 & c = 2 & c = 1 & c = 2 \\
c = 2 & c = 1 & c = 2
\end{array}
\]
We assign to each of $\ell$ abacus presentations with $n$ runners a $q$-wedge product of level 1. In fact, straightening rules in each “sector” are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see Example 2.5 below)

We introduce some notation.

**Definition 2.4.** For an integer $k$, let $c, d$ and $m$ be the unique integers satisfying (4), and write

$$u_k = u_{c-nm}^{(d)}. \quad \text{(5)}$$

Also we write $u_{c-nm_1} > u_{c-nm_2}$ if $k_1 > k_2$, where $k_i = c_i + n(d_i - 1) - n\ell m_i$, $i = 1, 2$.

We regard $u_{c-nm}$ as $u_{c-nm_0}$ in the case of $\ell = 1$.

**Example 2.5.** If $n = 2$, $\ell = 3$, then we have

$$u_{-10} \wedge u_1 = -q^{-1}u_1 \wedge u_{-10}+(q^{-2}-1)u_{-4} \wedge u_{-5},$$

that is,

$$u_{-12}^{(1)} \wedge u_2^{(1)} = -q^{-1}u_1^{(1)} \wedge u_{-12}^{(1)}+(q^{-2}-1)u_0^{(1)} \wedge u_{-1}. \quad \text{(6)}$$

On the other hand, in the case of $n = 2$, $\ell = 1$,

$$u_{-2} \wedge u_1 = -q^{-1}u_1 \wedge u_{-2}+(q^{-2}-1)u_0 \wedge u_{-1}.$$

2.3. $\ell$-tuples of Young diagrams. Another indexation of the ordered $q$-wedge products is given by the set of pairs $(\lambda, s)$ of $\ell$-tuples of Young diagrams $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(\ell)})$ and integer sequences $s = (s_1, \cdots, s_\ell)$ summing up to $s$. Let $k = (k_1, k_2, \cdots) \in P^{++}(s)$, and write

$$k_r = c_r + n(d_r - 1) - n\ell m_r, \quad 1 \leq c_r \leq n, \quad 1 \leq d_r \leq \ell, \quad m_r \in \mathbb{Z}. \quad \text{(7)}$$

For $d \in \{1, 2, \cdots, \ell\}$, let $k_1^{(d)}, k_2^{(d)}, \cdots$ be integers such that

$$\beta^{(d)} = \{c_r - nm_r | d_r = d\} = \{k_1^{(d)}, k_2^{(d)}, \cdots\} \quad \text{and} \quad k_1^{(d)} > k_2^{(d)} > \cdots$$

Then we associate to the sequence $(k_1^{(d)}, k_2^{(d)}, \cdots)$ an integer $s_d$ and a partition $\lambda^{(d)}$ by

$$k_r^{(d)} = s_d - r + 1 \quad \text{for sufficiently large } r \quad \text{and} \quad \lambda^{(d)}_r = k_r^{(d)} - s_d + r - 1 \quad \text{for } r \geq 1. \quad \text{(8)}$$

In this correspondence, we also write

$$u_k = |\lambda; s\rangle \quad (k \in P^{++}(s)). \quad \text{(9)}$$

**Example 2.6.** If $n = 2$, $\ell = 3$, $s = 0$, and $k = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots)$, then

$$k_1 = 6 = 2 + 2(3-1) - 6 \cdot 0, \quad k_2 = 3 = 1 + 2(2-1) - 6 \cdot 0, \quad k_3 = 2 = 2 + 2(1-1) - 6 \cdot 0, \quad \cdots \quad \text{and so on.}$$

Hence,

$$\beta^{(1)} = \{2, 1, 0, -1, -2, \cdots\}, \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \cdots\}, \quad \beta^{(3)} = \{2, -3, -4, -5, \cdots\}.$$

Thus, $s = (2, 0, -2)$ and $\lambda = (\emptyset, (1, 1), (4)).$

Note that we can read off $s = (2, 0, -2)$ and $\lambda = (\emptyset, (1, 1), (4))$ from the abacus presentation. (see Example 2.3)
2.4. The $q$-deformed Fock spaces of higher levels.

**Definition 2.7.** For $s \in \mathbb{Z}^\ell$, we define the $q$-deformed Fock space $F_q[s]$ of level $\ell$ to be the subspace of $\Lambda^s$ spanned by $|\lambda; s\rangle$ ($\lambda \in \Pi^\ell$):

$$F_q[s] = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}(q)|\lambda; s\rangle.$$  

We call $s$ a multi charge.

2.5. The bar involution.

**Definition 2.8.** The involution $\overline{\cdot}$ of $\Lambda^s$ is the $\mathbb{Q}$-vector space automorphism such that $\overline{q} = q^{-1}$ and

$$\overline{u_k} = \overline{u_{k_1} \wedge \cdots \wedge u_{k_r} \wedge \cdots} = (-q)^{c_1 \cdots c_r} q^{-\kappa(c_1, \cdots, c_r)} (u_{k_r} \wedge \cdots \wedge u_{k_1} \wedge \cdots),$$

where $c_i, d_i$ are defined by $k_i$ as in (4), $r$ is an integer satisfying $k_r = s - r + 1$. And $\kappa(a_1, \cdots, a_r)$ is defined by

$$\kappa(a_1, \cdots, a_r) = \# \{(i, j) | i < j, a_i = a_j\}.$$  

**Remarks** (i) The involution is well defined, i.e. it doesn’t depend on $r$ [Ugl00].

(ii) The involution comes from the bar involution of affine Hecke algebra $H_r$. (see [Ugl00] for more detail.)

(iii) The involution preserves the $q$-deformed Fock space $F_q[s]$ of higher level.

2.6. The dominance order. We define a partial ordering $|\lambda; s\rangle \geq |\mu; s\rangle$. For $|\lambda; s\rangle$ and $|\mu; s\rangle$, we define multi-sets $\overline{\lambda}$ and $\overline{\mu}$ as

$$\overline{\lambda} = \{\lambda_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(J^{l^{(d)}}))\},$$

$$\overline{\mu} = \{\mu_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}.$$  

We denote by $\overline{(\lambda_1, \lambda_2, \cdots)}$ (resp. $\overline{\mu_1, \mu_2, \cdots}$) the sequence obtained by rearranging the elements in the multi-set $\overline{\lambda}$ (resp. $\overline{\mu}$) in decreasing order.

**Definition 2.9.** Let $|\lambda; s\rangle = u_{k_1} \wedge u_{k_2} \wedge \cdots$ and $|\mu; s\rangle = u_{g_1} \wedge u_{g_2} \wedge \cdots$. We define $|\lambda; s\rangle \geq |\mu; s\rangle$ if $|\lambda| = |\mu|$ and

$$\begin{cases} (a) & \overline{\lambda} \neq \overline{\mu}, & \sum_{j=1}^r \lambda_j \geq \sum_{j=1}^r \mu_j \quad \text{(for all } r = 1, 2, 3, \cdots), \quad \text{or} \quad \\
(b) & \overline{\lambda} = \overline{\mu}, & \sum_{j=1}^r k_j \geq \sum_{j=1}^r g_j \quad \text{(for all } r = 1, 2, 3, \cdots) \end{cases}.$$  

**Remark.** The order in Definition 2.9 is different from the order in [Ugl00] (see Example 2.10 below). However, the unitriangularity in (11) holds for both of them.

**Example 2.10.** Let $n = \ell = 2$, $s = (1, -1)$, $\lambda = ((1, 1), \emptyset)$, and $\mu = (\emptyset, (2))$. Then, $|\lambda; s\rangle = u_2 \wedge u_1 \wedge u_{-1} \wedge u_{-3} \wedge \cdots$ and $|\mu; s\rangle = u_3 \wedge u_1 \wedge u_{-2} \wedge u_{-3} \wedge \cdots$. In Uglov's order, $|\mu; s\rangle$ is greater than $|\lambda; s\rangle$. However, $|\lambda; s\rangle > |\mu; s\rangle$ under our order since $\{\lambda_1, \lambda_2, \lambda_3\} = \{2, 2, -1\}$ and $\{\mu_1, \mu_2, \mu_3\} = \{1, 1, 1\}$.  

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We define a matrix $(a_{\lambda,\mu}(q))_{\lambda,\mu}$ by
\begin{equation}
|\lambda; s\rangle = \sum_{\mu} a_{\lambda,\mu}(q) |\mu; s\rangle.
\end{equation}

Then the matrix $(a_{\lambda,\mu}(q))_{\lambda,\mu}$ is unitriangular with respect to $\geq$, that is
\begin{equation}
\begin{cases}
(a) & \text{if } a_{\lambda,\mu}(q) \neq 0, \text{ then } |\lambda; s\rangle \geq |\mu; s\rangle, \\
(b) & a_{\lambda,\lambda}(q) = 1.
\end{cases}
\end{equation}

Thus, by the standard argument, the unitriangularity implies the following theorem.

**Theorem 2.11.** [Ugl00] There exist unique bases $\{G^{+}(\lambda; s) | \lambda \in \Pi^{\ell} \}$ and $\{G^{-}(\lambda; s) | \lambda \in \Pi^{l} \}$ of $F_{q}[s]$ such that
\begin{align*}
(i) & \quad \overline{G^{+}(\lambda; s)} = G^{+}(\lambda; s), \\
(ii) & \quad G^{+}(\lambda; s) \equiv |\lambda; s\rangle \mod q \mathcal{L}^{+}, \\
& \quad G^{-}(\lambda; s) \equiv |\lambda; s\rangle \mod q^{-1} \mathcal{L}^{-}
\end{align*}
where
\begin{align*}
\mathcal{L}^{+} = \bigoplus_{\lambda \in \Pi^{\ell}} \mathbb{Q}[q] |\lambda; s\rangle, \\
\mathcal{L}^{-} = \bigoplus_{\lambda \in \Pi^{l}} \mathbb{Q}[q^{-1}] |\lambda; s\rangle.
\end{align*}

**Definition 2.12.** Define matrices $\Delta^{+}(q) = (\Delta^{+}_{\lambda,\mu}(q))_{\lambda,\mu}$ and $\Delta^{-}(q) = (\Delta^{-}_{\lambda,\mu}(q))_{\lambda,\mu}$ by
\begin{equation}
G^{+}(\lambda; s) = \sum_{\mu} \Delta^{+}_{\lambda,\mu}(q) |\mu; s\rangle, \\
G^{-}(\lambda; s) = \sum_{\mu} \Delta^{-}_{\lambda,\mu}(q) |\mu; s\rangle.
\end{equation}

The entries $\Delta^{\pm}_{\lambda,\mu}(q)$ are called $q$-decomposition numbers. Note that $q$-decomposition numbers $\Delta^{\pm}(q)$ depend on $n, \ell$ and $s$. The matrices $\Delta^{+}(q)$ and $\Delta^{-}(q)$ are also unitriangular with respect to $\geq$.

It is known [Ugl00, Theorem 3.26] that the entries of $\Delta^{-}(q)$ are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type $A$, and that they are polynomials in $p = -q$ with non-negative integer coefficients (see [KT02]).

### 3. A comparison of $q$-decomposition numbers

#### 3.1. Sufficiently large and sufficiently small.

**Definition 3.1.** Let $s = (s_{1}, s_{2}, \cdots, s_{\ell}) \in \mathbb{Z}^{\ell}$ be a multi charge and $1 \leq j \leq \ell$.

(i). We say that the $j$-th component $s_{j}$ of the multi charge $s$ is **sufficiently large** for $|\lambda; s\rangle \in F_{q}[s]$ if
\begin{equation}
s_{j} - s_{i} \geq \lambda_{i}^{(1)} \quad \text{for all } i = 1, 2, \cdots, \ell.
\end{equation}

More generally, we say that $s_{j}$ is sufficiently large for a $q$-wedge $u_{k}$ if
\begin{equation}
s_{j} \geq c_{r} - nm_{r} \quad \text{for all } r = 1, 2, \cdots,
\end{equation}
where $k_{r} = c_{r} + n(d_{r} - 1) - 2nm_{r}, (r = 1, 2, \cdots), 1 \leq c \leq n$ and $1 \leq d \leq \ell$ (see §2).

(ii). We say that $s_{j}$ is **sufficiently small** for $|\lambda; s\rangle$ if
\begin{equation}
s_{i} - s_{j} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \quad \text{for all } i \neq j.
\end{equation}
Note that the definition of sufficiently small depends only on the size of $\lambda$ and the multi charge $s$. When we fix the multi charge $s$, we say that $s_j$ is sufficiently small for $N$ if

\begin{equation}
    s_i - s_j \geq N \quad \text{for all } i \neq j.
\end{equation}

Remark. If $|\lambda; s|$ is 0-dominant in the sense of [Ugl00], that is

\[ s_i - s_{i+1} \geq |\lambda| = |\lambda^{(i)}| + \cdots + |\lambda^{(\ell)}| \quad \text{for all } i = 1, 2, \cdots, \ell - 1, \]

then $s_1$ is sufficiently large for $|\lambda; s|$ and $s_\ell$ is sufficiently small for $|\lambda; s|$.

**Lemma 3.2.** If $s_j$ is sufficiently large for $|\lambda; s|$ and $|\lambda; s| \geq |\mu; s|$, then

(i) $\lambda^{(j)} = \emptyset$,

(ii) $s_j$ is also sufficiently large for $|\mu; s|$.

In particular, $\mu^{(j)} = \emptyset$.

**Proof.** It is clear that $\lambda^{(j)} = \emptyset$ by the definition.

Note that

$s_j$ is sufficiently large for $|\lambda; s|$ \iff $s_j - s_i \geq \lambda^{(i)}_1$ \quad \text{for all } i = 1, 2, \cdots, \ell$

\[ \Rightarrow s_j \geq \max\{s_1 + \lambda_1^{(1)}, \cdots, s_\ell + \lambda_\ell^{(\ell)}\} = \lambda_1. \]

If $|\lambda; s| \geq |\mu; s|$, then $\lambda_1 \geq \mu_1$ and so $s_j \geq \mu_1$. It means that $s_j$ is sufficiently large for $|\mu; s|$. \qed

**Lemma 3.3.** Suppose that $s_j$ is sufficiently small for $|\lambda; s|$. If $|\lambda; s| \geq |\mu; s|$ and $\mu^{(j)} = \emptyset$, then $\lambda^{(j)} = \emptyset$.

**Proof.** Suppose that $l(\lambda^{(j)}) \geq 1$. Then $s_j$ is the minimal integer in the set $\{\mu_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ because $\mu^{(j)} = \emptyset$ and $s_j$ is the minimal integer in $s$. On the other hand, the minimal integer in the set $\{\lambda_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ is greater than $s_j$ because $s_j$ is sufficiently small for $|\lambda; s|$. Therefore $|\lambda; s| \ngeq |\mu; s|$. This is a contradiction. \qed

### 3.2. Main Results

Now, we are ready to state our main theorems.

**Theorem 3.4 ([Iij]).** Let $\epsilon \in \{+, -\}$. If $s_j$ is sufficiently large for $|\lambda; s|$, then

\begin{equation}
    \Delta^\epsilon_{\lambda, \mu; s}(q) = \Delta^\epsilon_{\lambda, \mu; s}(q),
\end{equation}

where $\lambda$ (resp. $\mu$, $s$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu$, $s$).

**Theorem 3.5 ([Iij]).** Let $\epsilon \in \{+, -\}$. If $s_j$ is sufficiently small for $|\mu; s|$ and $\mu^{(j)} = \emptyset$, then

\begin{equation}
    \Delta^\epsilon_{\lambda, \mu; s}(q) = \Delta^\epsilon_{\lambda, \mu; s}(q),
\end{equation}

where $\lambda$ (resp. $\mu$, $s$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu$, $s$).

**Example 3.6.** (i) If $n = \ell = 2$, $s = (3, -3)$ and $\lambda = (\emptyset, (6)), \mu = (\emptyset, (5, 1))$, then $s_1$ is sufficiently large for $|\lambda; s|$. Hence

\[ \Delta^+_{\lambda, \mu; s}(q) = \Delta^+_{\lambda, \mu; s}(q) = \Delta^+_{(6), (5, 1); (-3)}(q) = -q^{-1}. \]

(ii) If $n = \ell = 2$, $s = (3, -3)$ and $\lambda = ((6), \emptyset), \mu = ((5, 1), \emptyset)$, then $s_2$ is sufficiently small for $|\mu; s|$. Hence
\[ \Delta_{\lambda\mu;\nu}^{-}(q) = \Delta_{\check{\lambda}\check{\mu};\check{\nu}}^{-}(q) = \Delta_{(6),(5,1);(-3)}^{-}(q) = -q^{-1}. \]