A CORRESPONDENCE OF CANONICAL BASES IN THE $q$-DEFORMED HIGHER LEVEL FOCK SPACES
(Combinatorial Representation Theory and its Applications)

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A CORRESPONDENCE OF CANONICAL BASES IN THE $q$-DEFORMED HIGHER LEVEL FOCK SPACES

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ABSTRACT. The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. The $q$-decomposition matrix is a transition matrix from the standard basis to the canonical basis defined by Uglov in the $q$-deformed Fock space. In this paper, we show that parts of $q$-decomposition matrices of level $\ell$ coincide with that of level $\ell - 1$ under certain conditions of multi charge.

1. Introduction

The $q$-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMMO91]. For a multi charge $s = (s_1, \ldots, s_\ell) \in \mathbb{Z}_\ell^\ell$, the $q$-deformed Fock space $F_q[s]$ of level $\ell$ is the $\mathbb{Q}(q)$-vector space whose basis are indexed by $\ell$-tuples of Young diagrams, i.e. $\{\lambda_s \mid \lambda \in \Pi^\ell\}$, where $\Pi$ is the set of Young diagrams.

The canonical bases $\{G^+(\lambda; s) \mid \lambda \in \Pi^\ell\}$ and $\{G^-(\lambda; s) \mid \lambda \in \Pi^\ell\}$ are bases of the Fock space $F_q[s]$ that are invariant under a certain involution $^-$ [UgI00]. Define matrices $\Delta^+(q) = (\Delta^+_{\lambda,\mu}(q))_{\lambda,\mu}$ and $\Delta^-(q) = (\Delta^-_{\lambda,\mu}(q))_{\lambda,\mu}$ by

$$G^+(\lambda; s) = \sum_\mu \Delta^+_{\lambda,\mu}(q) |\mu; s\rangle, \quad G^-(\lambda; s) = \sum_\mu \Delta^-_{\lambda,\mu}(q) |\mu; s\rangle.$$ 

We call $\Delta^+_{\lambda,\mu}(q)$ and $\Delta^-_{\lambda,\mu}(q)$ $q$-decomposition numbers. These $q$-decomposition matrices play an important role in representation theory. However, it is difficult to compute $q$-decomposition matrices.

In the case of $\ell = 1$, Varagnolo-Vasserot [VV99] proved that $\Delta^+(q)$ coincides with the decomposition matrix of $v$-Schur algebra. For $\ell \geq 2$, Yvonne [Yvo07] conjectured that the matrix $\Delta^+(q)$ coincides with the $q$-analogue of the decomposition matrix of cyclotomic Schur algebras at a primitive $n$-th root of unity under a suitable condition of multi charge. Rouquier [Rou08, Theorem 6.8, §6.5] conjectured that, for arbitrary multi charge, the multiplicities of simple modules in standard modules in the category $O$ of rational Cherednik algebras are equal to the corresponding coefficients $\Delta^+_{\lambda,\mu}(q)$.

We say that the $j$-th component $s_j$ of the multi charge is sufficiently large for $|\lambda; s\rangle$ if $s_j - s_i \geq \lambda^{(i)}_1$ for any $i = 1, 2, \ldots, \ell$, and that $s_j$ is sufficiently small for $|\lambda; s\rangle$ if $s_i - s_j \geq |\lambda^{(i)}| + \cdots + |\lambda^{(\iota)}|$ for any $\iota = 1, 2, \ldots, \ell$ (see Definition 3.1). If $s_j$ is sufficiently large for $|\lambda; s\rangle$ and $|\lambda; s\rangle > |\mu; s\rangle$, then the $j$-th components of $\lambda$ and $\mu$ are both the empty Young diagram $\emptyset$ (Lemma 3.2). On the other hand, if $s_j$ is sufficiently small for $|\lambda; s\rangle$ and $|\lambda; s\rangle > |\mu; s\rangle$, then $\mu^{(i)} = \emptyset$ implies $\lambda^{(i)} = \emptyset$ (Lemma 3.3).

Our main results are as follows.

**Theorem A.** (Theorem 3.4) [Iij]

Let $\epsilon \in \{+,-\}$. If $s_j$ is sufficiently large for $|\lambda; s\rangle$, then

$$\Delta^\epsilon_{\lambda,\mu}(q) = \Delta^\epsilon_{\lambda,\mu}(q).$$
where $\lambda$ (resp. $\mu, s$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu, s$), $\Delta^e_{\lambda, \mu, s}(q)$ is the $q$-decomposition number of level $\ell$ and $\Delta^e_{\lambda, \mu, s}(q)$ is the $q$-decomposition number of level $\ell - 1$.

**Theorem B.** (Theorem 3.5) [Iij]

Let $\epsilon \in \{+,-\}$. If $s_j$ is sufficiently small for $|\mu; s\rangle$ and $\mu^{(j)} = 0$, then

$$\Delta^e_{\lambda, \mu, s}(q) = \Delta^e_{\lambda, \mu, s}(q),$$

where $\tilde{\lambda}$ (resp. $\tilde{\mu}, \tilde{s}$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu, s$).

This paper is organized as follows. In Section 2, we review the $q$-deformed Fock spaces of higher levels and its canonical bases. In Section 3, we state the main results.

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**Notations.** For a positive integer $N$, a partition of $N$ is a non-increasing sequence of non-negative integers summing to $N$. We write $|\lambda| = N$ if $\lambda$ is a partition of $N$. The length $l(\lambda)$ of $\lambda$ is the number of non-zero components of $\lambda$. And we use the same notation $\lambda$ to represent the Young diagram corresponding to $\lambda$. For an $\ell$-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(\ell)})$ of Young diagrams, we put $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(\ell)}|$.  

2. THE $q$-DEFORMED FOCK SPACES OF HIGHER LEVELS

2.1. $q$-WEDGE PRODUCTS AND STRAIGHTENING RULES. Let $n, \ell, s$ be integers such that $n \geq 2$ and $\ell \geq 1$. We define $P(s)$ and $P^{++}(s)$ as follows;

\begin{enumerate}
  \item $P(s) = \{ k = (k_1, k_2, \cdots) \in \mathbb{Z}^\infty \mid k_r = s - r + 1$ for any sufficiently large $r \}$,
  \item $P^{++}(s) = \{ k = (k_1, k_2, \cdots) \in P(s) \mid k_1 > k_2 > \cdots \}$.
\end{enumerate}

Let $\Lambda^x$ be the $\mathbb{Q}(q)$ vector space spanned by the $q$-wedge products

\begin{equation}
  u_k = \wedge_{k} u_{k_1} \wedge u_{k_2} \wedge \cdots \quad (k \in P(s))
\end{equation}

subject to certain commutation relations, so-called straightening rules. Note that the straightening rules depend on $n$ and $\ell$. [Ugl00, Proposition 3.16].

**Example 2.1.** (i) For every $k_1 \in \mathbb{Z}$, $u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1}$. Therefore $u_{k_1} \wedge u_{k_1} = 0$.  

(ii) Let $n = 2, \ell = 2, k_1 = -2$, and $k_2 = 4$. Then

$$u_{-2} \wedge u_4 = q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0.$$  

(iii) Let $n = 2, \ell = 2, k_1 = -1$, $k_2 = -2$ and $k_3 = 4$. Then

$$u_{-1} \wedge u_{-2} \wedge u_4 = u_{-1} \wedge u_{-2} \wedge u_4 = u_{-1} \wedge (u_{-2} \wedge u_4) = u_{-1} \wedge (q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0)$$

$$= q u_{-1} \wedge u_4 \wedge u_{-2} + (q^2 - 1) u_{-1} \wedge u_2 \wedge u_0$$

By applying the straightening rules, every $q$-wedge product $u_k$ is expressed as a linear combination of so-called ordered $q$-wedge products, namely $q$-wedge products $u_k$ with $k \in P^{++}(s)$. The ordered $q$-wedge products $\{u_k \mid k \in P^{++}(s)\}$ form a basis of $\Lambda^x$ called the **standard basis**.
2.2. **Abacus.** It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with $N$ runners labeled $1, 2, \ldots, N$ from left to right. The positions on the $i$-th runner are labeled by the integers having residue $i$ modulo $N$.

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
-N & -N+1 & -N+2 & \cdots & -1 & 0 & \\
-1 & 2 & \cdots & N-1 & N & \\
N+1 & N+2 & \cdots & 2N-1 & 2N & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

Each $k \in P^+(s)$ (or the corresponding $q$-wedge product $u_k$) can be represented by a bead-configuration on the abacus with $n\ell$ runners and beads put on the positions $k_1, k_2, \ldots$. We call this configuration the abacus presentation of $u_k$.

**Example 2.2.** If $n = 2$, $\ell = 3$, $s = 0$, and $k = (6, 3, 2, 1, -2, -5, -7, -8, -9, \ldots)$, then the abacus presentation of $u_k$ is

\[
\begin{array}{ccc}
d = 1 & d = 2 & d = 3 \\
\vdots & \vdots & \vdots \\
1^7 & 1^6 & 1^5 1^4 1^3 1^2 \\
1^6 & 1^5 & -9 -8 -7 -6 \\
-5 & -4 & -3 -2 -1 0 \\
1 & 2 & 3 4 5 6 \\
\vdots & \vdots & \vdots \\
c = 1 & c = 2 & c = 2 \\
\end{array}
\]

We use another labeling of runners and positions. Given an integer $k$, let $c, d$ and $m$ be the unique integers satisfying

\begin{equation}
k = c + n(d - 1) - n\ell m, \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell.
\end{equation}

Then, in the abacus presentation, the position $k$ is on the $c + n(d - 1)$-th runner (see the previous example). Relabeling the position $k$ by $c - nm$, we have $\ell$ abaci with $n$ runners.

**Example 2.3.** In the previous example, relabeling the position $k$ by $c - nm$, we have

\[
\begin{array}{ccc}
d = 1 & d = 2 & d = 3 \\
\vdots & \vdots & \vdots \\
-5 & -4 & -3 -5 -4 \\
-3 & -2 & -1 -3 -2 \\
-2 & 0 & -1 0 \\
1 & 2 & 1 2 \\
\vdots & \vdots & \vdots \\
c = 1 & c = 2 & c = 1 \\
\end{array}
\]
We assign to each of \( \ell \) abacus presentations with \( n \) runners a \( q \)-wedge product of level 1. In fact, straightening rules in each “sector” are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see Example 2.5 below)

We introduce some notation.

**Definition 2.4.** For an integer \( k \), let \( c, d \) and \( m \) be the unique integers satisfying (4), and write

\[
\begin{align*}
  u_k &= u_{c,\text{nm}}^{(d)}.
\end{align*}
\]

Also we write \( u_{c_1-\text{nm}_1}^{(d_1)} > u_{c_2-\text{nm}_2}^{(d_2)} \) if \( k_1 > k_2 \), where \( k_i = c_i + n(d_i - 1) - n\ell m_i, \) \( (i = 1, 2) \).

We regard \( u_{c,\text{nm}}^{(d)} \) as \( u_{c-nm} \) in the case of \( f = 1 \).

**Example 2.5.** If \( n = 2, \ell = 3 \), then we have

\[
\begin{align*}
  u_{-2} \wedge u_1 &= -q^{-1} u_1 \wedge u_{-2} + (q^{-2} - 1) u_0 \wedge u_{-1},
\end{align*}
\]

that is,

\[
\begin{align*}
  u_{-2}^{(1)} \wedge u_1^{(1)} &= -q^{-1} u_1^{(1)} \wedge u_{-2}^{(1)} + (q^{-2} - 1) u_0^{(1)} \wedge u_{-1}^{(1)}.
\end{align*}
\]

On the other hand, in the case of \( n = 2, \ell = 1 \),

\[
\begin{align*}
  u_{-2} \wedge u_1 &= -q^{-1} u_1 \wedge u_{-2} + (q^{-2} - 1) u_0 \wedge u_{-1}.
\end{align*}
\]

2.3. \( \ell \)-tuples of Young diagrams. Another indexation of the ordered \( q \)-wedge products is given by the set of pairs \((\lambda, s)\) of \( \ell \)-tuples of Young diagrams \( \lambda = (\lambda^{(1)}, \cdots, \lambda^{(\ell)}) \) and integer sequences \( s = (s_1, \cdots, s_\ell) \) summing up to \( s \). Let \( k = (k_1, k_2, \cdots) \in \mathbb{P}^{++}(s) \), and write

\[
\begin{align*}
  k_r &= c_r + n(d_r - 1) - n\ell m_r, \quad 1 \leq c_r \leq n, \quad 1 \leq d_r \leq \ell, \quad m_r \in \mathbb{Z}.
\end{align*}
\]

For \( d \in \{1, 2, \cdots, \ell\} \), let \( k_1^{(d)}, k_2^{(d)}, \cdots \) be integers such that

\[
\beta^{(d)} = \{c_r - nm_r \mid d_r = d\} = \{k_1^{(d)}, k_2^{(d)}, \cdots\} \quad \text{and} \quad k_1^{(d)} > k_2^{(d)} > \cdots
\]

Then we associate to the sequence \((k_1^{(d)}, k_2^{(d)}, \cdots)\) an integer \( s_d \) and a partition \( \lambda^{(d)} \) by

\[
\begin{align*}
  k_r^{(d)} &= s_d - r + 1 \quad \text{for sufficiently large } r \quad \text{and} \quad \lambda_r^{(d)} = k_r^{(d)} - s_d + r - 1 \quad \text{for } r \geq 1.
\end{align*}
\]

In this correspondence, we also write

\[
\begin{align*}
  u_k &= |\lambda; s\rangle \quad (k \in \mathbb{P}^{++}(s)).
\end{align*}
\]

**Example 2.6.** If \( n = 2, \ell = 3, s = 0, \) and \( k = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots) \), then

\[
\begin{align*}
  k_1 &= 6 = 2 + 2(3 - 1) - 6 \cdot 0, \quad k_2 = 3 = 1 + 2(2 - 1) - 6 \cdot 0, \\
  k_3 &= 2 = 2 + 2(1 - 1) - 6 \cdot 0, \quad \cdots \quad \text{and so on.}
\end{align*}
\]

Hence,

\[
\beta^{(1)} = \{2, 1, 0, -1, -2, \cdots\}, \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \cdots\}, \quad \beta^{(3)} = \{2, -3, -4, -5, \cdots\}.
\]

Thus, \( s = (2, 0, -2) \) and \( \lambda = (\emptyset, (1, 1), (4)) \).

Note that we can read off \( s = (2, 0, -2) \) and \( \lambda = (\emptyset, (1, 1), (4)) \) from the abacus presentation. (see Example 2.3)
2.4. The $q$-deformed Fock spaces of higher levels.

**Definition 2.7.** For $s \in \mathbb{Z}^t$, we define the $q$-deformed Fock space $F_q[s]$ of level $t$ to be the subspace of $\Lambda^s$ spanned by $|\lambda; s\rangle$ ($\lambda \in \Pi^f$):

\[
F_q[s] = \bigoplus_{\lambda \in \Pi^f} \mathbb{Q}(q)|\lambda; s\rangle.
\]

We call $s$ a multi charge.

2.5. The bar involution.

**Definition 2.8.** The involution $\overline{\cdot}$ on $\Lambda^s$ is the $\mathbb{Q}$-vector space automorphism such that $\overline{q} = q^{-1}$ and

\[
\overline{u_k} = \overline{u_{k_1} \wedge \cdots \wedge u_{k_r}} = (-q)^{\kappa(c_1, \cdots, c_r)}(u_{k_{r+1}} \wedge \cdots \wedge u_{k_1})\overline{u}_{k_r} \wedge \cdots,
\]

where $c_i, d_i$ are defined by $k_i$ as in (4), $r$ is an integer satisfying $k_r = s - r + 1$. And $\kappa(a_1, \cdots, a_r)$ is defined by

\[
\kappa(a_1, \cdots, a_r) = \# \{(i, j) | i < j, a_i = a_j\}.
\]

**Remarks** (i) The involution is well defined, i.e. it doesn’t depend on $r$ [Ugl00].

(ii) The involution comes from the bar involution of affine Hecke algebra $H_r$. (see [Ugl00] for more detail.)

(iii) The involution preserves the $q$-deformed Fock space $F_q[s]$ of higher level.

2.6. The dominance order. We define a partial ordering $|\lambda; s\rangle \geq |\mu; s\rangle$. For $|\lambda; s\rangle$ and $|\mu; s\rangle$, we define multi-sets $\overline{\lambda}$ and $\overline{\mu}$ as

\[
\overline{\lambda} = \{\lambda_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(J^{(d)}))\},
\]

\[
\overline{\mu} = \{\mu_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}.
\]

We denote by $(\overline{\lambda}_1, \overline{\lambda}_2, \cdots)$ (resp. $(\overline{\mu}_1, \overline{\mu}_2, \cdots)$) the sequence obtained by rearranging the elements in the multi-set $\lambda$ (resp. $\mu$) in decreasing order.

**Definition 2.9.** Let $|\lambda; s\rangle = u_{k_1} \wedge u_{k_2} \wedge \cdots$ and $|\mu; s\rangle = u_{g_1} \wedge u_{g_2} \wedge \cdots$. We define $|\lambda; s\rangle \geq |\mu; s\rangle$ if $|\lambda| = |\mu|$ and

\[
(9) \quad \begin{cases} (a) & \overline{\lambda} \neq \overline{\mu}, \quad \sum_{j=1}^r \overline{\lambda}_j \geq \sum_{j=1}^r \overline{\mu}_j \quad (\text{for all } r = 1, 2, 3, \cdots) \quad \text{or} \\ (b) & \overline{\lambda} = \overline{\mu}, \quad \sum_{j=1}^r \overline{k}_j \geq \sum_{j=1}^r \overline{g}_j \quad (\text{for all } r = 1, 2, 3, \cdots) \end{cases}.
\]

**Remark.** The order in Definition 2.9 is different from the order in [Ugl00] (see Example 2.10 below). However, the unitriangularity in (11) holds for both of them.

**Example 2.10.** Let $n = \ell = 2, s = (1, -1), \lambda = ((1, 1), 0)$, and $\mu = (\emptyset, (2))$. Then, $|\lambda; s\rangle = u_2 \wedge u_1 \wedge u_{-1} \wedge u_{-3} \wedge \cdots$ and $|\mu; s\rangle = u_3 \wedge u_1 \wedge u_{-2} \wedge u_{-3} \wedge \cdots$. In Uglov's order, $|\mu; s\rangle$ is greater than $|\lambda; s\rangle$. However, $|\lambda; s\rangle > |\mu; s\rangle$ under our order since $\{\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3\} = \{2, 2, -1\}$ and $\{\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3\} = \{1, 1, 1\}$. 
We define a matrix \((a_{\lambda,\mu}(q))_{\lambda,\mu}\) by

\[
|\lambda; s\rangle = \sum_{\mu} a_{\lambda,\mu}(q) |\mu; s\rangle.
\]

Then the matrix \((a_{\lambda,\mu}(q))_{\lambda,\mu}\) is unitriangular with respect to \(\geq\), that is

\[
\begin{cases}
(a) & \text{if } a_{\lambda,\mu}(q) \neq 0, \text{ then } |\lambda; s\rangle \geq |\mu; s\rangle, \\
(b) & a_{\lambda,\lambda}(q) = 1.
\end{cases}
\]

Thus, by the standard argument, the unitriangularity implies the following theorem.

**Theorem 2.11.** [Ugl00] There exist unique bases \(\{G^{+}(\lambda; s)|\lambda \in \Pi^{\ell}\}\) and \(\{G^{-}(\lambda; s)|\lambda \in \Pi^{l}\}\) of \(F_{q}[s]\) such that

(i) \(G^{+}(\lambda; s) = G^{+}(\lambda; s)\), \(G^{-}(\lambda; s) = G^{-}(\lambda; s)\)

(ii) \(G^{+}(\lambda; s) \equiv |\lambda; s\rangle \mod q \mathcal{L}^{+}\), \(G^{-}(\lambda; s) \equiv |\lambda; s\rangle \mod q^{-1} \mathcal{L}^{-}\)

where \(\mathcal{L}^{+} = \bigoplus_{\lambda \in \Pi^{\ell}} \mathbb{Q}[q]|\lambda; s\rangle\), \(\mathcal{L}^{-} = \bigoplus_{\lambda \in \Pi^{l}} \mathbb{Q}[q^{-1}]|\lambda; s\rangle\).

**Definition 2.12.** Define matrices \(\Delta^{+}(q) = (\Delta_{\lambda,\mu}^{+}(q))_{\lambda,\mu}\) and \(\Delta^{-}(q) = (\Delta_{\lambda,\mu}^{-}(q))_{\lambda,\mu}\) by

\[
G^{+}(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^{+}(q) |\mu; s\rangle, \quad G^{-}(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^{-}(q) |\mu; s\rangle.
\]

The entries \(\Delta_{\lambda,\mu}^{\pm}(q)\) are called \(q\)-decomposition numbers. Note that \(q\)-decomposition numbers \(\Delta^{\pm}(q)\) depend on \(n, \ell\) and \(s\). The matrices \(\Delta^{+}(q)\) and \(\Delta^{-}(q)\) are also unitriangular with respect to \(\geq\).

It is known [Ugl00, Theorem 3.26] that the entries of \(\Delta^{-}(q)\) are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type \(A\), and that they are polynomials in \(p = -q\) with non-negative integer coefficients (see [KT02]).

3. A COMPARISON OF \(q\)-DECOMPOSITION NUMBERS

3.1. Sufficiently large and sufficiently small.

**Definition 3.1.** Let \(s = (s_{1}, s_{2}, \ldots, s_{\ell}) \in \mathbb{Z}^{\ell}\) be a multi charge and \(1 \leq j \leq \ell\).

(i). We say that the \(j\)-th component \(s_{j}\) of the multi charge \(s\) is sufficiently large for \(|\lambda; s\rangle \in F_{q}[s]\) if

\[
s_{j} - s_{i} \geq \lambda^{(0)}_{i} \quad \text{for all } i = 1, 2, \ldots, \ell.
\]

More generally, we say that \(s_{j}\) is sufficiently large for a \(q\)-wedge \(u_{k}\) if

\[
s_{j} \geq c_{r} - nm_{r} \quad \text{for all } r = 1, 2, \ldots,
\]

where \(k_{r} = c_{r} + n(d_{r} - 1) - ntm_{r}, (r = 1, 2, \ldots), 1 \leq c \leq n\) and \(1 \leq d \leq \ell\) (see §2).

(ii). We say that \(s_{j}\) is sufficiently small for \(|\lambda; s\rangle\) if

\[
s_{i} - s_{j} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \quad \text{for all } i \neq j.
\]
Note that the definition of sufficiently small depends only on the size of $\lambda$ and the multi charge $s$. When we fix the multi charge $s$, we say that $s_j$ is sufficiently small for $N$ if

$$s_i - s_j \geq N \quad \text{for all } i \neq j. \quad (16)$$

**Remark.** If $|\lambda; s|$ is 0-dominant in the sense of [Ugl00], that is

$$s_i - s_{i+1} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(f)}| \quad \text{for all } i = 1, 2, \ldots, \ell - 1,$$

then $s_1$ is sufficiently large for $|\lambda; s|$ and $s_{\ell}$ is sufficiently small for $|\lambda; s|$.

**Lemma 3.2.** If $s_j$ is sufficiently large for $|\lambda; s|$ and $|\lambda; s| \geq |\mu; s|$, then

(i) $\lambda^{(j)} = \emptyset$,

(ii) $s_j$ is also sufficiently large for $|\mu; s|$. In particular, $\mu^{(j)} = \emptyset$.

**Proof.** It is clear that $\lambda^{(j)} = \emptyset$ by the definition.

Note that

$s_j$ is sufficiently large for $|\lambda; s| \iff s_j - s_i \geq \lambda^{(s)}_i \quad \text{for all } i = 1, 2, \ldots, \ell$

$$\iff s_j \geq \max(\lambda^{(1)} + s_1, \ldots, \lambda^{(\ell)} + s_\ell) = \lambda_1.$$ If $|\lambda; s| \geq |\mu; s|$, then $\lambda_1 \geq \mu_1$ and so $s_j \geq \mu_1$. It means that $s_j$ is sufficiently large for $|\mu; s|$.

**Lemma 3.3.** Suppose that $s_j$ is sufficiently small for $|\lambda; s|$. If $|\lambda; s| \geq |\mu; s|$ and $\mu^{(j)} = \emptyset$, then $\lambda^{(j)} = \emptyset$.

**Proof.** Suppose that $l(\lambda^{(j)}) \geq 1$. Then $s_j$ is the minimal integer in the set $\{\mu^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ because $\mu^{(j)} = \emptyset$ and $s_j$ is the minimal integer in $s$. On the other hand, the minimal integer in the set $\{\lambda^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ is greater than $s_j$ because $s_j$ is sufficiently small for $|\lambda; s|$. Therefore $|\lambda; s| \not\geq |\mu; s|$. This is a contradiction.

3.2. Main results. Now, we are ready to state our main theorems.

**Theorem 3.4** ([Iij]). Let $\epsilon \in \{+,-\}$. If $s_j$ is sufficiently large for $|\lambda; s|$, then

$$\Delta_{\lambda, \mu, s}^\epsilon(q) = \Delta_{\check{\lambda}, \check{\mu}, \check{s}}^\epsilon(q), \quad (17)$$

where $\check{\lambda}$ (resp. $\check{\mu}$, $\check{s}$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu$, $s$).

**Theorem 3.5** ([Iij]). Let $\epsilon \in \{+,-\}$. If $s_j$ is sufficiently small for $|\mu; s|$ and $\mu^{(j)} = \emptyset$, then

$$\Delta_{\lambda, \mu, s}^\epsilon(q) = \Delta_{\check{\lambda}, \check{\mu}, \check{s}}^\epsilon(q). \quad (18)$$

where $\check{\lambda}$ (resp. $\check{\mu}$, $\check{s}$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu$, $s$).

**Example 3.6.** (i) If $n = \ell = 2$, $s = (3,-3)$ and $\lambda = (0, (6))$, $\mu = (\emptyset, (5,1))$, then $s_1$ is sufficiently large for $|\lambda; s|$. Hence

$$\Delta_{\lambda, \mu, s}^+(q) = \Delta_{\check{\lambda}, \check{\mu}, \check{s}}^+(q) = \Delta_{(6), (5,1), (-3)}^+(q) = -q^{-1}.$$

(ii) If $n = \ell = 2$, $s = (3,-3)$ and $\lambda = ((6), \emptyset)$, $\mu = ((5,1), \emptyset)$, then $s_2$ is sufficiently small for $|\mu; s|$. Hence
\[\Delta_{\lambda,\mu;\nu}(q) = \Delta_{\check{\lambda},\check{\mu};\check{\nu}}(q) = \Delta_{(6),(5,1);(-3)}(q) = -q^{-1}.\]

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